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Effective operators in the relativistic meson-nucleon system

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Taking the meson-nucleon system below threshold as an example, it is shown that it can formally be described by nucleonic degrees of freedom alone. In the Fock space of nucleons one has eigenvalue equations with different optional versions of effective interactions. The Hermitian version explicitly exhibits full relativistic invariance. The non-Hermitian form can be decomposed into one, two, three, . . . , particle operators. The latter requires the solution of the vacuum, one, two, . . . , nucleon problems. The techniques proposed are known from coupled cluster many body theory and do not invoke perturbation theory.

[NUCLEAR STRUCTURE Nucleon-nucleon interaction, meson exchange.]

I. INTRODUCTION

It is often desirable to introduce effective operators which exactly describe in principle a given number of physical nucleons without mesonic degrees of freedom. Most of low energy nuclear physics assumes that such a description is possible. The nucleus is considered as a set of $A=N+Z$ nucleons interacting via two (maybe three, four, . . .,) body forces. Indeed it is possible to eliminate the mesonic degrees of freedom, leading to a theory which divides the Hilbert space into two separate parts: a Fock space with nucleons only and an "effective Hamiltonian." The other part contains additional mesons. The separation traditionally is made—using a method due to Okubo[1]—by block diagonalizing the relevant operators with respect to the two subspaces. In this form it has often been used together with perturbation theory. Of course the effective operators are determined by both subspaces, such that a complete solution requires the use of the whole Hilbert space. Still the use of effective operators, i.e., the elimination of mesonic degrees of freedom, offers so many evident advantages that it is worthwhile to explore its structure and possibilities further.

One question is whether these methods are tied to perturbation theory; another is whether a fully relativistic description is possible. Finally, one might wish to explore the structure of the effective Hamiltonian. Is there a decomposition into genuine two body, three body, etc. operators? It is known that, for instance, the usual Hermitian effective two body operator in a three nucleon system depends on the c.m. of the two particles. It thus is influenced by the third nucleon; it is not a genuine two body quantity.

In this paper we shall construct relativistic, nonperturbative effective operators, thereby allowing for a fully relativistic description of the nucleon system below meson threshold without mesonic degrees of freedom. This can be done without invoking perturbation theory[2] and in a much more general form than Okubo's. The ten generators of the Poincaré-Lie group will be considered and the corresponding effective operators will be constructed guaranteeing relativistic covariance. This scheme is necessarily restricted to energies below meson threshold. The procedure is not unique. In particular there are non-Hermitian generators (which can be made Hermitian afterwards, if desirable). These operators do not satisfy the standard Lie algebra; however, they yield eigenvalues and eigenfunctions of the four-momentum. This is the version allowing for a decomposition into effective one, two, three, . . ., body operators. We shall demonstrate this in Sec. III.

The ideas behind this presentation are well known in nonrelativistic many body theory.[3–9] In this paper we adapt them to quantum field theory, which requires the consideration of relativistic covariance as well as the detachment from perturbation theory. The latter aspect is important in view of the fact that nonperturbational methods recently have become important.[10] The many body theory used here is the coupled cluster version,[4–9] which also can be used to derive nonperturbative results.

II. CONSTRUCTION OF EFFECTIVE OPERATORS AND RELATIVISTIC INVARIANCE

Let $\mathscr{H}$ be the Hilbert space of the interacting field theory. We are dealing with the subspace $\mathscr{H}_{\mathscr{P}} \in \mathscr{H}$ of N interacting nucleons with no mesons present; i.e., $\mathscr{H}_{\mathscr{P}}$ contains only states below meson threshold. $\mathscr{H}_{\mathscr{P}}$ can be represented by an infinite set of orthonormal eigenstates of the four-momentum

$$P_\mu \mid \psi(\alpha)\rangle = \pi_\mu(\alpha) \mid \psi(\alpha)\rangle \ , \tag{2.1}$$

with

$$\langle \psi(\alpha) \mid \psi(\beta)\rangle = \delta_{\alpha\beta} \ . \tag{2.2}$$

These states $\psi(\alpha)$ are the (bound or scattering) states of N physical nucleons. The Hilbert space is decomposed:

$$\mathscr{H} = \mathscr{H}_{\mathscr{P}} \oplus \mathscr{H}_{\mathscr{Q}} \ , \tag{2.3}$$

with

$$\mathscr{P} = \sum_\alpha \mid \psi(\alpha)\rangle\langle \psi(\alpha) \mid \tag{2.4}$$

as projection onto $\mathscr{H}_{\mathscr{P}}$ and $\mathscr{Q} = 1 - \mathscr{P}$.

