Separable approximations of two-body interactions

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We perform a critical discussion of the efficiency of the Ernst-Shakin-Thaler method for a separable approximation of arbitrary two-body interactions by a careful examination of separable $^3S_1$-$^3D_1$ N-N potentials that were constructed via this method by Pieper. Not only the on-shell properties of these potentials are considered, but also a comparison is made of their off-shell characteristics relative to the Reid soft-core potential. We point out a peculiarity in Pieper's application of the Ernst-Shakin-Thaler method, which leads to a resonant-like behavior of his potential 3SD1D. It is indicated where care has to be taken in order to circumvent drawbacks inherent in the Ernst-Shakin-Thaler separable approximation scheme.

[NUCLEAR REACTIONS Critical discussion of the Ernst-Shakin-Thaler separable approximation method. Pieper's separable N-N potentials examined on shell and off shell.]

I. INTRODUCTION

In recent years it became necessary to include the vast complexity of two-body interactions into calculations of few-body systems in order to arrive at reasonable and conclusive results. Therefore one had to struggle with the difficulties associated with the introduction of realistic two-body forces, which in general are quite complicated. For example, with respect to the N-N system it is rather troublesome, if not impossible, to use in few-body applications modern interaction models, which were derived from first dynamical principles and have commonly been desired because of their appealing features. Separable approximation schemes always served as an essential tool to overcome these difficulties. Unfortunately the resulting separable potentials did not always resemble the properties of "true" two-body interactions in a satisfactory way. They did not only fall short in their on-shell results, but above all suffered from unrealistic off-shell characteristics.[1] Therefore separable representations that allowed one to reproduce the on- as well as off-shell properties of the model potential became important. A particular method of that type is provided by the approach of Ernst, Shakin, and Thaler[2] (EST). In the literature this method was already used for practical applications, e.g., by Pieper.[3] Unfortunately, like other separable approximation techniques[4,5] the EST method is also not free of deficiencies which might lead to spurious results. Indeed, in Pieper's work not enough care was taken in the construction of the separable potentials. This led to unphysical properties, particularly in his 3SD1D model. This observation made us perform a critical examination of the abilities of various separable approximation schemes.

In the present work we will concentrate on the EST method. In Sec. II we give a brief account of the formalism generalized to arbitrary (spin-dependent) forces acting in coupled partial waves. There a few hints are already made concerning a competent application of the method. In Sec. III a discussion of the results (re)calculated for Pieper's potentials is presented. We also consider the half-off-shell behavior of his separable potentials and compare them to the Reid soft-core (SC) potential.[6] In the final section we summarize the abilities of the EST method and point out at which stages care has to be taken in order to avoid unreasonable results.

II. THE EST METHOD

A. Outline of the formalism

Consider the general partial-wave decomposition of the Lippmann-Schwinger equation for the wave function belonging to some N-N potential V with spin-dependent forces (tensor force, spin-orbit force, etc.). For coupled partial waves with angular momenta $l_< = J-1$ and $l_> = J+1$ (J being the total angular momentum), this equation reads

$$|\psi_{EIL}^{(S)}\rangle = |\phi_{EL}\rangle\delta_{IL} + \sum_{L'=l_<,l_>} G_{0,L}^{P}(E)V_{LL'}|\psi_{EIL'}^{(S)}\rangle \quad (2.1)$$

for $L=l_<,l_>$. The label l, which can assume the values $l_<$ and $l_>$, designates the initial-state angular momentum. Though in Eq. (2.1) and throughout this paper we adopt standing-wave boundary conditions—according to the use of the principal-value Green's operator $G^P_{0,L}(E)$—we emphasize that the method and the results do not depend on the specific boundary conditions employed.

It is convenient to introduce a matrix notation so that one can combine the coupled wave functions $|\psi^{(S)}_{ElL}\rangle$ two by two into the kets $|\Psi^{(S)}_{El}\rangle$, which then represent two-dimensional column matrices. The potential and resolvent operators become (2×2) matrices denoted by $\mathscr{V}$ and $\mathscr{G}^P_0$, respectively.

