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Test of three-body separable pole expansions at negative and positive energies

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We compare the performance of separable pole expansions for the three-body amplitude in reproducing three-body data at both negative and positive energies. In addition to the generalized unitary pole expansion and the energy dependent pole expansion, we present results for two new expansions. As benchmarks for our calculations, we use the three-boson results of Aaron, Amado, and Yam. At negative energies, we give off-shell scattering amplitudes and investigate how well the expansions are capable of generating the second bound state of the three-boson problem, if the first is put explicitly into the expansions. At positive energies, we calculate phase shifts and inelasticities. We find that the convergence properties of the expansions investigated change rather markedly above the three-particle breakup threshold and only one of the four expansions is capable of reproducing the data in this energy region. In general, the results get better the more energy dependence is explicitly contained in the expansion functions. The relevance of these findings for four-body calculations, which employ separable three-body input, is discussed briefly.

[NUCLEAR REACTIONS Calculations of three-boson bound state and scattering data with separable expansions. Comparison of generalized unitary pole expansion, energy dependent pole expansion, and two new expansions.]

I. INTRODUCTION

The usefulness of separable approximations of the three-body amplitude is largely due to the fact that, when used as input in four-body calculations, they allow one to reduce the four-body problem to an effective two-body problem and thus render it manageable numerically.[1,2] The advantage of exact expansion schemes over separable approximations introduced by other, more phenomenologically motivated means is evident: Exact expansion schemes allow one to improve the approximations in a systematic way, thereby making it possible to define criteria for the convergence of the numerical results.

Although it ought to be quite clear that the final judgment on the applicability of a three-body separable expansion in the four-body problem can only be passed if one actually does a four-body calculation, one can nevertheless gain some insight into the quality of such expansions by investigating their performance in reproducing three-body data *before* one undertakes costly and time-consuming four-body calculations. This is the attitude adopted in the present paper. The expansions to be investigated are the energy dependent pole expansion (EDPE) of Sofianos *et al.*[3] and the generalized unitary pole expansion (GUPE) of Casel *et al.*[4] Furthermore, we introduce in Sec. II two new expansions with certain features which, in the course of the investigation, we found desirable for fast convergence. The EDPE and the GUPE have both already been used successfully in four-body binding energy calculations and were found to produce reliable results.[5] As stated previously, here we want to find out how well they do in reproducing three-body data at negative and positive energies.

Both EDPE and GUPE expansions of the three-body transition amplitude, as well as the two new expansions derived in Sec. II, have the property that, if one three-body bound state exists, they allow the corresponding pole in the amplitude to be put into the expansions explicitly. However, if more than one bound state exist, still only one pole can be explicitly included. Hence, the first question to which we address ourselves in the present paper is whether the expansions are capable of generating the other poles with a reasonable number of terms in the expansion series. This property is of particular relevance for the four-body problem since it is often

argued that it is the pole dominance of the three-body subsystem amplitudes that governs the effect of these amplitudes in four-body calculations.

Next, we investigate the performance of the expansions in reproducing three-body scattering data. In particular, we are interested in the off-shell scattering amplitudes, at various energies, and in phase shifts and inelasticity parameters for s and p waves. From the four-body point of view, these findings become relevant if one thinks of extending existing scattering calculations[2,6–8] to the energy region above the breakup threshold. So far, calculations in this energy region have only been done with simple pole approximations[2,6,7] but never with a full expansion series. The existing calculations employing expansion methods are restricted to energies below the breakup threshold.[8] Our results for the phase shifts and inelasticity parameters indicate that, at positive energies, the performance of the separable expansions on the three-body level is drastically different from what one is used to at negative energies. However, whether this behavior leads to corresponding effects on the four-body level requires further investigations.

The exact three-body model, which we use as a benchmark for our calculations, is the one of Aaron, Amado, and Yam.[9] The treatment of the three-boson case by these authors is based on the nonrelativistic field-theoretical three-body model of Amado.[10] As is well known, this model is identical to the exact three-body theory by Alt, Grassberger, and Sandhas (AGS) (Ref. 11) for the special case of separable two-body interactions. Hence, the AGS four-body theory[1,2] may be thought of as being the one in which the three-body result of Aaron, Amado, and Yam could be used as input (for a field-theoretical four-body model, see Ref. 6, and for its relation to the AGS theory, see Ref. 12).

This paper is organized as follows: In Sec. II, we introduce the GUPE, the EDPE, and the two new expansions. We assume that the reader is familiar with the details of the three-body problem and give all derivations only in a shorthand matrix notation. The results of our calculations are presented in Sec. III. Conclusions are given in Sec. IV. In order to avoid the singularities on the positive real axis, we used the method of contour rotation. Our choice for the path of integration, which is different from the one usually employed,[13] is explained in the Appendix. There we also give the explicit expression for the evaluation of the partial wave projected expansion functions.

