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Virtual photon spectrum in second-order Born approximation

P. Durgapal and D. S. Onley
Ohio University, Athens, Ohio 45701
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We have evaluated expressions for $E1$ and $M1$ virtual photon spectra for relativistic electron scattering in second order Born approximation and included the effects of finite nuclear size and charge. The scattering amplitude can be evaluated analytically provided the elastic and inelastic form factors contain only poles. We have observed that the virtual photon spectrum is insensitive to details of both form factors over a wide range of electron energy and depends only on the nuclear rms radius and transition radius. The integral over the physical momentum transfer is performed numerically. We have evaluated spectra for electric dipole and magnetic dipole radiation for electron energies in the range 10—200 MeV and compared with distorted wave calculations. Except for low energy and high Z, our results compare well. A range-of-validity plot is given for each multipole, distinguishing regions of electron energy and mass number where the second order calculation is adequate and also those where the first order term alone would be sufficient and where the conventional long wavelength limit is applicable. Comparison with data on the excitation of the 16.3 MeV isobaric analog state in ${}^{90}_{40}$Zr yields excellent agreement with no adjustable parameters.

NUCLEAR REACTIONS Inelastic electron scattering, virtual photon method, second order Born approximation.

I. INTRODUCTION

The idea of virtual photons[1] is that the effect of the electromagnetic field of a charged particle passing near a nucleus or other target is considered to be equivalent to the incidence of a burst of radiation. The spectrum of these virtual quanta is widely used in the analysis of electrodisintegration data to relate them to the corresponding photodisintegration processes. If $\sigma_\gamma^{\tau l}(\omega)$ is the photoabsorption cross section for photons of energy ω and multipolarity τl (τ being the label E or M for electric or magnetic), then the integrated cross section for electrons of kinetic energy E_e is assumed to be of the form

$$\sigma_e(E_e)=\int_0^{E_e}\sum_{\tau l}\sigma_\gamma^{\tau l}(\omega)N^{\tau l}(E_e,\omega)\frac{d\omega}{\omega}\ . \qquad (1)$$

In order to use this expression we need to know the virtual photon spectrum $N^{\tau l}(E_e,\omega)$.

The earliest quantum mechanical calculations[2] of the virtual photon spectrum were done using the plane wave Born approximation (PWBA) and the long wavelength limit. The authors of Ref. 2 give explicit expressions for $E1$, $E2$, $E3$, and $M1$ spectra which have been widely used. We refer to these as the conventional spectra. Coulomb distortion effects on the electron wave function were taken into account by Gargaro and Onley,[3] who showed that these result in an enhancement of the virtual photon spectrum in comparison to the conventional spectrum, which increases with the atomic number Z and also with multipolarity l. Reference 3 also uses the long wavelength limit or its equivalent in that the nucleus is considered to be of negligible extension.

The assumption of a point nucleus in the distorted wave calculations becomes questionable where the wavelength of either electron or photon would become comparable with the physical nuclear radius. Some estimates of corrections due to finite nuclear size were made within the framework of PWBA by Barber[4] and Schotter,[5] and the results indicate that the effect of finite size is to decrease the virtual photon spectrum; these calculations are limited to electric type transitions only. Further, the use of PWBA in first order means that Coulomb distortion effects are neglected, and since the two effects may clearly interfere, a simple sequential application of a finite size correction from one calculation and a finite charge (Coulomb) effect from the other would be incorrect.

In order to correct for both finite size and Coulomb distortion we have included form factors for both elastic and inelastic scattering and carried out calculations in second order Born approximation[6] (SOBA). The method adopted is similar to the one developed by Cutler[7] for Coulomb excitation and later used by Bergstrom[8] for magnetic transitions. Both these authors have neglected the mass and energy loss of the electron in their calculations, which would be out of the question for our calculations.

In Sec. II we have derived an expression for the differential cross section applicable with any electron wave function. In Sec. III we consider first order calculations. It consists of two parts; in the first we derive expressions for the conventional spectrum for a general value of the multipole and in the second we make a study of finite size effects of the nucleus to determine to what extent the shape of the spectrum depends on the specifics of the inelastic form factor. In Sec. IV we derive an expression for the virtual photon spectrum for $E1$ and $M1$ radiation in SOBA, which is compared with distorted wave calculations and with some experimental results in Sec. V. The Appendix contains a number of integrals necessary to obtain the results of Sec. IV.