The EST method aims at the construction of a separable potential $\tilde{\mathscr{V}}$ that resembles the on- as well as off-shell properties of some arbitrary given potential $\mathscr{V}$. For the general case of coupled partial waves it is designed to reproduce exactly the half-off-shell elements of the reaction matrix $\mathscr{R}(E)$ pertaining to the potential $\mathscr{V}$ at selected energies. Let the ensemble $\alpha_i=\{E_i l_i\}$ denote an energy point E_i in the particular channel characterized by the initial angular momentum l_i. Then the equality of the half-off-shell elements of the reaction matrix components $\tilde{R}_{Ll_i}(E_i)$ and $R_{Ll_i}(E_i)$ corresponding to the potentials $\tilde{\mathscr{V}}$ and $\mathscr{V}$, respectively, is achieved by demanding that the form factors of the separable potential consist of the objects $\mathscr{V}|\Psi_{\alpha_i}\rangle$, where $|\Psi_{\alpha_i}\rangle$ is a solution of the Lippmann-Schwinger equation with the model potential $\mathscr{V}$. Hence the separable potential of the form

$$\tilde{\mathscr{V}}=\sum_{i,j=1}^{N}\mathscr{V}|\Psi_{\alpha_i}\rangle M_{ij}\langle\Psi_{\alpha_j}|\mathscr{V} \tag{2.2}$$

guarantees that for N combinations α_i $(i=1,\ldots,N)$

$$\tilde{\mathscr{V}}|\Psi_{\alpha_i}\rangle=\mathscr{V}|\Psi_{\alpha_j}\rangle=\mathscr{R}(E_i)|\Phi_{\alpha_i}\rangle\ , \tag{2.3}$$

if the coupling parameters M_{ij} fulfill the condition

$$\sum_{j=1}^{N}M_{ij}\langle\Psi_{\alpha_j}|\mathscr{V}|\Psi_{\alpha_k}\rangle=\delta_{ik}\ . \tag{2.4}$$

We remark that E_i can as well be chosen to be a bound-state energy, of the deuteron, say. This case is contained in the formalism if the inhomogeneous term is omitted in Eq. (2.1). The EST method would then lead to a separable potential $\tilde{\mathscr{V}}$ with bound-state wave functions $|\tilde{\Psi}_{E_i}\rangle$ being exactly the same as $|\Psi_{E_i}\rangle$ pertaining to the potential $\mathscr{V}$. Note the similarity of this special case (namely, if $\tilde{\mathscr{V}}$ is given only rank 1) to the unitary pole approximation (UPA).

From Eq. (2.4) it is already seen that some care has to be taken in the selection of the ensembles $\{E_i l_i\}$. Since evaluating the coupling parameters M_{ij} requires inversion of the matrix $\langle\Psi_{\alpha_j}|\mathscr{V}|\Psi_{\alpha_k}\rangle$, one must beware of choosing such α_i, for which this matrix becomes singular. We will come back to this and similar drawbacks in the next section when discussing the practical application of the EST method.

The objects needed for the evaluation of the (off-shell) matrix elements of $\tilde{\mathscr{R}}(E)$ (and equivalently of $\tilde{\mathscr{V}}$) can be calculated using the half-off-shell R matrices belonging to the potential $\mathscr{V}$ taken at the fixed preselected energies E_i.[7] The final solution for the matrix elements of $\tilde{\mathscr{R}}(E)$ reads:

$$\begin{aligned}\tilde{R}_{L'L''}(E',E'';E)&=\langle\Phi_{E'L'}|\tilde{\mathscr{R}}(E)|\Phi_{E''L''}\rangle\\&=\sum_{i,j=1}^{N}\langle\Phi_{E'L'}|\mathscr{V}|\Psi_{\alpha_i}\rangle\\&\quad\times D_{ij}(E)\langle\Psi_{\alpha_j}|\mathscr{V}|\Phi_{E''L''}\rangle\end{aligned} \tag{2.5}$$

for $L',L''=l_<,l_>$. The matrix elements $D_{ij}(E)$ are defined by

$$\mathscr{D}(E)=[\mathscr{M}^{-1}-\mathscr{G}(E)]^{-1}=[1-\mathscr{M}\mathscr{G}(E)]^{-1}\mathscr{M}\ , \tag{2.6}$$

where

$$G_{ij}(E)=\langle\Psi_{\alpha_i}|\mathscr{V}\mathscr{G}^P_0(E)\mathscr{V}|\Psi_{\alpha_j}\rangle\ . \tag{2.7}$$

B. Discussion of the method

In this subsection we will briefly explain how the formalism just outlined can be employed for various purposes. Of course, we will mainly have in mind application to the N-N system, since we will later on examine in detail the work by Pieper. By the way, we will thus also have occasion to make evident the potency as well as the limitations of the EST method.