II. EXPANSIONS

For the derivation of the separable expansions employed in the present paper, we follow the approach of Ref. 4 and use a shorthand matrix notation. In this matrix notation (its details can be found, e.g., in Refs. 1 and 4; they will not be recapitulated here), the three-body equation by Alt, Grassberger, and Sandhas[11] (AGS) takes the form of a two-body Lippmann-Schwinger (LS) equation[4]:

$$\mathscr{T}(z)=\mathscr{V}(z)+\mathscr{V}(z)\mathscr{G}_0(z)\mathscr{T}(z)\ . \tag{2.1}$$

For simplicity, we assume that the basic two-body forces are separable and act in s waves only (cf. Sec. III). It is well known that, for this special case, the three-body AGS equation is identical to the equation obtained by Amado[10] in a field-theoretical model. Hence, we may use the results of the latter model for comparison. [For the momentum space matrix elements of the effective potential $\mathscr{V}(z)$ and the effective free Green's function $\mathscr{G}_0(z)$ for this special case, see Eqs. (2.20) and (2.18), respectively.]

All the expansions used here are based on Sturmian functions (for other possibilities, see, e.g., Ref. 14). The Sturmian functions $|\,\Gamma^\nu(B)\rangle$ are given as solutions of the LS kernel eigenvalue equation

$$\mathscr{V}(B)\mathscr{G}_0(B)\,|\,\Gamma^\nu(B)\rangle=\eta_\nu(B)\,|\,\Gamma^\nu(B)\rangle\ . \tag{2.2}$$

In principle, the fixed energy B may be any energy below the two-body threshold. In practice, however, if a bound state exists, B is usually chosen to be equal to the corresponding binding energy, the reason being that with this choice any of the following separable expansions for the transition operator $\mathscr{T}(z)$ will reproduce the pole of $\mathscr{T}(z)$ at this bound state within only a single term. (For a detailed investigation of the pole behavior of transition operators, see Ref. 4.) An interesting question in this context is which choice is to be favored if more than one bound state exist. This will be one of the questions dealt with in this paper.

Normalizing the $|\,\Gamma^\nu(B)\rangle$ of Eq. (2.2) according to

$$-\langle\,\Gamma^\mu(B)\,|\,\mathscr{G}_0(B)\,|\,\Gamma^\nu(B)\,\rangle=\delta_{\mu\nu}\ , \tag{2.3}$$

it may be shown that the following very convenient unit operator expansions hold true,[4]

$$1=-\sum_\nu\,|\,\Gamma^\nu(B)\rangle\langle\,\Gamma^\nu(B)\,|\,\mathscr{G}_0(B)\ , \tag{2.4}$$

$$1=-\sum_\nu\,\mathscr{G}_0(B)\,|\,\Gamma^\nu(B)\rangle\langle\,\Gamma^\nu(B)\,|\ . \tag{2.5}$$

The generalized unitary pole expansion (GUPE) of Ref. 4 is now immediately obtained upon multiplying Eq. (2.1) by (2.4) from the left and by (2.5) from the right, viz.,

$$\mathscr{T}(z)=\sum_{\nu,\mu}\,|\,\Gamma^\nu(B)\rangle\tau_{\nu\mu}(B;z)\langle\,\Gamma^\mu(B)\,|\ , \tag{2.6}$$

with the propagator

$$\{\tau^{-1}(B;z)\}_{\nu\mu}=\{\Lambda^{-1}(B;z)\}_{\nu\mu} - \langle \Gamma^{\nu}(B) \mid \mathscr{G}_0(z) \mid \Gamma^{\mu}(B)\rangle \ , \tag{2.7}$$

where

$$\Lambda_{\nu\mu}(B;z)=\langle \Gamma^{\nu}(B) \mid \mathscr{G}_0(B)\mathscr{V}(z)\mathscr{G}_0(B) \mid \Gamma^{\mu}(B)\rangle \ . \tag{2.8}$$

From the representation (2.6), we get the energy-dependent pole expansion (EDPE) of Sofianos *et al.*[3] by rewriting it with the help of Eqs. (2.4) and (2.5) to yield

$$\mathscr{T}(z)=\sum_{\nu,\mu} \mid \tilde{\Gamma}^{\nu}(B;z)\rangle \tilde{\tau}_{\nu\mu}(B;z)\langle \tilde{\Gamma}^{\mu}(B;z) \mid \ , \tag{2.9}$$

with the EDPE propagator

$$\{\tilde{\tau}^{-1}(B;z)\}_{\nu\mu}=\Lambda_{\nu\mu}(B;z) - \langle \tilde{\Gamma}^{\nu}(B;z) \mid \mathscr{G}_0(z) \mid \tilde{\Gamma}^{\mu}(B;z)\rangle \ , \tag{2.10}$$

where we have used the abbreviation

$$\mid \tilde{\Gamma}^{\nu}(B;z)\rangle = \mathscr{V}(z)\mathscr{G}_0(B) \mid \Gamma^{\nu}(B)\rangle \ . \tag{2.11}$$

Both of these expansions have already been used to represent the three-body off-shell input in four-body binding energy calculations[5] with good to very good results. As explained in the Introduction, here we want to find out how good these expansions are in reproducing certain three-body data at positive energies. From the results of Sec. III, it will become clear that both expansions perform very poorly above the three-body breakup threshold. We have therefore investigated two more expansions.

