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Model dependence of the $^1S_0$ *pp* scattering length

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We argue that the strong interaction model dependence of the electromagnetically corrected $^1S_0$ *pp* scattering length is less than about 2 fm.

[NUCLEAR REACTIONS Electromagnetic effects in proton-proton scattering.]

I. INTRODUCTION

The search for charge symmetry breaking (CSB) of nuclear forces has been long and difficult.[1] In addition to the expected small sizes of CSB effects there is also a problem in removing irrelevant purely electromagnetic effects. A typical case in point has been the comparison of the scattering lengths of the proton-proton (*pp*) and neutron-neutron (*nn*) system. Once the purely electromagnetic effects, such as the Coulomb force, are removed, charge symmetry predicts the equality of the *pp* and *nn* scattering lengths. However, the uncertainty in our knowledge of the strong force between nucleons has led to great difficulties in subtracting electromagnetic effects.[2–4] The trouble is that the size of the electromagnetic correction seems to depend crucially on the model used to describe the short separation distance nucleon-nucleon wave function. In particular, Sauer and Walliser (SW) have argued that the uncertainties in the $^1S_0$ scattering length due to the model dependence of the proton-proton strong interaction can be large. This is a conclusion different than that of earlier workers (see, e.g., Henley's 1969 review, Ref. 1).

Such a large theoretical uncertainty in the purely hadronic phase shifts presents obvious problems for tests of fundamental calculations (e.g., Ref. 5) of the difference between the proton-proton and neutron-neutron scattering lengths and effective ranges by comparisons with data. Because basic theoretical calculations of charge symmetry breaking and of charge independence violation, based on the mass difference between up and down quarks (or equivalent tadpole effects), are of immediate interest, it is worthwhile to examine just how SW determine the model dependence of the electromagnetic subtraction procedure.

The uncertanties obtained by Sauer and Walliser are arrived at by constructing families of $^1S_0$ hadronic potential ($\tilde{V}$) which, when combined with the electromagnetic interaction, W, yield the same phase shifts as some chosen hadronic reference potential V. That is, $V+W$ and $\tilde{V}+W$ yield the same "experimental" phase shifts. The potentials $\tilde{V}$ differ at short distances and so yield values of the purely hadronic (computed without W) scattering lengths $\tilde{a}$ that are different than the one, a, obtained from V.

In addition, Sauer and Walliser seek to impose another requirement on the set $\tilde{V}$: namely, that any potential employed should be consistent with the nonrelativistic quantum mechanical analysis used to remove the electromagnetic corrections as well as the nonrelativistic treatment of the one photon exchange and vacuum polarization interactions needed to obtain W. This condition is imposed by requiring that the expectation value of the nonrelativistic kinetic energy operator $[p^2/(2\mu)]$ (where μ is the reduced mass of the *pp* system) in the zero energy scattering eigenstate be small compared to $2M_pc^2$. However, even with this limitation, SW find large model dependence in $\tilde{a}-a$.

In the present paper we extend the analysis of Sauer and Walliser by imposing the new condition that the expectation value of the p^4/μ^3 term in the relativisitc kinetic energy operator be small com-

pared with that of the $p^2/2\mu$ term. Such a condition is necessary to ensure the integrity of the nonrelativistic quantum mechanical analysis. Were the eigenfunctions plane waves, this criterion would be satisfied immediately if the p^2 term were small. In that case the ratio of the two terms would be of the order p^2/μ^2, which would be small if the ratio of the nonrelativistic kinetic energy to the rest mass energy of the system were small. Such is not the case, so the use of the p^4 term represents a new limitation on the hadronic potential model. The addition of our nonrelativistic condition to the one of SW leads to a drastic reduction in the model dependence of $\tilde{a}-a$. Indeed for all potentials $\tilde{V}$ we employ 2.0 fm $>\tilde{a}-a>-0.27$ fm. Requiring R_1 to be smaller can have a similar effect.