Let us first have a look at uncoupled partial waves $L=L'=l$. If $\tilde{V}_L$ is required to reproduce the half-off-shell R matrix at N energy points E_i $(i=1,\ldots,N)$, it follows from Eq. (2.3) that the separable potential has to be of rank N (Ref. 8):

$$\tilde{V}_L(k',k)=\sum_{i,j=1}^{N}g_{Li}(k')M_{ij}g_{Lj}(k) \tag{2.8a}$$

with form factors

$$g_{Ln}(k)=\langle\phi_{EL}|V_L|\psi_{E_nL}\rangle\quad E=\frac{k^2\hbar^2}{2\mu}\ . \tag{2.8b}$$

From the latter equation it is clearly evident that the form factors of separable potentials constructed by

means of the EST method consist of the half-off-shell elements of the R matrix of the model potential V taken at the selected energies. If the coupling parameters M_{ij} are determined via Eq. (2.4), it is guaranteed that

$$\tilde{R}_L(E,E_n;E_n)=R_L(E,E_n;E_n)$$

for all E.

For the case of coupled partial waves, in addition to selecting energies E_i it is necessary to discern between the possible initial configurations l_i. To demonstrate this let us first consider the simplest choice for $\tilde{\mathscr{V}}$ allowed by Eq. (2.2), namely, with only one ensemble $\alpha_1=\{E_1 l_1\}$. From Eq. (2.2) one sees immediately that two form factors occur, which are given by

$$\begin{aligned} g_{l_< 1}(k)&=\langle \Phi_{El_<} \mid \mathscr{V} \mid \Psi_{\alpha_1}\rangle \\ &=\langle \phi_{El_<} \mid V_{l_< l_<} \mid \psi_{E_1 l_1 l_<}\rangle \\ &\quad+\langle \phi_{El_<} \mid V_{l_< l_>} \mid \psi_{E_1 l_1 l_>}\rangle , \end{aligned} \tag{2.9a}$$

$$\begin{aligned} g_{l_> 1}(k)&=\langle \Phi_{El_>} \mid \mathscr{V} \mid \Psi_{\alpha_1}\rangle \\ &=\langle \phi_{El_>} \mid V_{l_> l_<} \mid \psi_{E_1 l_1 l_<}\rangle \\ &\quad+\langle \phi_{El_>} \mid V_{l_> l_>} \mid \psi_{E_1 l_1 l_>}\rangle \end{aligned}$$

$$E=\frac{\hbar^2 k^2}{2\mu} . \tag{2.9b}$$

These form factors enter into the separable partial-wave potential in the following way:

$$\begin{aligned} &\tilde{V}_{LL'}(k,k')=g_{L1}(k) M g_{L'1}(k') ; \\ &L,L'=l_<,l_> . \end{aligned} \tag{2.9c}$$

Here, according to Eq. (2.4), the coupling parameter M, which is independent of the orbital angular momenta L,L', takes the value

$$M=(\langle \Psi_{\alpha_1} \mid \mathscr{V} \mid \Psi_{\alpha_1}\rangle)^{-1} .$$

However, such a choice for $\tilde{\mathscr{V}}$ amounts to the particular case where only two of the four R-matrix elements, viz., the ones corresponding to the same initial angular momentum l_1, are reproduced at the energy E_1. Then the equality

$$\tilde{R}_{LL'}(E,E_1;E_1)=R_{LL'}(E,E_1;E_1)$$

holds for only one particular value $L'=l_1$.

Thus for coupled partial waves a rank-1 separable potential as in Eqs. (2.9) is in general not able to reproduce all the off-shell information available at a single (scattering) energy.

In view of the correspondence between the form factors and the half-off-shell matrix elements [cf. Eqs. (2.9)] it is evident that four form factors are needed if one wants the separable potential to reproduce all four R-matrix elements of the model potential at a single energy E_1. Therefore $\tilde{\mathscr{V}}$ has to be at least of rank 2. The ensembles α_i to be selected for this particular case are

$$\alpha_1=\{E_1,l_1=l_<\} ,$$

$$\alpha_2=\{E_1,l_2=l_>\} .$$

It turns out that in practical applications, especially for the N-N system, the rank N can be kept relatively low. Thus it is often not necessary to demand all four R matrices to be reproduced at every energy desired. It happens that fixing a half-off-shell $\tilde{R}_{LL'}$ matrix element at some particular energy already guarantees that it is close to $R_{LL'}$ over a large energy range (above all when it is a function slowly varying with the on-shell energy). So one can exploit the freedom in choosing α_i by requesting that (pairs of) $\tilde{R}_{LL'}$ elements agree with (pairs of) $R_{LL'}$ at some crucial energies. For instance, for the $^3S_1$-$^3D_1$ N-N channel the ensembles α_i could be staggered like $\{E_1,l_1=0\}$, $\{E_2,l_2=2\}$, $\{E_3,l_3=2\},\ldots$, and thus lead to a reasonable result.