The first (called SE1 in the following) of these two new expansions is given by

$$\mathscr{T}(z)=\sum_{\nu,\mu} \mid F^{\nu}(B;z)\rangle \Delta_{\nu\mu}(B;z)\langle F^{\mu}(B;z) \mid \ , \tag{2.12}$$

with

$$\{\Delta^{-1}(B;z)\}_{\nu\mu}=\langle \Gamma^{\nu}(B) \mid \mathscr{G}_0^P(z)\mathscr{V}(z)\mathscr{G}_0^P(z) \mid \Gamma^{\mu}(B)\rangle - \langle F^{\nu}(B;z) \mid \mathscr{G}_0(z) \mid F^{\mu}(B;z)\rangle \ , \tag{2.13}$$

where

$$\mid F^{\nu}(B;z)\rangle = \mathscr{V}(z)\mathscr{G}_0^P(z) \mid \Gamma^{\nu}(B)\rangle \ . \tag{2.14}$$

The second expansion (called SE2) reads

$$\mathscr{T}(z)=\sum_{\nu,\mu} \mid \tilde{F}^{\nu}(B;z)\rangle \tilde{\Delta}_{\nu\mu}(B;z)\langle \tilde{F}^{\mu}(B;z) \mid \ , \tag{2.15}$$

with

$$\{\tilde{\Delta}^{-1}(B;z)\}_{\nu\mu}=\langle \tilde{\Gamma}^{\nu}(B;z) \mid \mathscr{G}_0^P(z)\mathscr{V}(z)\mathscr{G}_0^P(z) \mid \tilde{\Gamma}^{\mu}(B;z)\rangle - \langle \tilde{F}^{\nu}(B;z) \mid \mathscr{G}_0(z) \mid \tilde{F}^{\mu}(B;z)\rangle \tag{2.16}$$

and the abbreviated notation

$$\mid \tilde{F}^{\nu}(B;z)\rangle = \mathscr{V}(z)\mathscr{G}_0^P(z) \mid \tilde{\Gamma}^{\nu}(B;z)\rangle \ . \tag{2.17}$$

Both of these new expansions can easily be verified by repeated applications of the unit operator expansions (2.4) and (2.5). As compared with the GUPE (2.6), the expansion SE1 (2.12) is obtained by pulling out factors of $\mathscr{V}(z)\mathscr{G}_0^P(z)$ and $\mathscr{G}_0^P(z)\mathscr{V}(z)$ explicitly to the left and right, respectively. [By $\mathscr{G}_0^P(z)$, we denote the principle value part of $\mathscr{G}_0(z)$; see the following.] The same is true for the EDPE (2.9) and SE2 (2.15).

In principle, we could have pulled out explicitly the full LS kernel $\mathscr{V}(z)\mathscr{G}_0(z)$ instead of just $\mathscr{V}(z)\mathscr{G}_0^P(z)$. Let us briefly explain why we have not done so. To this end, we note (see also Ref. 4) that the effective free Green's function $\mathscr{G}_0(z)$ is essentially given by the two-body propagator $t(s)$, i.e., in momentum space we have

$$\langle \vec{q}\,' \mid \mathscr{G}_0(z) \mid \vec{q}\rangle = \delta(\vec{q}\,'-\vec{q})t(z-\tfrac{3}{4}q^2) \ , \tag{2.18}$$

with

$$t^{-1}(s)=\lambda^{-1}-\int_0^{\infty} dp\, p^2 \frac{\mid g(p)\mid^2}{s-p^2} \ , \tag{2.19}$$

where $g(p)$ is the two-body form factor of the separable two-body potential $V=\mid g\rangle\lambda\langle g\mid$. In the three-body Hilbert space, $t(s)$ gives rise to the two-body cut starting at $z=E_d$ (E_d is the deuteron binding energy). Furthermore, due to the integral in Eq. (2.19), $t(s)$ also contributes to the three-body breakup cut starting at $z=0$. This, however, is not the only contribution to this cut; there is also a contribution from the effective potential

$$\langle \vec{q}\,' \mid \mathscr{V}(z) \mid \vec{q}\rangle = \frac{g(\mid \frac{1}{2}\vec{q}\,'+\vec{q}\mid)g(\mid \frac{1}{2}\vec{q}+\vec{q}\,'\mid)}{z-q^2-q'^2-\vec{q}\cdot\vec{q}\,'} \ . \tag{2.20}$$