II. CALCULATIONS

Following Sauer[2] we construct a set of two-nucleon $^1S_0$ interactions $\tilde{V}$, via the formula

$$\tilde{V}=U(K+W+V)U^{\dagger}-K-W\ . \tag{1}$$

Here W is the electromagnetic interaction obtained by the Hermitian nonrelativistic reduction of the Breit interaction by means of the Foldy-Wouthuysen transformation.[6] It includes Coulomb, orbit-orbit, Darwin-Foldy, and spin-spin contact terms. For W we use the expressions given by SW. The operator K is the nonrelativistic kinetic energy operator $K=p^2/2\mu$, and V is a hadronic reference potential. We use the separable Yamaguchi interaction of SW because the corresponding families of $\tilde{V}$ duplicate[7] the particulars of the model dependence of the Coulomb subtraction of Ref. 2. The term U is a unitary operator which reduces to 1 for large pp separations. Therefore $\tilde{V}$ differs from V only at small relative distances. It has long been known that the interaction $\tilde{V}+W$ gives the same prediction of experimental scattering data as the interaction $V+W$. This is because the eigenfunction of the first interaction is obtained by the action of U on the eigenfunction of the second. U is given by[2]

$$\begin{aligned} U&=1-2\,|\,g\rangle\langle g\,|\ , \\ \langle r\,|\,g\rangle&=Nr(1-\beta r)e^{-\alpha r}\ , \\ \langle g\,|\,g\rangle&=1\ . \end{aligned} \tag{2}$$

The model dependence of $\tilde{a}$ is studied by computing scattering lengths derived from the purely hadronic Hamiltonians $\tilde{H}=K_h+\tilde{V}$ as a function of the parameters α and β. The operator K_h is the nonrelativistic kinetic energy term p^2/M_h computed using the hadronic mass M_h which is arbitrarily chosen by SW and by us to be the neutron mass.[8]

The difference between the scattering lengths $\tilde{a}$ (derived from $\tilde{H}$) and a (derived from $H=K_h+V$), is obtained using[2,4]

$$\tilde{a}^{-1}=a^{-1}+\frac{M_h}{\hbar^2}\langle\psi(0)\,|\,U^{\dagger}(K-K_h+W)U-(K-K_h+W)\,|\,\psi(0)\rangle\ . \tag{3}$$

Note that if $K-K_h$ and W were to vanish, then $\tilde{a}=a$. The same holds if $U=1$.

Sauer[2] and Sauer and Walliser[4] compute and compare scattering lengths $\tilde{a}$ for a variety of values of α and β, and thus drastic differences between $\tilde{a}$ and a are obtained. Thus it seems that the purely hadronic scattering length is very sensitive to the interior part of the proton-proton wave function. The largest contributions to $\tilde{a}-a$ are due to the orbit-orbit interaction of W and the operator $K-K_h$. Each of these terms has a strong momentum dependence. Thus large differences in $\alpha-a$ are due to high momentum components of the zero energy wave function $|\,\tilde{\psi}(0)\rangle$ of $\tilde{H}$.

Indeed, SW argue that such large momentum are extraneous because $|\,\tilde{\psi}(0)\rangle$ contains significant components with kinetic energy as large as the rest mass $2M_pc^2$ of the system. To assess the kinetic energy content of $|\,\tilde{\psi}(0)\rangle$ the following method[4] was employed. Consider first the quantity

$$\langle U\psi(0)\,|\,K\,|\,U\psi(0)\rangle(\equiv\langle K\rangle)\ .$$

Because $\psi(0)$ is a zero energy wave function, $\langle K\rangle$ receives contributions only from the interaction region. Furthermore, over this same region, the integral of

$$|\,\langle r\,|\,U\psi(0)\rangle\,|^2$$

is about one half of the effective range r_0 if the effective range normalization is used for $\psi(0)$. Thus one may define a matrix element

$$\begin{aligned} &\langle U\psi(0)\,|\,\bar{K}\,|\,U\psi(0)\rangle \\ &\quad\equiv\langle\psi(0)\,|\,K\,|\,U\psi(0)\rangle/(r_0/2)\ , \end{aligned} \tag{4}$$