II. THE DIFFERENTIAL CROSS SECTION

The interaction matrix element is given by[9]

$$H_{\rm fi}=\langle J_fM_f\phi_f \mid \int d^3r\, d^3r'[\hat{\rho}_n(\vec{r})G_C(\vec{r}-\vec{r}\,')\hat{\rho}_e(\vec{r}\,') - \hat{\vec{j}}_n(\vec{r})\cdot\overleftrightarrow{G}_T(\vec{r}-\vec{r}\,')\cdot\hat{\vec{j}}_e(\vec{r}\,')] \mid \phi_iJ_iM_i\rangle\ , \tag{2}$$

where the caret indicates an operator, J_iM_i (J_fM_f) are the angular momentum quantum numbers of the initial (final) nuclear states, ϕ_i (ϕ_f) designate the initial (final) electron states, $\hat{\rho}_n(\vec{r})$ and $\hat{\vec{j}}_n(\vec{r})$ are the nuclear charge and current operators, and $\hat{\rho}_e(\vec{r}\,')$ and $\hat{\vec{j}}_e(\vec{r}\,')$ are the same for the electron. G_C and $\overleftrightarrow{G}_T$ are the instantaneous scalar and retarded tensor Green's functions:

$$G_C(\vec{r}-\vec{r}\,')=\frac{1}{2\pi^2}\int \frac{d^3q}{q^2}e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}\ , \tag{3}$$

$$\overleftrightarrow{G}_T(\vec{r}-\vec{r}\,')=\overleftrightarrow{I}\frac{1}{2\pi^2}\int \frac{d^3q}{q^2-\omega^2}e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}\ , \tag{4}$$

where $\overleftrightarrow{I}$ is a unit dyadic and ω is the energy exchanged. On substituting

$$\hat{\rho}_e(\vec{r}\,')=-e\delta(\vec{r}_e-\vec{r}\,') \tag{5}$$

and

$$\hat{\vec{j}}_e(\vec{r}\,')=-e\vec{\alpha}\delta(\vec{r}_e-\vec{r}\,')\ , \tag{6}$$

where $\vec{\alpha}$ is the Dirac spinor operator, in Eq. (2), and integrating over the variable r', we get

$$H_{\rm fi}=-\frac{e}{2\pi^2}\langle J_fM_f\phi_f \mid \int d^3q \int d^3r \left[\frac{\hat{\rho}_n(\vec{r})e^{i\vec{q}\cdot(\vec{r}-\vec{r}_e)}}{q^2}-\frac{\hat{\vec{j}}_n(\vec{r})\cdot\vec{\alpha}e^{i\vec{q}\cdot(\vec{r}-\vec{r}_e)}}{q^2-\omega^2}\right] \mid \phi_iJ_iM_i\rangle\ . \tag{7}$$

For the first or scalar term we employ the expansion

$$\int d^3r\,\hat{\rho}_n(\vec{r})e^{i\vec{q}\cdot\vec{r}}=4\pi\sum_{lm}\hat{M}_{lm}(q)Y_l^{m*}(\hat{q})\ , \tag{8}$$

and for the second or vector term

$$\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\vec{\alpha}\cdot\hat{\vec{j}}_n(\vec{r})=4\pi\sum_{lm}\{-i[\vec{\alpha}\cdot\hat{q}\times\vec{Y}_{ll}^{m*}(\hat{q})]\hat{T}_{lm}^E(q)+[\vec{\alpha}\cdot\vec{Y}_{ll}^{m*}(\hat{q})]\hat{T}_{lm}^M(q)\}\ , \tag{9}$$

where $\vec{Y}_{ll}^m(\hat{q})$ are the vector spherical harmonics, and $\hat{M}_{lm}(q)$ and $\hat{T}_{lm}^{\tau}(q)$ are the Coulomb and the transverse nuclear multipole operators:

$$\hat{M}_{lm}(q)=i^l\int d^3r\,\hat{\rho}_n(\vec{r})j_l(qr)Y_l^m(\hat{r})\ , \tag{10}$$

$$\hat{T}_{lm}^E(q)=\frac{i^l}{q}\int d^3r\,\hat{\vec{j}}_n(\vec{r})\cdot\nabla\times[j_l(qr)\vec{Y}_{ll}^m(\hat{r})]\ , \tag{11}$$

$$\hat{T}_{lm}^M(q)=i^l\int d^3r\,\hat{\vec{j}}_n(\vec{r})\cdot[j_l(qr)\vec{Y}_{ll}^m(\hat{r})]\ . \tag{12}$$