Taken together appropriately, both of these latter

contributions are required to provide the correct unitarity relation. However, any separable approximation of $\mathscr{T}(z)$ automatically implies via Eq. (2.1) a corresponding approximation of $\mathscr{V}(z)$ also (and vice versa). As a consequence, the contributions to the free cut from (2.18) and from the *separable approximation* of (2.20) will in general not add up appropriately to yield the correct unitarity relation above the breakup threshold. Elastic unitarity, on the other hand, is already given correctly via one factor of $\mathscr{G}_0(z)$ in all of the propagators above. Any other factor of $\mathscr{G}_0(z)$ would lead to a violation of elastic unitarity. We therefore have taken $\mathscr{V}(z)\mathscr{G}_0^P(z)$ instead of $\mathscr{V}(z)\mathscr{G}_0(z)$, because $\mathscr{G}_0^P(z)$ is free of cuts, and we have thus preserved the exact unitarity relation below the three-particle threshold not only for the GUPE and the EDPE, but also for the new expansions (2.12) and (2.15).

III. RESULTS

As described in the Introduction, the aim of this paper is to study how the EDPE and the GUPE, as well as the two new expansions SE1 and SE2 developed in Sec. II, reproduce three-body data at both positive and negative energies. For this purpose, we choose a system of three identical spinless bosons interacting via s-wave separable potentials with Yamaguchi form factors $g(p)=(p^2+\beta^2)^{-1}$, with a range parameter $\beta=1.408$ fm^{-1}. The coupling strength of the potential is taken such that the two-body binding energy is equal to the deuteron binding energy $E_d=-2.226$ MeV. This model three-body problem was first studied by Aaron, Amado, and Yam[9]; it was chosen here because it is both simple to solve numerically and sufficiently rich to provide a good test for the separable expansions introduced in Sec. II.

The exact results for the model three-body problem are obtained by solving the integral equation (2.1) in each partial wave L by inversion. These are then compared with the corresponding results calculated from finite rank versions of the four expansions derived in the preceding section. [In the following, we mean by an N-term expansion that all eigenfunctions of Eq. (2.2) (for its explicit, partial wave decomposed form, see the Appendix), corresponding to the first N eigenvalues in decreasing order of magnitude, are taken into account in the double sums of Eqs. (2.6), (2.9), (2.12), and (2.15).] The presentation of this comparison is given separately for three-body energies E below and above the two-body threshold at $E=E_d$.

A. $E<E_d$

In the s wave the model problem has two three-body bound states, at $E_1=-24.84$ MeV and at $E_2=-2.373$ MeV, and, correspondingly, the exact t matrix acquires two poles at $E=E_1$ and $E=E_2$. An interesting and important question is now how the separable approximations investigated here are able to reproduce *both* poles.

Clearly, in those three-body problems where only one bound state exists, choosing the energy parameter B in Eq. (2.2) equal to the binding energy, automatically guarantees, by construction, that the corresponding pole of the t matrix is already reproduced exactly in the $N=1$ approximation.[4] In the model three-boson case considered here, the same is true for the pole at the ground state energy E_1, if we choose $B=E_1$. In this latter case, the largest eigenvalue in Eq. (2.2) is just the one pertaining to this ground state E_1 and hence the respective first separable term in all four expansions contains the ground state pole explicitly. In the first two columns of Table I, we have listed the values E_2' at which, if at all, the EDPE and the GUPE exhibit a second pole. [It should be noted that this second pole will not be contained explicitly in any of the separable terms of the expansions, but must be generated by *all* eigenfunctions via the inversion of the propagator matrices (2.7) and (2.10).] For a maximum number of six terms, we have found no second pole in the GUPE. The situation is better for the EDPE, where a pole is generated with four terms. However, one needs six terms to come close to the true excited state at E_2.

For the other possibility $B=E_2$, the last two columns of Table I give the values E_1' at which the ground state pole is found. Again, similar to the previous choice, the pole of the excited state E_2 is contained explicitly in the expansions due to the fact that $B=E_2$. Now, however, this pole appears only for $N=2$ because the $N=1$ terms of the separable expansions are based on the eigenfunction of the ground state E_1, but evaluated at $B=E_2$, where its

TABLE I. Position of the second bound state pole, generated by the EDPE and the GUPE, as a function of the number of terms N in the expansions. The first bound state pole is built in explicitly at $E=B$. For $N=1$ and $B=-2.373$ MeV, the values with an asterisk give the position of the only pole in this case (cf. Sec. III). All values are in MeV.