which has the dimensions of energy and where r_0 is the effective range. We use $r_0=2.75$ fm but SW use $r_0=2.80$ fm. [Our value of r_0 is closer to the experimental value of r_0 for the neutron-proton system ($^1S_0$ state), and this is known better than the proton-proton effective range.] The dimensionless ratio

$$R_1=\langle\psi(0)\,|\,\bar{K}\,|\,\psi(0)\rangle/(2Mc^2)$$

is then studied as a function of α and β. If Sauer and Walliser eliminate all transformations which lead to values of R_1 greater than 0.1, then the model dependence of the electromagnetically subtracted scattering length is reduced to about 8 fm. If SW take $R_1 < 0.05$, a 3 fm uncertainty is obtained.

Our purpose here is to improve the nonrelativistic criteria by requiring that the p^4 term in the relativistic correction to the nonrelativistic kinetic energy operator be small compared with $p^2/2\mu$. To do this consider the relativistic kinetic energy operator T expressed in the center-of-mass frame,

$$T=[p^2+M_p{}^2]^{1/2}+[p^2+M_p{}^2]^{1/2}-2M_p \approx p^2/2\mu+\tfrac{3}{4}p^4/M_p{}^3 . \quad (5)$$

We then compute the ratio R_2 as

$$R_2 \equiv \frac{\left\langle U\psi(0)\left|\frac{3}{4}\frac{p^4}{M_p{}^3c^2}\right|U\psi(0)\right\rangle}{\langle U\psi(0)\,|\,p^2/2\mu\,|\,U\psi(0)\rangle} . \quad (6)$$

The evaluation of the matrix element

$$\langle U\psi(0)\,|\,p^4\,|\,U\psi(0)\rangle$$

must be treated with some care. The operator p^4 is Hermitian since

$$\langle U\psi(0)\,|\,p^4U\psi(0)\rangle=\langle p^4U\psi(0)\,|\,U\psi(0)\rangle .$$

However, because of the long range nature of $\psi(0)$ it is true that

$$\langle U\psi(0)\,|\,p^4\,|\,U\psi(0)\rangle\neq\langle p^2U\psi(0)\,|\,p^2U\psi(0)\rangle .$$

Nevertheless $|p^4U\psi(0)\rangle$ is well defined and is used in computing the matrix element in (6). A similar problem occurs in evaluating the zero-energy matrix element of

$$p^2=\frac{-\partial^2}{\partial r^2} .$$

Integration by parts also fails, so that

$$\int dr\,\psi_0\frac{\partial^2}{\partial r^2}\psi_0\neq-\int dr\left[\frac{\partial\psi_0}{\partial r}\right]\left[\frac{\partial\psi_0}{\partial r}\right] .$$

Our procedure is to eliminate from consideration all unitary transformations that have values of R_2 that are too large. All values of R_2 greater than about 0.2 are deemed to be too large. To assess whether this twenty percent criterion is reasonable we compute R_2 for several wave functions. For example, with $|\psi(0)\rangle$ equal to the zero-energy wave function of H, we find $R_2=0.043$. Further insight can be obtained by examining R_2 as obtained from the bound state deuteron wave function $|\psi_d\rangle$. To do this $|\tilde{\psi}(0)\rangle$ is replaced in (6) by $|\psi_d\rangle$ obtained with three different potential models. For the Reid[9] soft core wave function we find $R_2=0.20$. The use of a simple square well (of depth 31.28 MeV and range 2.21 fm) in the s wave leads to a value of $R_2=0.003$. The Hulthein wave function (using the parameters of Ref. 10) has $R_2=0.08$. These values of R_2 indicate that the procedure of eliminating unitary transformations with R_2 greater than about 20% is a sensible one—reasonable wave functions do satisfy this criterion.

We now search for a set of values of α and β which give both R_1 less than 0.10 and R_2 less than 0.20. This is done by varying α and β within the range

$$3\ \text{fm}^{-1}\leq\alpha\leq 11.6\ \text{fm}^{-1} , \quad -10\ \text{fm}^{-1}\leq\beta\leq(0\ \text{fm}^{-1}) , \quad (7)$$

with α changing in steps of 0.1 fm^{-1} and β in steps of 0.02 fm^{-1}. Only a discrete set of pairs (α,β) satisfies our restrictions. These values and the corresponding values of $\tilde{a}-a=\Delta a$ are shown in Fig. 1.