Here $j_l(qr)$ is the usual notation for the spherical Bessel functions. Substituting the expansions of Eqs. (8) and (9) in Eq. (7), and separating nuclear and electron parts in each term of the interaction matrix element, we have

$$H_{\rm fi}=-\frac{2e}{\pi}\int d^3q\sum_{lm}C(J_i,J_f,l;M_i,M_f,m)\left\{\frac{F_C^l(q)}{q^2}[R_C(\vec{q})Y_l^{m*}(\hat{q})]+\frac{iF_T^{El}(q)}{q^2-\omega^2}[\vec{R}_T(\vec{q})\cdot\hat{q}\times\vec{Y}_{ll}^{m*}(\hat{q})]-\frac{F_T^{Ml}(q)}{q^2-\omega^2}[\vec{R}_T(\vec{q})\cdot\vec{Y}_{ll}^{m*}(\hat{q})]\right\}, \quad (13)$$

where $C(J_i,J_f l;M_i,M_f,m)$ are the Clebsch-Gordan coefficients.[10] In Eq. (13) we have introduced the following notation for those matrix elements which involve the electron wave functions and electron operators:

$$R_C(\vec{q})=\sum_{\rm spins}\int d^3r_e\phi_f^\dagger e^{-i\vec{q}\cdot\vec{r}_e}\phi_i\ , \quad (14)$$

$$\vec{R}_T(\vec{q})=\sum_{\rm spins}\int d^3r_e\phi_f^\dagger\vec{\alpha}e^{-i\vec{q}\cdot\vec{r}_e}\phi_i\ , \quad (15)$$

and the nuclear form factors $F_C^l(q)$ and $F_T^{\tau l}(q)$ are related to the reduced matrix elements

$$F_C^l(q)=\frac{1}{(2J_i+1)^{1/2}}\langle J_f||\hat{M}_l(q)||J_i\rangle \quad (16)$$

and

$$F_T^{\tau l}(q)=\frac{1}{(2J_i+1)^{1/2}}\langle J_f||\hat{T}_l^\tau(q)||J_i\rangle\ . \quad (17)$$

The labels C and T in Eqs. (13)—(17) distinguish Coulomb (or longitudinal) from transverse contributions. For the differential cross section we need the square of the interaction matrix element averaged over incident electron and nuclear spin directions and summed over final spin directions, i.e.,

$$\overline{\sum_{\rm spins}}|H_{\rm fi}|^2=\frac{1}{2J_i+1}\sum_{M_iM_f}\frac{1}{2}\sum_{s_is_f}|H_{\rm fi}|^2\ . \quad (18)$$

The sum over nuclear spins can be done using closure relations[10] and the sum (18) takes on the form

$$\overline{\sum_{\rm spins}}|H_{\rm fi}|^2=\frac{4e^2}{\pi^2(2l+1)}\int dq\,dq'q^2q'^2\left\{\frac{F_C^l(q)F_C^l(q')}{q^2q'^2}X_C^l+\frac{F_T^{El}(q)F_T^{El}(q')}{(q^2-\omega^2)(q'^2-\omega^2)}X_T^{El}+\frac{F_T^{Ml}(q)F_T^{Ml}(q')}{(q^2-\omega^2)(q'^2-\omega^2)}X_T^{Ml}+2\,{\rm Re}\left[\frac{iF_C^l(q')F_T^{El}(q)}{q'^2(q^2-\omega^2)}X_{EC}^l\right]-2\,{\rm Re}\left[\frac{F_C^l(q')F_T^{Ml}(q)}{q'^2(q^2-\omega^2)}X_{MC}^l\right]+2\,{\rm Re}\left[\frac{iF_T^{El}(q')F_T^{Ml}(q)}{(q^2-\omega^2)(q'^2-\omega^2)}X_{ME}^l\right]\right\}. \quad (19)$$

The dependence on the angular coordinates, electron spins, and the magnetic quantum number m has been collected in the kernels X which are labeled to distinguish the Coulomb, transverse, and interference contributions:

$$\begin{aligned}
X_C^l&=\int d\Omega_q d\Omega_{q'}\sum_m[R_C(\vec{q})Y_l^{m*}(\hat{q})][R_C^*(\vec{q}\,')Y_l^m(\hat{q}\,')]\ ,\\
X_T^{\tau l}&=\int d\Omega_q d\Omega_{q'}\sum_m[\vec{R}_T(\vec{q})\cdot\vec{Y}_\tau^*(\hat{q})][R_T^*(\vec{q}\,')\cdot\vec{Y}_\tau(\hat{q}\,')]\ ,\\
X_{\tau C}^l&=\int d\Omega_q d\Omega_{q'}\sum_m[\vec{R}_T(\vec{q})\cdot\vec{Y}_\tau^*(\hat{q})][R_C^*(\vec{q}\,')Y_l^m(\hat{q}\,')]\ ,\\
X_{\tau\tau'}^l&=\int d\Omega_q d\Omega_{q'}\sum_m[\vec{R}_T(\vec{q})\cdot\vec{Y}_\tau^*(\hat{q})][\vec{R}_T^*(\vec{q}\,')\cdot\vec{Y}_{\tau'}(\hat{q}\,')]\ .
\end{aligned} \quad (20)$$