We also consider another set of states

$$\langle \phi(\alpha) \mid \phi(\beta)\rangle = \delta_{\alpha\beta} \ , \tag{2.5}$$

which we later chose to be states of N noninteracting (bare) nucleons, and define

$$\mathscr{P}_0 = \sum_\alpha \mid \phi(\alpha)\rangle\langle \phi(\alpha) \mid \ , \tag{2.6}$$

$$\mathscr{Q}_0 = 1 - \mathscr{P}_0 \ . \tag{2.7}$$

Now we define the partial isometry G by

$$G = \sum_\alpha \mid \psi(\alpha)\rangle\langle \phi(\alpha) \mid \ . \tag{2.8}$$

Thus

$$\delta_{\alpha\beta} = \langle \psi(\alpha) \mid \psi(\beta)\rangle = \langle \phi(\alpha) \mid G^\dagger G \mid \phi(\beta)\rangle \ . \tag{2.9}$$

Note that we do not require G to be unitary. We have

$$\mid \psi(\alpha)\rangle = G \mid \phi(\alpha)\rangle \tag{2.10}$$

and

$$\delta_{\alpha\beta} = \langle \phi(\alpha) \mid \psi(\beta)\rangle \ . \tag{2.9'}$$

Thus

$$GG^\dagger = \sum_\alpha \mid \psi(\alpha)\rangle\langle \psi(\alpha) \mid = \mathscr{P} \ . \tag{2.11}$$

$$G^\dagger G = \sum_\alpha \mid \phi(\alpha)\rangle\langle \phi(\alpha) \mid = \mathscr{P}_0 \ , \tag{2.12}$$

$$[GG^\dagger, P_\mu] = 0 \ . \tag{2.13}$$

Then we define Hermitian effective operators by

$$0_p^{\text{eff}} = G^\dagger 0_p G \ . \tag{2.14}$$

Multiplying from the left by $G^\dagger$ and using (2.8) and (2.12) we obtain from (2.1)

$$P_\mu^{\text{eff}} \mid \phi(\alpha)\rangle = \pi_\mu(\alpha) \mid \phi(\alpha)\rangle \ . \tag{2.15}$$

This is an eigenvalue equation in the nucleon space.

We also need an equation that determines G. This is obtained by projecting (2.1) onto $\mathscr{Q}_0 = 1 - \mathscr{P}_0$:

$$\langle \mathscr{Q}_0 \mid P_\mu G \mid \phi(\alpha)\rangle - \sum_\beta \langle \mathscr{Q}_0 \mid G \mid \phi(\beta)\rangle\langle \phi(\beta) \mid P_\mu^{\text{eff}} \mid \phi(\alpha)\rangle = 0 \ . \tag{2.16}$$

Equations (2.15) and (2.16) are equivalent to the original equation (2.1). So far everything is quite general and there is as of yet no explicit construction of G. We have introduced this general form only to show that Lorentz covariance in the nucleon space does not hinge on special forms of effective interactions.

To see this we recall that there are ten Poincaré-Lie operators L_i with the commutation relations

$$[L_i, L_j] = \sum a_{ij}^k L_k \ , \tag{2.17}$$

where the a_{ij}^k are known coefficients. The construction of the L_i from the stress tensor is standard.[11] Four of the L_i are the four-momenta P_μ (generating translations). There are six rotations, three of them being Lorentz boosts. Consider the mass operator M,

$$M^2 = -P_\mu P_\mu \ , \tag{2.18}$$

which as a scalar commutes with all the L_i. Returning to our problem we recall that we are below meson threshold. Thus we are in a subspace with eigenvalues of M below the lowest eigenvalue of a state with one additional meson. The operators L_i—commuting with M—therefore cannot connect states in $\mathscr{H}_{\mathscr{P}}$ with $\mathscr{H}_{\mathscr{Q}}$ [see Eqs. (2.3) and (2.4)]. Thus

$$[L_i, \mathscr{P}] = [L_i, GG^\dagger] = 0 \ . \tag{2.19}$$

Here we have used (2.11). From (2.14) and (2.17) it follows that

$$[L_i^{\text{eff}}, L_j^{\text{eff}}] = [G^\dagger L_i G, G^\dagger L_j G] = \sum_k a_{ij}^k L_k^{\text{eff}} \ . \tag{2.20}$$

This establishes the desired commutation relations of the effective operators in the nucleon space and thus leads to a relativistic description even after the mesonic degrees of freedom have (formally) been eliminated. Neither perturbation theory nor unitarity of G was necessary for this result.

To obtain an explicit construction of G in terms of known practical methods, we make the following assumptions:

(1) *All* wave functions $\psi(\alpha)\in\mathscr{H}_{\mathscr{P}}$ have a nonvanishing component $\phi(\alpha)$ in the Fock space of N nucleons.