	$B=E_1=-24.84$		$B=E_2=-2.373$	
	E_2'	E_2'	E_1'	E_1'
N	(EDPE)	(GUPE)	(EDPE)	(GUPE)
1			−18.23*	−9.68*
2			−24.75	−23.31
4	−2.228		−24.84	−24.83
6	−2.314		−24.84	−24.84

eigenvalue is larger than the one belonging to the excited state E_2 (which is unity at $B=E_2$, of course). This latter fact, namely that the eigenfunction of the ground state has an eigenvalue larger than unity at $B=E_2$, implies that the one-term approximations must already have a pole at an energy $E_1' < B=E_2$ (see the respective first value of the last two columns of Table I). The next term to be added is based on the eigenfunction of the excited state *at its binding energy* E_2 and hence, as stated previously, contains the corresponding pole by construction. Its effect on the other pole at E_1' is only to shift it closer to the true value E_1. Additional expansion terms further improve the values of E_1', but we see from Table I that the two-term approximations already yield reasonable results; this is particularly true for the EDPE.

In Table II, we show the corresponding results obtained with the new expansions SE1 and SE2. Comparing with the findings listed in Table I, we observe that, in general, fewer terms are needed in order to reproduce the positions of all bound state poles. Note, in particular, the great accuracy with which one is able to generate the ground state pole for the choice $B=E_2$ and $N=2$.

Next, in Figs. 1 and 2, we show the off-shell s-wave scattering amplitude $f_0(E;q,Q)$ as a function of momentum q for different parametric values of energy E and off-shell momentum Q. In general, we find that the choice $B=E_2$ provides better results over a wider energy range than the choice $B=E_1$. Furthermore, the EDPE, for $N=2$, is a better approximation than the GUPE for the same N. With increasing N (and $B=E_2$), the difference between EDPE and GUPE decreases, and for $N=6$ both expansions practically coincide with the exact result. For the choice $B=E_1$, both EDPE and GUPE are similar in quality as long as $E \lesssim E_1$, but differ substantially as E approaches E_2, where the EDPE converges faster. Since four-boson bound states occur at energies below the three-boson bound state E_1, these results seem to indicate that both EDPE and GUPE are good representations of the three-body input for four-body calculations and that in this case the choice $B=E_1$ should be used for better convergence.

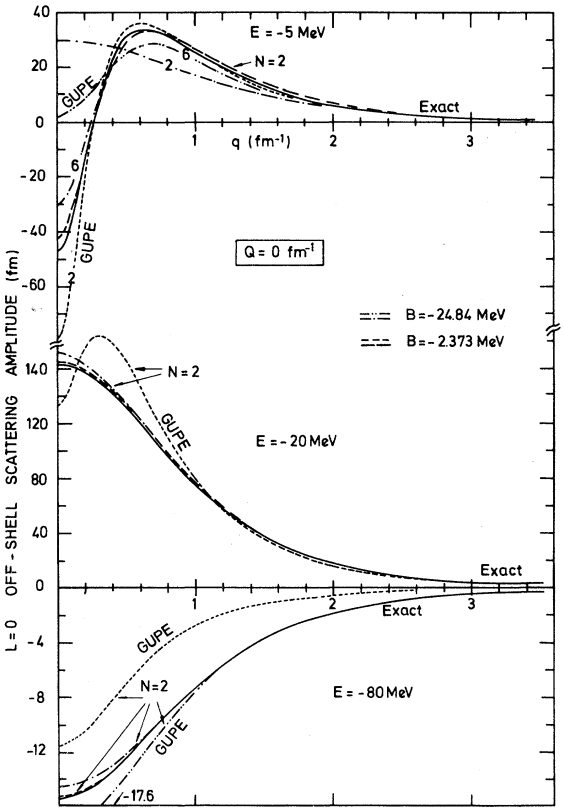

FIG. 1. S-wave off-shell scattering amplitude $f_0(E;q,Q)$, for $Q=0$ fm^{-1}, and various negative energies E, versus off-shell momentum q. The solid lines represent the exact results of Ref. 9. For two different expansion energies B and various numbers of terms in the separable approximations as indicated, the long-dashed and the dashed-dotted lines are EDPE results while the short-dashed and dashed-double-dotted lines are GUPE results.

Similar calculations performed with SE1 and SE2 have not been shown; they coincide with the exact results already for $N=2$, with $B=E_2$ being the better choice.

TABLE II. Same as Table I but for the new expansions SE1 and SE2.