A qualitative explanation of these results is as follows. To make R_1 and R_2 small, $|U\psi(0)\rangle$ should not have much high momentum content. However, the function

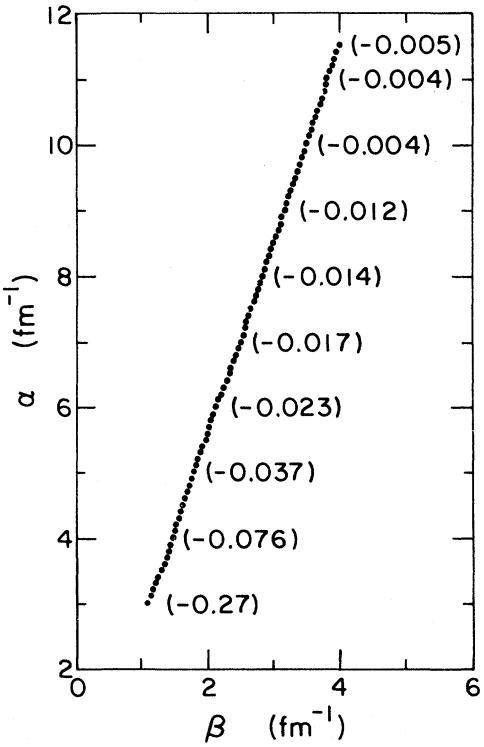

FIG. 1. Values of α and β, Eq. (2), which lead to nonrelativistic potentials $\tilde{V}$. The numbers in parentheses refer to the largest magnitude of $(\tilde{a}-a)$ in fm corresponding to the nearby (α,β) region.

$$\frac{d^2g(r)}{dr^2},$$

which is a measure of the momentum content of $|U\psi(0)\rangle$, has severe cancellations at the peak of $g(r)$ if $\alpha \approx 3\beta$. Because the line formed by the points of Fig. 1 satisfies $\alpha \approx 3\beta$, the effects of the p^2 operator are suppressed for the set shown. It is this reduction which gives the damping of the momentum dependent terms of $W+K-K_h$.

The largest value (in magnitude) of Δa shown in Fig. 1 is $\Delta a = -0.27$ fm which occurs for $\alpha = 3.1$ fm^{-1} and $\beta = 1.12$ fm^{-1}. Thus the model dependence of the electromagnetic subtraction procedure is almost totally removed by imposing the additional nonrelativistic criterion.

It should be noted that there is a set of "near misses" at $\alpha \approx 11$ fm^{-1} and $\beta \approx 7$ fm^{-1}. Such transformations have R_1 only slightly greater than 0.10 (some have R_2 less than 0.20) and values of $\tilde{a}-a$ of about -6 fm^{-1}. Since these transformations almost satisfy the nonrelativistic criteria we must consider the question of whether to keep or discard them. However, for such values of α and β the maximum value of $g(r)$ occurs at about $r=0.14$ fm, and for such small pp separations the description of the system as two individual nucleons must break down. Furthermore, for such large values of α and β it is expected that the p^6 terms in the kinetic energy are very large. For these reasons as well as the fact that the nonrelativistic criteria are not satisfied, we make no further considerations of such transformations.

In order to see if one can find a unitary transformation satisfying both nonrelativistic criteria and having a large value of Δa, we consider yet another infinite set of NN interactions defined by the unitary transformation U_f with

$$U_f = 1 - 2|f\rangle\langle f|, \tag{8a}$$

$$\langle r|f\rangle = N_f r[1+\alpha r/R - (2\alpha+3)(r/R)^2 + (\alpha+2)(r/R)^3] \quad r \le R, \tag{8b}$$

$$= 0 \quad r > R, \tag{8c}$$

with $\langle f|f\rangle = 1$.