(2) The components $\phi(\alpha)$ are complete in the Fock space of n nucleons, i.e.,

$$\mathscr{P}_0\equiv\sum_\alpha|\phi(\alpha)\rangle\langle\phi(\alpha)| = \int dp_1\cdots dp_N\,|\phi(p_1\cdots p_N)\rangle \times\langle\phi(p_1\cdots p_N)| \;. \tag{2.21}$$

Here

$$\phi(p_1\cdots p_N)=\alpha^\dagger(p_1)\cdots\alpha^\dagger(p_N)\phi_0 \tag{2.22}$$

[with ϕ_0 the bare vacuum; $\alpha^\dagger(p)$ creates a nucleon with momentum p_i; spin and isospin labels are suppressed]. We shall discuss these assumptions later on. With these assumptions we can write $\psi(\alpha)$ as

$$|\psi(\alpha)\rangle=(1+F)\frac{1}{(1+F^\dagger F)^{1/2}}|\phi(\alpha)\rangle \;. \tag{2.23}$$

The square root takes care of the normalization. Comparing with (2.10) we obtain

$$G=(1+F)\frac{1}{(1+F^\dagger F)^{1/2}}\mathscr{P}_0 \;, \tag{2.24}$$

with

$$F=\mathscr{Q}_0F\mathscr{P}_0 \;. \tag{2.25}$$

Thus F creates noninteracting mesons and nucleon-antinucleon pairs. We do not need to write the projection $\mathscr{Q}$ in front of the right-hand sides of (2.23) and (2.24). Owing to (2.1) this operator has no components connecting with $\mathscr{H}_{\mathscr{Q}}$. (In fact, G is now a part of Okubo's unitary operator.) By the same token, the effective operators

$$L_i^{\text{eff}}=\frac{\mathscr{P}_0}{(1+F^\dagger F)^{1/2}}(1+F^\dagger)L_i(1+F) \times\frac{\mathscr{P}_0}{(1+F^\dagger F)^{1/2}} \tag{2.26}$$

again do not need the projections $\mathscr{P}$. The states $\phi(\alpha)$ can be written as

$$|\phi(\alpha)\rangle\equiv\int dp_1\cdots dp\,|\phi(p_1\cdots p_N)\rangle \times\tau(\alpha{:}p_1\cdots p_N) \;. \tag{2.27}$$

The eigenvalue equation (2.15) becomes an equation for the "vector" $\tau(\alpha{:}p_1\cdots p_N)$ and the "equation for F" becomes

$$\mathscr{Q}_0P_\mu(1+F)\frac{1}{(1+F^\dagger F)^{1/2}}|\phi(p_1\cdots p_N)\rangle-\int dp_1'\cdots dp_N'\,\mathscr{Q}_0F\frac{1}{(1+F^\dagger F)^{1/2}}|\phi(p_1'\cdots p_N')\rangle \times\langle\phi(p_1'\cdots p_N')|P_\mu^{\text{eff}}|\phi(p_1\cdots p_N)\rangle=0 \;. \tag{2.28}$$

This equation explicitly exhibits the facts that F depends neither on the eigenvalues nor on the boundary condition of scattering states. Remember, there are two unitarily equivalent sets of states $\psi_+(\alpha)$ and $\psi_-(\alpha)$ in $\mathscr{H}_{\mathscr{P}}$, corresponding to outgoing and ingoing scattered waves. Owing to the second assumption we are allowed to use free nucleon states in $\mathscr{P}_0$ and $\mathscr{Q}_0$ and hence have no asymptotic behavior to watch. This is different in the eigenvalue equation (2.15). Indeed here we must pay due attention to the scattering conditions and have either ϕ_+,τ_+ or ϕ_-,τ_-.

The question may fairly be asked whether the two assumptions are really necessary. With some reservations the answer is yes. If we relax the first assumption, then some $\psi(\alpha)$ would have no component in $\mathscr{H}_{\mathscr{P}_0}$ and these states would be "lost." If the $\phi(\alpha)$ would not be complete the momentum integral in (2.18) [right-hand side of (2.21)] would have to be replaced by a sum over states $\phi(\alpha)$, which we do not know. This would make (2.28) useless and F dependent on the scattering boundary conditions. We expect that above meson threshold the completeness is lost as a new channel is open.

We cannot prove that in a given situation for a given field theory the assumptions are valid as in two body scattering theory, where the completeness is proven for a large class of potentials. However, in a reasonable field theory the $\psi(\alpha)$ and $\phi(\alpha)$—describing asymptotically free particles in a scattering situation—should have the desired features.