The explanations just made apply to the case where E_i are scattering energies. Nevertheless the EST method is also applicable to bound-state energies. The formalism is even simiplified, because it is not necessary to distinguish between different initial configurations l; the corresponding index can be omitted. At the location of the bound state the interest-

TABLE I. Ranks of the separable potentials constructed by Pieper under the assumption of the ensembles $\alpha_i=\{E_i l_i\}$ quoted.

Potential	3SD1A	3SD1B	3SD1C	3SD1D
rank	1	2	4	5
	$\{-2.227,-\}$	$\{-2.227,-\}$	$\{-2.227,-\}$	$\{-2.227,-\}$
		$\{200,2\}$	$\{125,0\}$	$\{50,0\}$
$\{E_i,l_i\}$			$\{125,2\}$	$\{50,2\}$
			$\{400,2\}$	$\{300,2\}$
				$\{400,0\}$

ing half-off-shell entities are represented by the (coupled) wave functions $|\psi_{EL}\rangle$, $(L=l_<,l_>)$. In order to reproduce both of them, i.e., all half-off-shell characteristics of the model $\mathscr{V}$, at a single bound state only one rank is needed in the separable potential $\tilde{\mathscr{V}}$. From this it is also evident that the EST method can include the possibilities offered by the UPA.

III. PIEPER'S POTENTIALS

We now come to discuss the application of the EST method performed by Pieper for the $^3S_1$-$^3D_1$ state of the N-N system.[3] Specifically we would like to demonstrate how reasonable a description of the on- and half-off-shell behavior of the N-N interaction is provided by the particular separable potentials he constructed. Clearly we will take the RSC potential[6] as a measure for comparison, since this potential was used as the input model. We will mainly concentrate on the off-shell behavior of the separable potentials, because these aspects were not considered by Pieper in his original paper. Still we will include the on-shell properties, since we will thereby have the occasion to correct some erroneous results given in Pieper's work. Furthermore this will shed light on the practicability of the EST method.

A. Representation of separable potentials

Pieper constructed four separable potentials of the form (2.2) each having a different rank N (cf. Table I). By applying a unitary transformation

$$\hat{\mathscr{M}}=\mathscr{U}\mathscr{M}\mathscr{U}^{-1} \tag{3.1}$$

to diagonalize the matrix of the coupling parameters, he cast the separable potentials into the form[9]

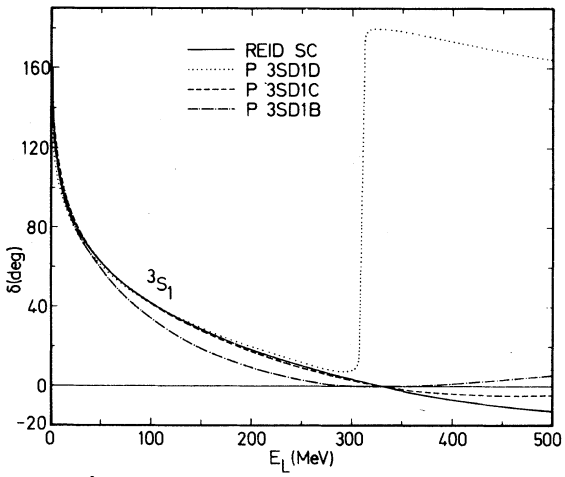

FIG. 1. $^3S_1$ phase shifts for Pieper's potentials as compared to the results of the Reid SC.

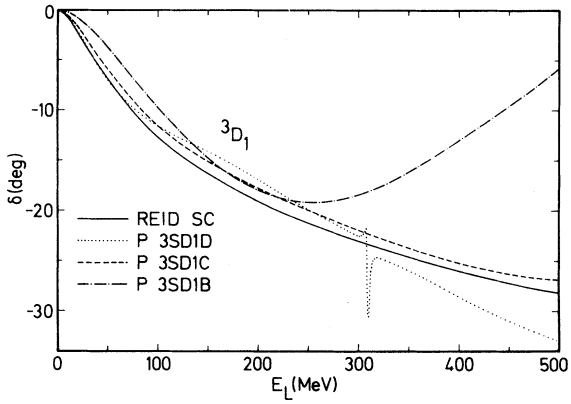

FIG. 2. $^3D_1$ phase shifts for Pieper's potentials as compared to the results of the Reid SC.

$$\tilde{\mathscr{V}}=\sum_{i=1}^{N}\mathscr{V}|\hat{\Psi}_i\rangle\hat{M}_{ii}\langle\hat{\Psi}_i|\mathscr{V} \tag{3.2a}$$

with

$$|\hat{\Psi}_i\rangle=\sum_{j=1}^{N}U_{ij}|\Psi_{\alpha_j}\rangle\ . \tag{3.2b}$$

The form factors

$$g_{Li}(k)=\langle\Phi_{EL}|\mathscr{V}|\hat{\Psi}_i\rangle$$

were approximated by the functions

$$h_{l_<i}(k)=\sum_{m=1}^{8}C_{l_<im}\frac{1}{(k^2+\beta_m{}^2)^2}\ , \tag{3.3a}$$

$$h_{l_>i}(k)=\sum_{m=1}^{8}C_{l_>im}\frac{k^2}{(k^2+\beta_m{}^2)^3}\ . \tag{3.3b}$$

While the parameters β_m were fixed to $\beta_m=3m/2$, the numerical values for C_{Lim} were obtained by a fitting procedure.