	$B=E_1=-24.84$		$B=E_2=-2.373$	
	E_2'	E_2'	E_1'	E_1'
N	(SE2)	(SE1)	(SE2)	(SE1)
1			−24.43*	−22.83*
2	−2.290		−24.84	−24.82
4	−2.370	−2.350	−24.84	−24.84

B. $E > E_d$

In the scattering region, we make use of the contour rotation method to evaluate the on-shell t matrix for elastic scattering. The path of integration in the momentum variables used in the calculation of the propagator matrix elements (2.7), (2.10), (2.13), and (2.16) is explained in the Appendix; it is different from the one which is usually employed.[13]

In Fig. 3, we show the real part of the $L=0$ elastic phase shifts δ_0 and the inelasticity parameter

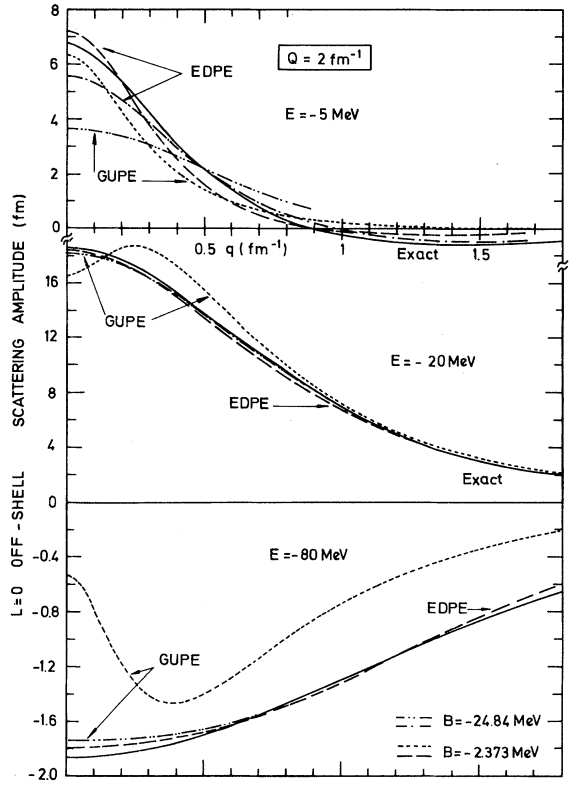

FIG. 2. Same as Fig. 1 with $Q=2$ fm^{-1}.

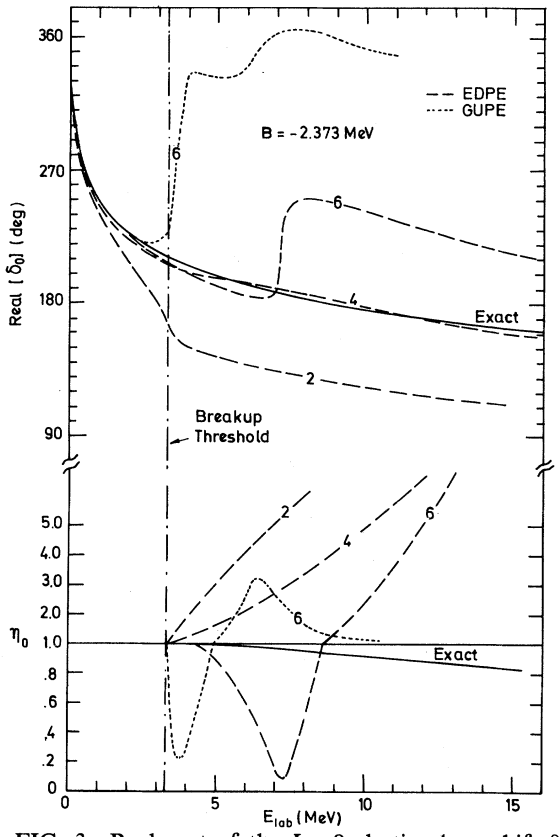

FIG. 3. Real part of the $L=0$ elastic phase shift δ_0 and inelasticity parameter η_0 versus laboratory energy, for the EDPE and the GUPE. The number on each curve gives the number of terms N used in the expansions.

$$\eta_0=\exp[-2\,\mathrm{Im}(\delta_0)]$$

for both the EDPE and the GUPE for different N, and $B=E_2$, as a function of the laboratory energy. The exact result of Ref. 9 is also given for comparison. We find that both expansions converge to the exact result below the three-body breakup threshold but behave poorly above, where the inelasticity parameter η_0 takes on values even greater than unity. This latter fact indicates that, even with as many as six terms, three-particle unitarity is satisfied neither by the EDPE nor the GUPE. In view of these findings, we have looked for more powerful expansion methods and come up with the expansions denoted SE1 and SE2 in Sec. II. In Fig. 4, we have plotted the real part of δ_0 and η_0 for these new expansions. We find that, while both give better results than GUPE or EDPE, only the expansion SE2 is able to reproduce the exact results with great accuracy: Although η_0 may be greater than unity for small N, one recovers the exact three-body result when six terms are included in SE2.

In Fig. 5, we show the results of a similar calculation for $L=1$. (Since there is no bound state in this partial wave, we have chosen $B=E_d$.) Again, the SE2 t matrix with $N=6$ is a good approximation, while the EDPE (and also the GUPE, which is not shown) behaves poorly above the breakup threshold.

We have not plotted any results for the other possible choice of B in the s wave, namely $B=E_1$. We only note that, in general, the convergence is worse than with the choice $B=E_2$.