Inspecting the zeroth and first order terms, it is found that both assumptions are made implicitly in perturbation theory. The zeroth order assumes that there is a $\phi(\alpha)$ in terms of nucleon states. Specifically, below meson threshold the energy denominators never vanish (no boundary condition). Although perturbation theory supports the theory presented here, it is hoped that the latter is more general, valid also for nonperturbative solutions such as coherent states.

A final remark concerning our first assumption. In all existing field theories—including those we do not need to renormalize—the bare nucleon component in the physical state is infinitesimally small if the normalization volume becomes infinite. This fact concerns very many papers using perturbation theory, especially those using Okubo's transformation. They all would be wrong without this assumption. The reason why—assuming we have a well-understood underlying field theory—they probably are correct is the following. For physical quantities—like the energy—only infinitesimal parts of the wave functions are important. The bare nucleon component is one major infinitesimal contribution among others. Leaving this component out would make the energy grossly incorrect. In other words, we need it, infinitesimally small or not. The fact that only infinitesimal parts of the exact wave function are relevant becomes even more evident if one considers the realities of measurement procedures. For instance, "measuring the energy," we "know" that our object is in the laboratory. In addition, there are some errors involved in the measurement process. In other words, the Hamiltonian and its eigenstates do *not* correspond to realistic measurements of energy. Some averaging, smearing, and/or coarse graining is required. Such a procedure will affect mainly the "complicated" contributions to the wave function. That is to say, only relatively few "simple" components are needed for a "realistic" wave function. Although we cannot prove it, it is assumed that the bare nucleon components now constitute a finite component of the realistic wave function and that we may compute it using the idealized Hamiltonian, wave functions, etc.

The preceding construction of effective operators is not unique. There exist several other versions. Here we write the simplest form we can imagine. It is adapted from the form employed mostly in many body theory.[3-5]

We replace the orthogonal set $\phi(\alpha)$ by a biorthogonal set

$$\mathscr{P}_0=\sum_\alpha |\hat{\phi}(\alpha)\rangle\langle\hat{\hat{\phi}}(\alpha)| \; , \qquad \langle\hat{\hat{\phi}}(\alpha)|\hat{\phi}(\beta)\rangle=\delta_{\alpha\beta} \; . \tag{2.29}$$

The wave function is written as

$$|\psi(\alpha)\rangle=\hat{G}\,|\hat{\phi}(\alpha)\rangle \; , \tag{2.30}$$

with

$$\hat{G}=\mathscr{P}(1+F)\mathscr{P}_0 \; . \tag{2.31}$$

Using essentially the same methods as before we arrive at the eigenvalue (Schrödinger) equation in the bare nucleon space

$$\hat{P}_\mu^{\text{eff}}\,|\hat{\phi}(\alpha)\rangle=\pi_\mu(\alpha)\,|\hat{\phi}(\alpha)\rangle \tag{2.32}$$

and the "equation for F"

$$\mathscr{Q}_0P_\mu(1+F)\,|\hat{\phi}(\alpha)\rangle-\sum_\beta \mathscr{Q}_0F\,|\hat{\phi}(\beta)\rangle\langle\hat{\hat{\phi}}(\beta)|\,\hat{P}_\mu^{\text{eff}}\,|\hat{\phi}(\alpha)\rangle=0 \; , \tag{2.33}$$

with the non-Hermitian effective operator

$$\hat{P}_\mu^{\text{eff}}=\mathscr{P}_0P_\mu(1+F)\mathscr{P}_0 \; . \tag{2.34}$$

$\hat{\phi}(\alpha)$ of course can be written similarly as in (2.34). For later use we rewrite this equation in a more explicit form:

$$\int dp_1'\cdots dp_N'\langle p_1\cdots p_N|\,\hat{P}_\mu^{\text{eff}}\,|p_1'\cdots p_N'\rangle\hat{r}(\alpha{:}p_1'\cdots p_N')=\pi_\mu(\alpha)\hat{r}(\alpha{:}p_1\cdots p_N) \tag{2.35}$$

and

$$\mathscr{Q}_0P_\mu(1+F)\,|p_1\cdots p_N\rangle-\int dp_1'\cdots dp_N'\mathscr{Q}_0F\,|p_1'\cdots p_N'\rangle\langle p_1'\cdots p_N'|\,\hat{P}_\mu^{\text{eff}}\,|p_1\cdots p_N\rangle=0 \; , \tag{2.36}$$

with

$$\langle p_1\cdots p_N|\,\hat{P}_\mu^{\text{eff}}\,|p_1'\cdots p_N'\rangle=\langle\phi(p_1\cdots p_N)|\,P_\mu(1+F)\,|\phi(p_1\cdots p_N)\rangle \; . \tag{2.37}$$

It is clear that the operator F in both versions discussed here is the same. Thus one may use the simpler equation (2.33) instead of (2.28) to obtain F and insert it into (2.15), to retain Hermitian operators $P_\mu^{\rm eff}$ and mutually orthogonal $\phi(\alpha)$. Of course (2.34) may also be made Hermitian. But this does not lead to any simplification.