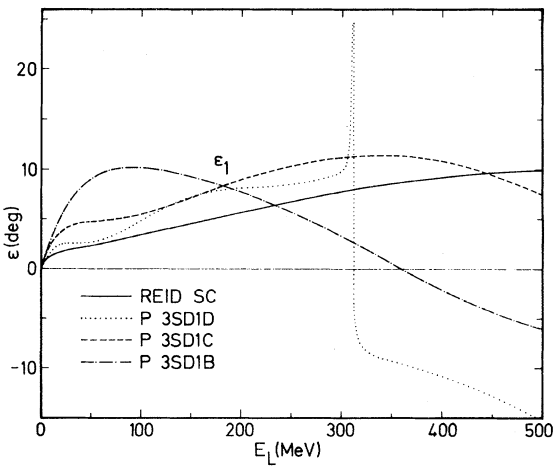

FIG. 3. Mixing parameters ϵ_1 for Pieper's potentials as compared to the results of the Reid SC.

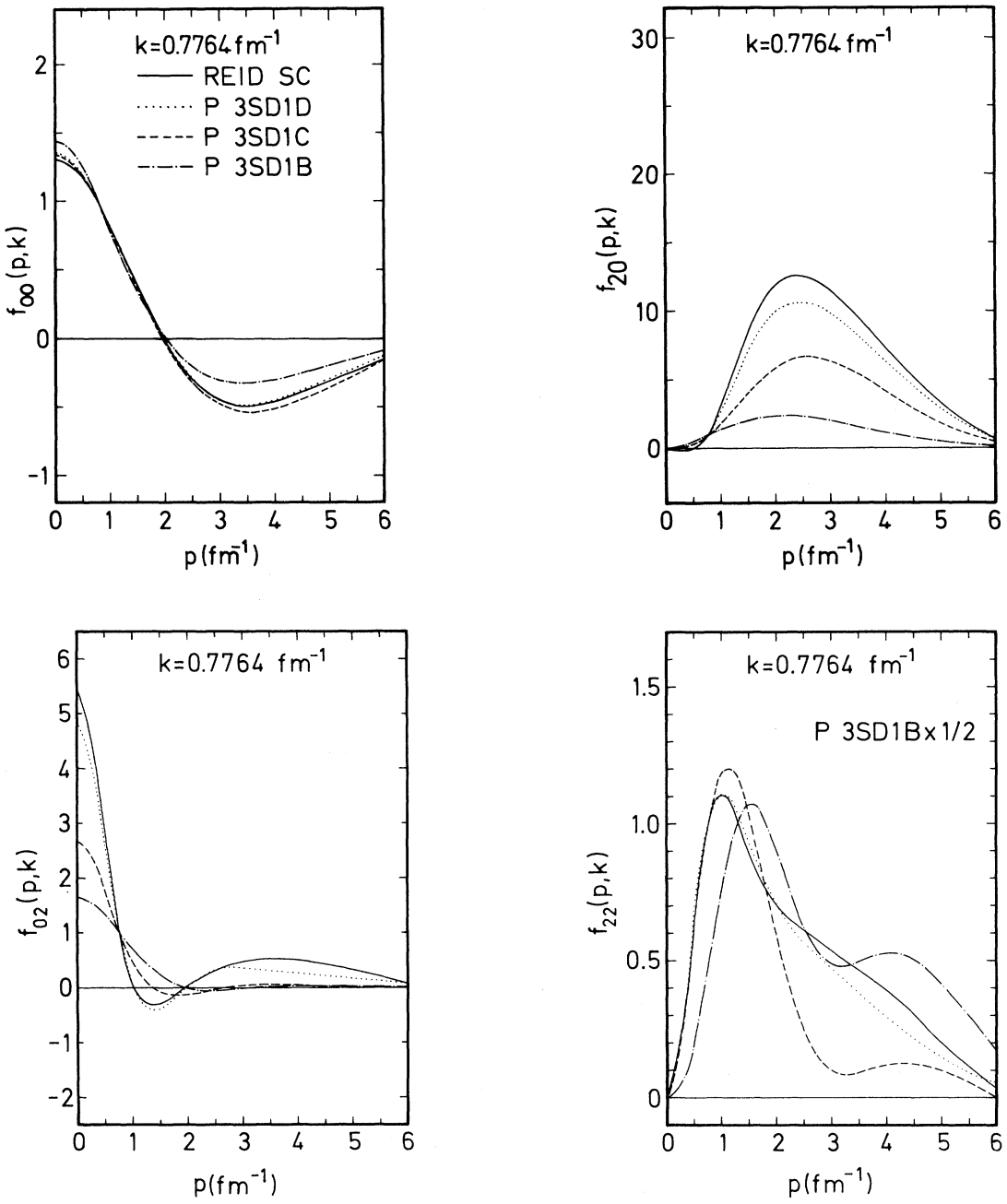

FIG. 4. Noyes-Kowalski half-off-shell functions $f_{LL'}(p,k)$ at $E_{\text{lab}}=50$ MeV for Pieper's potentials in comparison to the Reid SC predictions.

B. Discussion of separable potential properties

Among the four separable potentials that Pieper constructed via the EST method, we will consider here only the three more refined models P 3SD1B, P 3SD1C, and P 3SD1D. Except for the deuteron, the simplest model P 3SD1A does not provide a reasonable description of the N-N interaction; for instance, the corresponding $^3S_1$ phase shift does not pass through zero, while the $^3D_1$ phase shift even has the wrong sign, and the mixing parameter ϵ_1 is much too large. Therefore we decided to drop this model in the present work.

For the construction of his separable potentials, Pieper chose the ensembles α_i as quoted in Table I. That means that all models reproduce all the half-off-shell information at the deuteron bound state (i.e., they have the same deuteron properties E_D, p_D,