This transformation has the property that it is exactly unitary for $r > R$. Furthermore, $|\psi(0)\rangle$ and $|U\psi(0)\rangle$ and its first derivative are continuous at $r=R$. The higher derivatives of $|f\rangle\langle f|\psi(0)\rangle$ are not continuous. If $r \le R$ (8b) is used to compute

$$\frac{d^4}{dr^4}f(r),$$

and if $r > R$,

$$\frac{d^4 f}{dr^4} = 0.$$

The parameters to be varied are α and R, with R the range and α the shape parameter of the change in the wave function introduced by U_f.

To use (8) it is necessary to determine the maximum value of R to be employed. There are two considerations:

(1) Over what range is the NN interaction known? According to Vinh Mau[11] and Partovi and Lomon[12] the theoretically computed NN force is equal to the phenomenological ones at separation distances greater than about 0.8 fm. This suggests using a maximum value of R of 0.8 fm.

(2) The second derivative of $\tilde{\psi}(0)$ is discontinuous at $r=R$. If the Schrödinger equation is to be satisfied, there must be a discontinuity in $\tilde{V}$. For transformations that satisfy the nonrelativistic criteria this turns out to be about $-\hbar^2/\mu R^2$, which for $R=0.8$ fm is about -120 MeV. (The potential is deeper for $r<0.8$ fm.) There is no evidence or expectation that such a discontinuity exists. Nevertheless, in the spirit of this work we cannot rule out the use of U_f. It is true, though, that using $R>0.8$ fm is not valid. This is because such a discontinuity would certainly show up in the one-pion exchange range which is experimentally measurable.

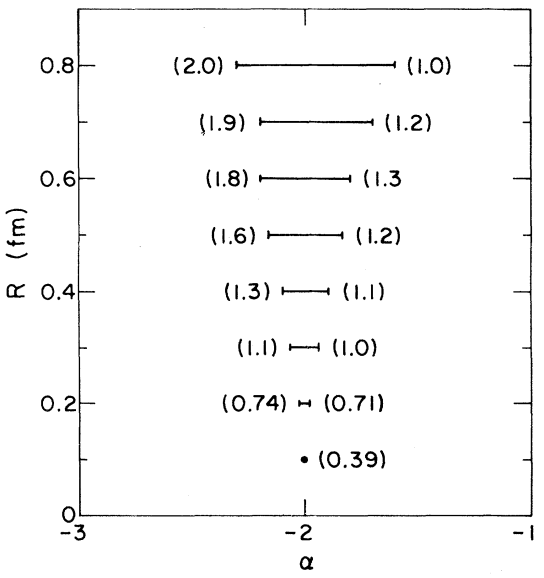

FIG. 2. Values of R and α, Eq. (8), which lead to nonrelativistic potentials $\tilde{V}$. The numbers in parentheses are the values of $\tilde{a}-a$ at the nearest end point of the adjacent line.

For values of

$$0.1 \text{ fm} \leq R \leq 0.8 \text{ fm} ,$$
$$-5 \leq \alpha \leq 5 , \quad (9)$$

with R and α varied in steps of 0.1 we determine a set of parameters (α,R) which satisfies the nonrelativistic criteria. These parameters and the corresponding values of Δa are shown in Fig. 2.

The values of α cluster about -2 because the r^3/R^3 term of $f(r)$ (which is the only term with a nonzero fourth derivative) vanishes for $\alpha=-2$. The largest value (in magnitude) of $\Delta a=2.0$ fm occurs for $R=0.8$ fm and $\alpha=-2.3$. Even this value of a, which is the largest we find, is much smaller than that obtained in Ref. 4.

III. SUMMARY

The use of the criteria that the fourth order term in the expansion of the relativistic kinetic energy operator in p/μ be small compared with the nonrelativistic kinetic energy (second order) operator eliminates many short distance behaviors of the hadronic interaction. An equivalent decrease in the size of the model dependence can also be achieved by using a stricter constraint on the matrix element of the nonrelativistic kinetic energy operator. For the nonrelativistic interactions that satisfy our criteria the model dependence of the electromagnetically subtracted scattering length is less than about 2 fm.

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