Equations (2.32) and (2.33) lead to the folded diagram expansion well known in many body theory.[3] The coupled cluster method[5] is another way to treat these same two equations. It is not tied to perturbation theory. The operators $\hat{L}_i^{\rm eff}$ defined in the same manner as $\hat{P}_\mu$ [Eq. (2.34)] do not have the same commutation relations as the L_i, in contrast to the Hermitian choice (2.26). Instead

$$[\hat{L}_i^{\rm eff},\hat{L}_j^{\rm eff}]=\sum_k a_{ij}^k \hat{L}_k^{\rm eff}\hat{G}^\dagger G \ . \tag{2.38}$$

The operators $\hat{L}_i^{\rm eff}$ still represent the translations and rotations in the biorthogonal representation $\hat{\phi}(\alpha)$ of the nucleon states. They seem not to be of any use, however.

III. DECOMPOSITION OF EFFECTIVE OPERATORS

We explicitly construct a series of non-Hermitian effective Hamiltonians for the 1,2, . . . , nucleon problem. We start by assuming that we have solved the vacuum problem, writing the vacuum state preferably in the form

$$|\psi_{\rm vac}\rangle=e^S|\phi_0\rangle \tag{3.1}$$

of the coupled cluster theory.[7] This form is closely related to the Gell-Mann–Low form of the ground state[12] and could be computed within the coupled cluster formalism, $S=\sum S_n$, where S_n creates n mesons and nucleon-antinucleon pairs. S is considered to be known, e.g., by somehow solving the eigenvalue equation in the form

$$e^{-S}He^S|\phi_0\rangle=E_{\rm vac}|\phi_0\rangle \ . \tag{3.2}$$

We now proceed to the one body problem, writing its state (using momentum conservation) as

$$|\psi_p(P)\rangle=e^S(1+F^{(1)})a^\dagger(p)|\phi_0\rangle \ . \tag{3.3}$$

$F^{(1)}$ is the operator to be determined. It can be written as

$$F^{(1)}=\sum_{n,m}F_{n,m}^{(1)}$$

with n/m as the number of created nucleon-antinucleon pairs/mesons. $F^{(1)}$ also changes the momentum of the one nucleon present "before" creation. The following example illustrates its structure:

$$F_{1,1}^{(1)}=\int dp_1dp_2dp_3dp_4dk\, b^\dagger(k)a^\dagger(p_1)\bar{a}^\dagger(p_2)a^\dagger(p_3)a(p_4)F_{1,1}^{(1)}(p_1p_2p_3p_4k) \ .$$

($b^\dagger$ resp. $\bar{a}^\dagger$ create a meson resp. an antinucleon.) $F_{1,1}^{(1)}(p_1\cdots k)$ is the yet unknown amplitude which must include momentum conservation. The Schrödinger equation is written as

$$e^{-S}He^S(1+F^{(1)})a^\dagger(p)|\phi_0\rangle =E_p(1+F^{(1)})a^\dagger(p)|\phi_0\rangle \ . \tag{3.4}$$

e^{-S} is applied to extract from this equation the vacuum state and energy. Projecting this equation onto a state $\phi(p)$ we obtain

$$\langle\phi(p')|e^{-S}He^S(1+F^{(1)})a^\dagger(p)\phi_0\rangle=E_p\delta(p-p') \ . \tag{3.5}$$

We now perform the contractions within the matrix element on the left-hand side of this equation. Collecting all contractions not connected to $F^{(1)}a^\dagger(p)$, we obtain the vacuum energy [compare with (3.2)]. All other contractions lead to completely linked terms, connecting operators directly or indirectly with each other. We denote this part by

$$\{e^{-S}He^S(1+F^{(1)})a^\dagger(p)\}_{\mathscr{L}} \ .$$

This leads to

$$\begin{aligned}\langle p'|\tilde{H}_{\rm eff}^{(1)}|p\rangle&=(E_p-E_{\rm vac})\delta(p-p')\\ &=\epsilon_p\delta(p-p') \ ,\end{aligned} \tag{3.6}$$

with

$$\langle p'|\tilde{H}_{\rm eff}^{(1)}|p\rangle=\langle\phi(p)|\tilde{H}_{\rm eff}^{(1)}|p\rangle$$

and

$$\tilde{H}_{\rm eff}^{(1)}|p\rangle=\{e^{-S}He^S(1+F^{(1)})a^\dagger(p)\}_{\mathscr{L}}|\phi_0\rangle \ . \tag{3.7}$$