IV. CONCLUSION

Comparing the GUPE form factors $|\Gamma^{\nu}(B)\rangle$ with the form factors $|\tilde{\Gamma}^{\nu}(B;z)\rangle$ of the EDPE [Eq. (2.11)], $|F^{\nu}(B;z)\rangle$ of the SE1 [Eq. (2.14)], and $|\tilde{F}^{\nu}(B;z)\rangle$ of the SE2 [Eq. (2.17)], we see that, in a systematic way, the latter have more energy dependence than the former. The calculations presented here indicate that, in general, the results become increasingly better the more explicit energy dependence is contained in the form factors of the expansions. Hence, for the purpose investigated here, namely to reproduce three-body data at positive *and* negative energies, expansion SE2 is the best of all four expansions. It is the only one that does not fail above the threshold for three-body breakup. Below

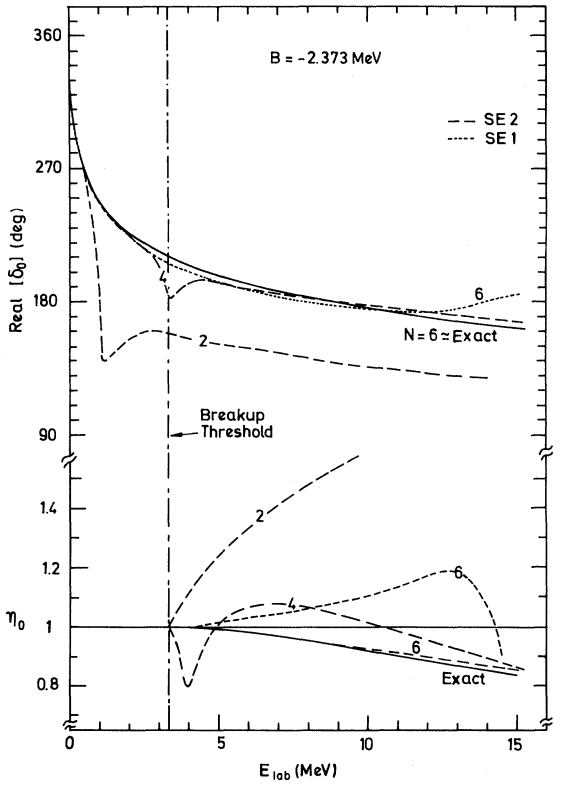

FIG. 4. Same as Fig. 3 for SE1 and SE2.

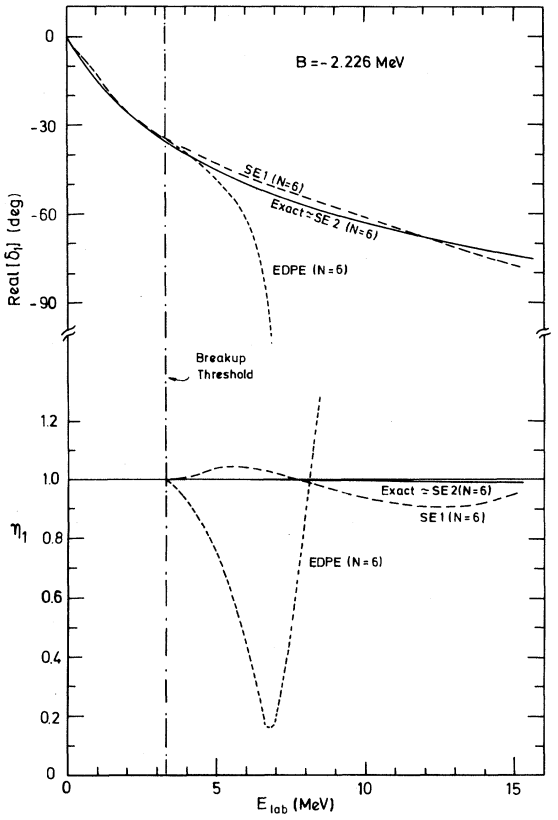

FIG. 5. Same as Fig. 3 for $L=1$ for EDPE, SE1, and SE2 with $N=6$.

this threshold, any of the methods mentioned above converges to the exact three-body t matrix.

Since we have found only one expansion which works above the breakup threshold, the results of our study seem rather discouraging in that energy region. The question is now how much of our findings is due to the model three-boson problem, and would the expansions perhaps behave differently (hopefully better) in the physical trinucleon system? This question can of course be answered finally only by actually doing calculations for the latter system. We have not done so because of the increased numerical difficulties as compared with a three-boson model. However, in the negative energy region the four-body binding energy results of Ref. 5 with three-body GUPE and EDPE input indeed show that the rate of convergence in these calculations improves the more realistic the nucleon-nucleon interaction becomes, and thus it may be hoped that this would also be the case in the three-nucleon system in the positive energy region.