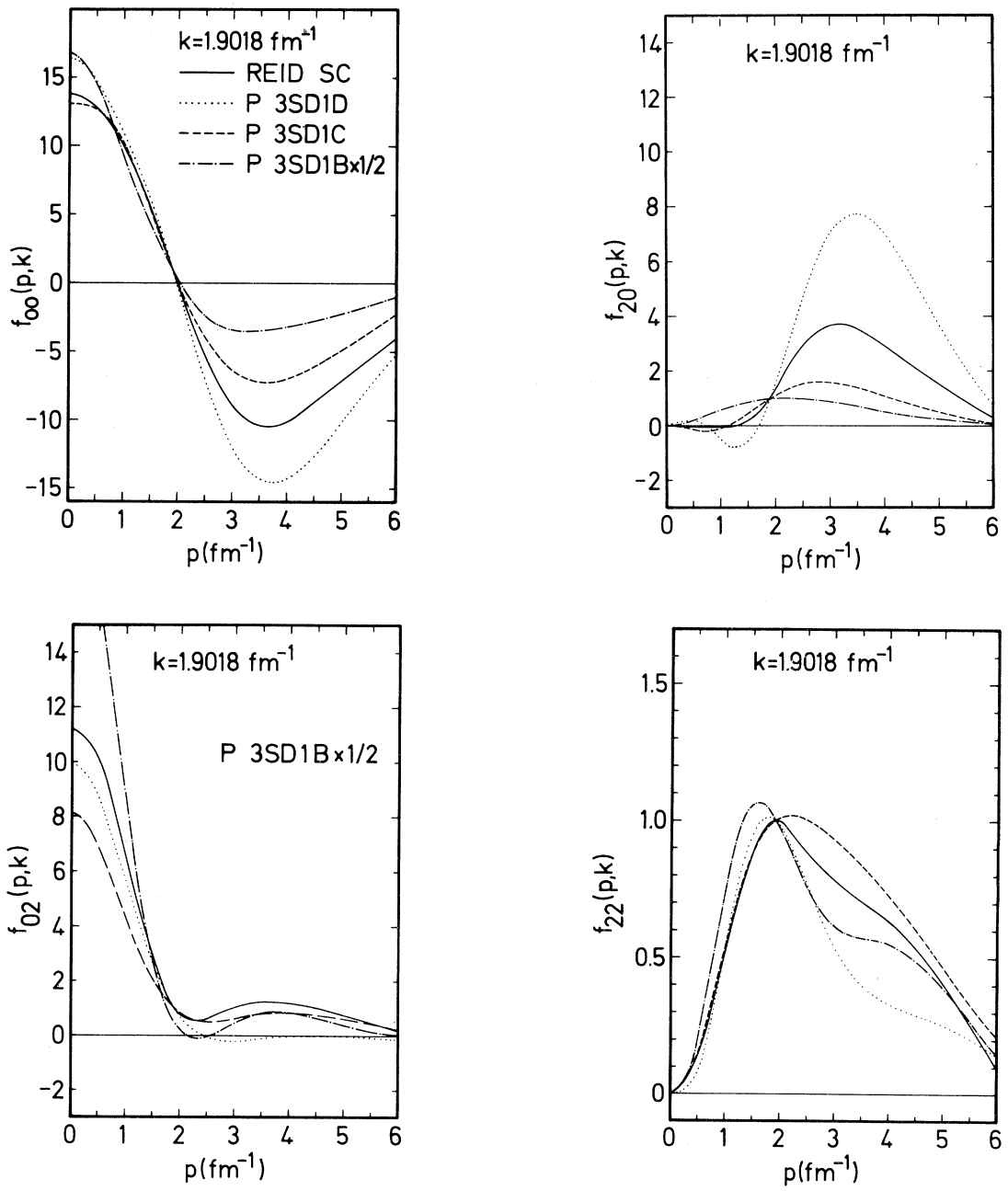

FIG. 5. Same as Fig. 4 at $E_{\text{lab}}=300$ MeV.

Q_D, η, etc., and the same wave function as the Reid SC). Specific R-matrix components $R_{LL'}$ were additionally fixed at energies $E_{\text{lab}}=50$, 125, 200, 300, and 400 MeV (cf. Table I).

Let us first have a look at the one-shell scattering data. As can be seen from Fig. 1, only the rank-4 potential P 3SD1C yields the $^3S_1$ phase shift in a satisfactory way. The rank-2 potential P 3SD1B only simulates the zero of $\delta(^3S_1)$ without changing the sign, while the rank-5 potential P 3SD1D even comes up with a phase-shift behavior which is completely wrong. At the position where this phase shift should pass through zero, it actually shows a resonantlike increase up to 180°. This fact was not observed by Pieper; rather he depicted a $\delta(^3S_1)$ curve with wiggles around this energy. Also in the $^3D_1$ phase shift the potential P 3SD1D yields unreasonable results (see Fig. 2). Again only P 3SD1C

can be considered to be a good approximation of the Reid SC. The same is true for the mixing parameter ϵ_1 in Fig. 3. The potential P 3SD1B produces an ϵ_1 much too high at low energies which furthermore passes through zero at $E_{\text{lab}} \approx 350$ MeV. The incorrect behavior of P 3SD1D has a tremendous influence on the mixing parameter and therefore leads to unacceptable results for ϵ_1.

The separable potential P 3SD1D, whose unphysical characteristics we have just seen in Figs. 1–3, is an example of an application of the EST method where not enough care was taken in the selection of the ensembles $\{E_i l_i\}$. The latter cannot be chosen completely arbitrarily. We have already remarked in subsection II A that one must refrain from allowing the matrix $\langle \Psi_{\alpha_i} | \mathscr{V} | \Psi_{\alpha_k} \rangle$ in Eq. (2.4) to become singular. Unfortunately there are further caltrops inherent in the EST method. So it could happen—as was actually the case with the potential P 3SD1D—that the matrix $[\mathbb{1} - \mathscr{M}\mathscr{G}(E)]$ occurring in Eq. (2.6) becomes singular at some particular energy E. When recalculating Pieper's potential 3SD1D we found that the determinant of the corresponding matrix $[\mathbb{1} - \mathscr{M}\mathscr{G}(E)]$ changes sign at $E_{\text{lab}} \approx 310$ MeV. This gives rise to a pole in the R-matrix elements $\tilde{R}_{LL'}$ whose effect is seen in the resonantlike structure of the scattering data (Figs. 1–3). Such a failure of the EST method happens more or less accidentally. It is due to the fact that half-off-shell R-matrix elements of the model potential are used as form factors of the separable potential $\tilde{\mathscr{V}}$ (cf. also the discussion in the final section). However, we would like to mention that other separable approximation methods too, e.g., the one by Oryu[4] or the expansion by Adhikari and Sloan,[5] are afflicted by similar difficulties.

Let us continue with the examination of the half-off-shell functions

$$f_{LL'}(p,k) = \frac{\tilde{R}_{LL'}(p,k;E)}{\tilde{R}_{LL'}(k,k;E)}, \qquad E = \frac{\hbar^2 k^2}{2\mu} \tag{3.4}$$

of Pieper's potentials. Firstly we state that all potentials adequately reproduce the off-shell behavior of the Reid SC for the channels and energies given in Table I. Only at intermediate energies and/or other channels do deviations occur. We may demonstrate this situation, e.g., at $E_{\text{lab}} = 50$ MeV. While for P 3SD1B and C all four functions $f_{LL'}$ differ significantly from the Reid SC, the potential P 3SD1D works well at this energy (see Figs. 4). Because the latter was designed to reproduce all half-off-shell characteristics of the Reid SC, the small discrepancies that can still be observed must be attributed to the facts that firstly, Pieper used a slightly modified model potential, namely, a so-called simplified Reid SC (SRSC) potential rather than the true one, and secondly, that the *a priori* form factors of the separable EST potential $\tilde{\mathscr{V}}$ were approximated by the functions of Eqs. (3.3).