Equation (3.6) is the eigenvalue equation for the one body problem. Whereas ϵ_p is known to be $(p^2+M^2)^{1/2}$, it is by no means trivial to calculate $F^{(1)}$. The equation for the latter is obtained by projection of Eq. (3.4) with $\mathscr{Q}_0$ [projection onto states different from the one nucleon states $\phi(p)$]:

$$\langle \mathcal{Q}_0 | \tilde{H}_{\rm eff}^{(1)} | P_0 \rangle - \langle \mathcal{Q}_0 | F^{(1)} | \phi(p) \rangle \langle p | \tilde{H}_{\rm eff}^{(1)} | p \rangle = 0 . \quad (3.8)$$

This equation corresponds to (2.36). It is clear that (3.8) must be solved by some method, either perturbation theory in QED or a scheme truncating beyond a certain number of meson and nucleon-antinucleon pairs. The latter corresponds to the subsystem approximation in coupled cluster theory.[4,5] One also could think of other nonperturbative methods. Here we take the position that we can deal with relativistic quantum field theory, which implies that we at least know what the physical vacuum and one particle states approximately look like. We are aware of the fact that in realistic field theories this is not the case. However, in field theoretical models the situation sometimes is much better.[14] We now proceed to the two nucleon problem, going very much along the same path as before. The wave function is written as

$$|\psi_2(\alpha)\rangle = \int dp_1 dp_2 e^S (1+F^{(1)}+\tfrac{1}{2}{:}F^{(1)2}{:}+F^{(2)}) | \phi(p_1p_2)\rangle \hat{r}(\alpha{:}p_1p_2) . \quad (3.9)$$

The idea behind this ansatz is, of course, that we want to incorporate vacuum fluctuations as well as the "clothing" of the nucleons, such that $F^{(2)}$ carries only those contributions which come from meson exchange between the two particles. The factor $\frac{1}{2}$ is needed to avoid overcounting; the normal product to avoid contractions. $F^{(2)}$ has the same structure as $F^{(1)}$, except that now two nucleons are involved. The eigenvalue equation again is written as

$$\int dp_1 dp_2 e^{-S} H e^S (1+F^{(1)}+\tfrac{1}{2}{:}F^{(1)2}{:}+F^{(2)}) | \phi(p_1p_2)\rangle \hat{r}(\alpha{:}p_1p_2) = E_\alpha \int dp_1 dp_2 (1+F^{(1)}+\tfrac{1}{2}{:}F^{(1)2}{:}+F^{(2)}) | \phi(p_1p_2)\rangle \hat{r}(\alpha{:}p_1p_2) . \quad (3.10)$$

By projecting onto a state $\phi(p_1p_2)$ of two nucleons we obtain the eigenvalue equation in the nucleon space in the form

$$\int dp_1' dp_2' \{ \tfrac{1}{2} \langle p_1p_2 | \tilde{H}_{\rm eff}^{(2)} | p_1'p_2' \rangle + \langle p_1 | \tilde{H}_{\rm eff}^{(1)} | p_1' \rangle \delta(p_2-p_2') + \langle p_2 | \tilde{H}_{\rm eff}^{(1)} | p_2' \rangle \delta(p_1-p_1') \} \hat{r}(\alpha{:}p_1'p_2') = E_2(\alpha) \hat{r}(\alpha{:}p_1p_2) , \quad (3.11)$$

where

$$\tilde{H}_{\rm eff}^{(2)} | p_1p_2 \rangle = \{ e^{-S} H e^S (1+F^{(1)}+\tfrac{1}{2}{:}F^{(1)2}{:}+F^{(2)}) a^\dagger(p_1) a^\dagger(p_2) \}_{\mathcal{L}} | \phi_0 \rangle . \quad (3.12)$$

Equation (3.12) has been obtained in a similar way as before: It contains all contractions producing completely linked structures. Those linking only with one $a^\dagger(p)$ produce the one particle operators and wave equations, those linking with no $a^\dagger(p)$ produce the vacuum. On the left-hand side a decomposition of the total effective two body operator $\hat{H}_{\rm eff}^{(2)}$ into a two body part and one body part has been achieved:

$$\hat{H}_{\rm eff}^{(2)} = \tilde{H}_{\rm eff}^{(2)} + \sum_{i=1,2} \tilde{H}_{\rm eff}^{(1)}(i) . \quad (3.13)$$