Moreover, as said already in the Introduction, we do not view the separable expansions of the three-body amplitude as a means to be used predominantly for the calculation of three-body data, but as a convenient tool to help numerically solve the four-body problem. Thus, it seems likely that the SE2 will perform well, too, if it is used as input for four-body calculations above the breakup threshold

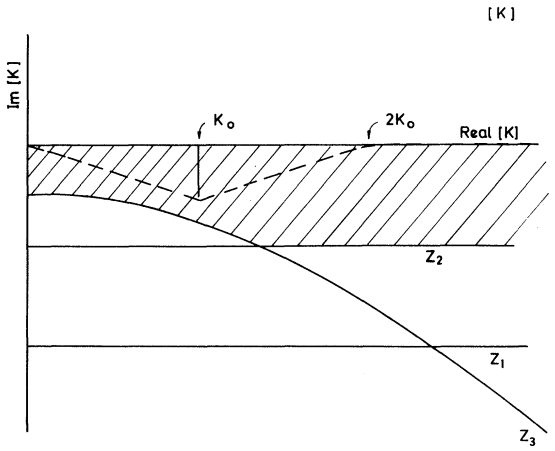

FIG. 6. Singularities of $\mathscr{V}_L(B;k',k)$ in the fourth quadrant of the complex k plane (see the Appendix). The dashed line represents the path of integration used in evaluating the matrix elements in the propagators of the various expansions ($k_0=[\frac{4}{3}(E-E_d)]^{1/2}$ is the elastic on-shell momentum).

(although, of course, this requires numerical tests). However, since the environment in which the three-body amplitude appears in four-body calculations is different (i.e., the three-body amplitude never occurs alone but is always being integrated over within the kernel of the four-body equations), it may be that the expansion SE2 actually is too complicated and that one of the less complicated expansions will do equally well with less numerical effort.

APPENDIX

Equation (2.2) takes in each partial wave L the form (in units of $\hbar=m=1$)

$$\int_0^\infty dk'k'^2\mathscr{V}_L(B;k,k')t(B-\tfrac{3}{4}k'^2)\Gamma_L^\nu(B;k')=\eta_L^\nu(B)\Gamma_L^\nu(B;k)\ , \tag{A1}$$

where the partial wave projection of the effective potential (2.20) is given by

$$\mathscr{V}_L(B;k,k')=\tfrac{1}{2}\int_{-1}^{+1}P_L(x)\frac{g(q)g(q')}{B-k^2-k'^2-kk'x}dx\ , \tag{A2}$$

with

$$\begin{aligned} q&=(\tfrac{1}{4}k^2+k'^2+kk'x)^{1/2}\ ,\\ q'&=(\tfrac{1}{4}k'^2+k^2+kk'x)^{1/2}\ ; \end{aligned} \tag{A3}$$

the $P_L(x)$ is the Legendre polynomial of order L. With the explicit expression for the coupling strength λ, the two-body propagator (2.19) may be rewritten as

$$t^{-1}(s)=(s-E_d)\int_0^\infty dp\,p^2\frac{|g(p)|^2}{(E_d-p^2)(s-p^2)}\ . \tag{A4}$$

In order to evaluate the matrix elements in Eqs. (2.7), (2.10), (2.13), and (2.16) above threshold, one needs to perform momentum variable integrations over the singularities of $\mathscr{G}_0$ and $\mathscr{V}$. This is easily done if one deforms the path of integration away from the singularities into the fourth quadrant of the complex momentum plane. To this end, one has to know the form factors $\Gamma_L^\nu(B;k)$ for complex k. This analytic continuation is achieved numerically by employing Eq. (A1) as a *defining equation* for *complex* k with k' being real. The necessary input, namely $\Gamma_L^\nu(B;k')$ with real k', is obtained by first solving Eq. (A1) as an *integral equation* for *real* k' *and* k. In this procedure, the only limitations on the allowed complex values of k stem from the singularities of $\mathscr{V}_L(B;k,k')$ which depend on B, k', and the range β of the two-body Yamaguchi form factors $g(p)=(p^2+\beta^2)^{-1}$. In the fourth quadrant of the complex k plane, the branch points of $\mathscr{V}_L$ move with k' according to

$$\begin{aligned} z_1(k')&=2k'-i2\beta\ ,\\ z_2(k')&=\tfrac{1}{2}k'-i\beta\ ,\\ z_3(B;k')&=\tfrac{1}{2}k'-i\tfrac{1}{2}\sqrt{|4B-3k'^2|}\ . \end{aligned} \tag{A5}$$

Since k' runs from zero to infinity in Eq. (A1), the complex momentum k can only take on values in the hatched region shown in Fig. 6. Our final choice of path of integration for the evaluation of the propagator matrix elements is given by the dashed line [$k_0=[\frac{4}{3}(E-E_d)]^{1/2}$ is the elastic on-shell momentum].

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