Similar results could be expected at $E_{\text{lab}} = 300$ MeV (cf. Table I). Unfortunately, here also the potential P 3SD1D falls short, though an ensemble $\{E_i = 300,\ l_i = 2\}$ was selected (see Figs. 5). This may be an effect of the R-matrix pole, which already lies nearby.

We calculated half-off-shell functions at many other energies ranging from $E_{\text{lab}} = 0$–500 MeV. The observations we made were always consistent with the exemplifying results given above: Except for configurations quoted in Table I there often occur considerable deviations from the off-shell behavior of the Reid SC.

IV. CONCLUSIONS

In the present paper we examined the efficiency of the EST method in its generalization to coupled channels. For this purpose we employed Pieper's potentials, stressing the comparison of their off-shell behavior to the Reid SC potential. We demonstrated that by means of the EST method an accurate separable approximation of the on- as well as off-shell properties of the model potential can be achieved at fixed energies. If the rank of the separable potential is taken sufficiently high, a reasonable description of the model interaction is possible over a wide energy range. Particularly in applications to the N-N system, a potential of not too high a rank could suffice if the energies $\{E_i l_i\}$ are chosen adequately. Thus the EST method can compete with other separable approximation schemes or might even turn out to be superior in ability.

However, as became clear in our discussion of the EST method and by accurately (re)calculating the results of Pieper's potentials, some care has to be taken in the application of this method. In particular, one has to check whether the matrices $\langle \Psi_{\alpha_j} | \mathscr{V} | \Psi_{\alpha_k} \rangle$ as well as $[\mathbb{1} - \mathscr{M}\mathscr{G}(E)]$ are well behaved under the choice adopted for the ensembles $\{E_i l_i\}$. Above all, a careful inspection of the energy dependence of the matrix $[\mathbb{1} - \mathscr{M}\mathscr{G}(E)]$ over the whole energy range is necessary in order to avoid unphysical poles of the R matrix. Consequently the ensembles α_i cannot be chosen arbitrarily. Therefore, if one determines for physical reasons that the α_i contain some crucial energies, there is no guarantee that $[\mathbb{1} - \mathscr{M}\mathscr{G}(E)]$ will not become singular. Indeed, with Pieper's potential 3SD1D it incidentally happens that the determinant of $[\mathbb{1} - \mathscr{M}\mathscr{G}(E)]$

passes through zero at a laboratory kinetic energy of about 310 MeV, thus giving rise to the unreasonable resonant structure observed. We emphasize that such limitations for the choice of α_i are not always peculiar to the special case of coupled channels. In a similar study of the EST method in uncoupled partial waves of the N-N system, we found much the same characteristics of the matrices in question. The principal reason for these drawbacks of the EST method lies in the fact that the form factors of the separable approximation are furnished by the half-off-shell R-matrix elements of the model potential. The latter are in general uncorrelated functions with a complicated (oscillatory) momentum dependence.

Though neither one of Pieper's potentials proves adequate in all respects, one must admit that separable potentials constructed by the EST method usually do better than separable models of comparable rank which exist in the literature and which were produced by conventional techniques.[10] Above all with respect to the off-shell behavior, EST potentials are superior. Therefore they will be of good use in few-body applications, where off-shell sensitivities play an important role.[11] Thus the method is predestined to remedy a long-standing shortcoming of separable potentials, viz., that they suffer from unrealistic off-shell properties. If enough care is taken in the application of the EST method, it represents an efficient tool for the separable approximation or abitrary potentials.

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[7]We emphasize that the separable potential (2.2) constructed via the EST method is not energy dependent. The only energy dependence of the reaction matrix $\bar{\mathscr{R}}(E)$ lies in the free resolvent $\mathscr{G}_0^p(E)$.

[8]Remember that the rank of a separable potential in some (coupled or uncoupled) partial-wave state is defined as the rank of the matrix of the coupling parameters. (See Ref. 1.)

[9]As a consequence of this diagonalization procedure the direct relation of the form factors of the resulting separable potential to the half-off-shell R matrix elements of the model potential no longer holds. Rather they are turned into linear combinations of these matrix elements [cf. Eq. (3.2b)].

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