The equation for $F^{(2)}$ is obtained as before by projecting onto states different from $\phi(p_1p_2)$:

$$\langle \mathcal{Q}_0 | \tilde{H}_{\rm eff}^{(2)} | p_1p_2 \rangle - \int dp_1' dp_2' \{ \tfrac{1}{2} \langle \mathcal{Q}_0 | F^{(1)}+\tfrac{1}{2}{:}F^{(1)2}{:}+F^{(2)} | \phi(p_1'p_2') \rangle \langle p_1'p_2' | \tilde{H}_{\rm eff}^{(2)} | p_1p_2 \rangle + \langle \mathcal{Q}_0 | F^{(2)} | p_1'p_2' \rangle [\langle p_1' | \tilde{H}_{\rm eff}^{(1)} | p_1 \rangle \delta(p_2-p_2') + \langle p_2' | \tilde{H}_{\rm eff}^{(1)} | p_2 \rangle \delta(p_1-p_1')] \} = 0 . \quad (3.14)$$

We may proceed to the three particle problem; as no new ideas and methods enter, we just state the main results. The wave function now is

$$|\psi_3(\alpha)\rangle = \int dp_1 dp_2 dp_3 e^S (1+F^{(1)}+\tfrac{1}{2}{:}F^{(1)2}{:}+{:}F^{(1)}F^{(2)}{:}+\frac{1}{3!}{:}F^{(1)3}{:}+F^{(3)}) | \phi(p_1p_2p_3)\rangle \hat{r}(\alpha{:}p_1p_2p_3) . \quad (3.15)$$

The total effective interaction becomes

$$\hat{H}_{\rm eff}^{(3)} = \tilde{H}_{\rm eff}^{(3)} + \tfrac{1}{2} \sum_{i,j}^{3} \tilde{H}_{\rm eff}^{(2)}(i,j) + \sum_{i=1}^{3} \tilde{H}_{\rm eff}^{(1)} , \quad (3.16)$$

with

$$\widetilde{H}^{(3)}_{\rm eff}\,|p_1p_2p_3\rangle=\{e^{-S}He^{S}(1+F^{(1)}+\tfrac{1}{2}{:}F^{(1)^2}{:}+F^{(2)}{:}\frac{1}{3!}{:}F^{(1)^3}{:}+F^{(3)})a^\dagger(p_1)a^\dagger(p_2)a^\dagger(p_3)\}_{\mathscr{L}}\,|\phi_0\rangle\ .$$

We may extract from $\hat{H}^{(3)}_{\rm eff}$ via (3.16) and (3.6) the one particle energies obtaining

$$\int dp_1'dp_2'dp_3'\,\langle p_1p_2p_3\,|\,\widetilde{H}^{(3)}_{\rm eff}+\tfrac{1}{2}\sum_{ij}\widetilde{H}^{(2)}_{\rm eff}(ij)\,|\,p_1'p_2'p_3'\rangle\hat{\tau}(\alpha{:}p_1'p_2'p_3')=[E_3(\alpha)-\epsilon_{p_1}-\epsilon_{p_2}-\epsilon_{p_3}]\hat{\tau}(\alpha{:}p_1p_2p_3)\ . \tag{3.17}$$

Again the desired decomposition is obtained. $\widetilde{H}^{(3)}_{\rm eff}$ consists of a three body part, two body parts, and (trivial) one body parts, where the two body part is known if the two body problem has been solved.

It is clear that one may go on to higher nucleon numbers in the same way. A general theory for the corresponding many body problem is described in Ref. 5. We could have used the same procedure for the momentum operators as well. We remark here that we have used this special non-Hermitian version— closely related to (2.34)—because in the Hermitian version with operators (2.26) the factor $(1+F^\dagger F)^{-1/2}$ with

$$F=F^{(1)}+\tfrac{1}{2}{:}F^{(1)^2}{:}+F^{(2)}$$

mixes two and one body effects in such a way that they cannot be disentangled. In this case the nice feature that, e.g., the two body force to be used in the three body problem is the force of the two body problem, is lost. As all versions are equivalent, finally it must be possible to reshuffle, e.g., the perturbation series such that the decomposition is recovered—but this is a long way to go. For obvious reasons it is desirable to extract from, e.g., the three body equation, what we "knew" already about the vacuum, and the one and two body problems. This can be achieved only with the non-Hermitian form.

It is important to recall that the operators F do not depend on the state to be considered. They do depend only on the number of nucleons (or the eigenvalue of the mass operator). Thus the effective operators also do not depend on the eigenvalues $\pi_\mu(\alpha)$. Knowing that, for instance, the center of mass momentum is a good quantum number, we immediately realize that the effective operators do not depend on it. In the case of non-Hermitian operators [nonorthogonal $\hat{\phi}(\alpha)$] this leads to desirable properties. Take, for instance, the two body case

$$\psi_2(\alpha)\rangle=e^{S}(1+F)\hat{\phi}(\alpha)\rangle\ , \tag{3.18}$$

with F given by (3.9). It is easily seen that the $\hat{\tau}(\alpha{:}p_1p_2)$ of

$$|\,\hat{\phi}(\alpha)\rangle=\int dp_1dp_2\,|\,\phi(p_1p_2)\rangle\hat{\tau}(\alpha{:}p_1p_2) \tag{3.19}$$

have the same translational symmetry as if there were no mesons, i.e.,

$$\hat{\tau}[\pi_i(\alpha){:}p_1p_2]=\delta[p_1+p_2-\pi_i(\alpha)]\times\hat{\tau}[\pi_i(\alpha){:}p_1-p_2]\ , \tag{3.20}$$

where $\pi_i(\alpha)$ is the center of mass momentum. We use the fact that in the interaction representation at $t=0$ the momentum can be written as

$$P_i=P_i(N)+P_i(m)\ , \tag{3.21}$$

i.e., a sum of "free" nucleon and meson momentum operators. Inserting this into the eigenvalue equation for the momentum

$$\langle\phi(p_1p_2\,|\,e^{-S}P_i\psi_2(\alpha)\rangle=\pi_i(\alpha)\hat{\tau}[\pi_i(\alpha){:}p_1p_2] \tag{3.22}$$

we obtain for all (p_1p_2)

$$\pi_i(\alpha)\hat{\tau}[\pi_i(\alpha){:}p_1p_2]=(p_1+p_2)\hat{\tau}[\pi_i(\alpha){:}p_1p_2]\ ,$$

which yields (3.20). (The introduction of the vacuum amplitude S is not essential, such that this result is quite general.) The orthogonal $\phi(\alpha)$ or their expansion coefficients $\tau(\alpha{:}p_1p_2)$ are related to the nonorthogonal ones via

$$\hat{\tau}(\alpha{:}p_1p_2)=\int dp_1'dp_2'\,\langle\phi(p_1p_2)\,|\,\frac{1}{(1+F^\dagger F)^{1/2}}\,|\,\phi(p_1'p_2')\rangle\tau(\alpha{:}p_1'p_2') \tag{3.23}$$

and therefore do not have property (3.20). This is not surprising, as the $\hat{\phi}(\alpha)$ are the direct bare nucleon components of $\psi(\alpha)$ [see (2.30)], whereas the $\phi(\alpha)$ are not [see (2.23)]. In this way it is made clear why the non-Hermitian effective operators have somewhat simpler features, allowing for a decomposition into genuine one, two, . . ., body operators.

IV. SUMMARY AND CONCLUSIONS

We have obtained a (formally exact) relativistic representation of the meson-nucleon system below meson threshold in the space of nucleonic degrees of freedom only. One has either Hermitian effective operators (interactions) acting in this space with all desirable transformation properties under the Poincaré group, or non-Hermitian ones with the other desirable feature of allowing for a decomposition into one, two, three, etc., particle operators. No perturbation theory has been used. This leads to a program requiring the solution of the vacuum, one body, two body, etc., problem. The underlying ideas borrowed from the coupled cluster many body theory seem to be helpful. Solution of the set of coupled cluster equations via perturbation theory leads to the well-known folded diagram expansion.[4–6] Indeed the recent paper by Li *et al.*[13] is a special case of the general method described here.

A recent study of the Goldstone boson[14] has shown that the methods proposed in this work can be applied sucessfully. In this model the renormalization problems easily can be overcome. The one particle state can be computed using the standard coupled cluster truncation. The results are encouraging up to large coupling constants. It is remarkable that the violation of relativistic invariance due to this truncation is consistent with the errors introduced via this approximation. Note that one should not expect exact Lorentz invariance. On the contrary, forcing an approximate wave function into an invariant form may make the wave function worse.

It is clear that for realistic quantum field theories there are several obstacles we have to overcome. First, one has to reorganize standard renormalization procedures away from perturbative methods to coupled cluster structures. This is not easy, but we believe that it can be done. Second, some good arguments are needed to justify the coupled cluster truncation scheme. There are several such arguments in nonrelativistic many body theory. None of them seem to apply to relativistic field theories—except numerical evidence for good convergence (which may be deceiving).

Although many problems have still to be solved, the knowledge that effective operators with desirable features exist has helped to guide calculations. Hopefully it will do the same in future applications.

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