In this expression we have introduced the abbreviation $\vec{Y}_\tau$ which is interpreted as follows:

$$\vec{Y}_M(\hat{q})=\vec{Y}^m_{ll}(\hat{q}) ,$$
$$\vec{Y}_E(\hat{q})=\hat{q}\times\vec{Y}^m_{ll}(\hat{q}) . \tag{21}$$

The differential cross section is now given in terms of the sum (19) by

$$\frac{d\sigma_l}{d\Omega}=\frac{E_2p_2}{(2\pi)^2}\frac{E_1}{p_1}\overline{\sum_{\text{spins}}}|H_{\text{fi}}|^2 . \tag{22}$$

III. THE SPECTRUM OF VIRTUAL PHOTONS IN PWBA

If we assume the electrons to be represented by plane waves, then the electron wave function will have the form

$$\phi_s(\vec{r})=e^{i\vec{p}_j\cdot\vec{r}}\left[\frac{m_e}{E_jV}\right]^{1/2}u_s(\vec{p}_j) , \tag{23}$$

where m_e is the electron mass, E_j (p_j) its energy (momentum), and $u_s(\vec{p}_j)$ represents a four component column matrix corresponding to the two positive energy solutions of the Dirac equation.[11] In our case we shall distinguish initial and final states by using the subscripts i (f) instead of s and the quantities E_1,p_1 (E_2,p_2) for the energy and momentum. On inserting the wave function of Eq. (23) in the Eqs. (20) for the kernels we get

$$X_C^l=\frac{m_e^2}{E_1E_2}\frac{(2\pi)^6}{V^2}\frac{\delta(\Delta-q)\delta(\Delta-q')}{\Delta^4}\sum_m Y_l^{m*}(\hat{\Delta})Y_l^m(\hat{\Delta})\,|u_f^\dagger(\vec{p}_2)u_i(\vec{p}_1)|^2 , \tag{24a}$$

$$X_T^{\tau l}=\frac{m_e^2}{E_1E_2}\frac{(2\pi)^6}{V^2}\frac{\delta(\Delta-q)\delta(\Delta-q')}{\Delta^4}\sum_m u_f^\dagger\vec{\alpha}\cdot\vec{Y}_\tau^*(\hat{\Delta})u_iu_i^\dagger\vec{\alpha}\cdot\vec{Y}_\tau(\hat{\Delta})u_f . \tag{24b}$$

The sums over final electron spins f and average over initial spin i are straightforward. Further, the sums over the magnetic quantum numbers m cause the interference terms to vanish and the differential cross sections for the Coulomb and transverse contributions are then found to be

$$\frac{d\sigma^{Cl}}{d\Omega}=4\pi\alpha\frac{p_2}{p_1}\frac{|F_C^l(\Delta)|^2}{\Delta^4}(E_p^2-\Delta^2) , \tag{25a}$$

$$\frac{d\sigma_T^{\tau l}}{d\Omega}=4\pi\alpha\frac{p_2}{p_1}\left[\frac{(\Delta_{\max}^2-\Delta^2)(\Delta^2-\Delta_{\min}^2)}{2\Delta^2(\Delta^2-\omega^2)^2}+\frac{1}{(\Delta^2-\omega^2)}\right]|F_T^{\tau l}(\Delta)|^2 , \tag{25b}$$

where $\Delta_{\max}$ and $\Delta_{\min}$ are the kinematical limits to the momentum transfer Δ, $E_p=E_1+E_2$, and α is the fine structure constant.

The cross section integrated over solid angle is then related to the virtual photon spectrum $N^{\tau l}(E_e,\omega)$ by dividing by the photoabsorption cross section,[12] and the result is

$$N^{\tau l}(E_e,\omega)=\sigma^{\tau l}(\omega)\Bigg/\left[\frac{(2\pi)^3}{\omega^2}|F_T^{\tau l}(\omega)|^2\right] . \tag{26}$$

While the spectrum for magnetic transitions is purely transverse, that for electric transitions contains both a Coulomb and a transverse part:

$$N^{Ml}(E_e,\omega)=N_T^{Ml}(E_e,\omega) , \tag{27a}$$

$$N^{El}(E_e,\omega)=N_T^{El}(E_e,\omega)+N^{Cl}(E_e,\omega) . \tag{27b}$$

In order to find $N^{Cl}(E_e,\omega)$ and $N_T^{\tau l}(E_e,\omega)$ we substitute Eqs. (25) and (27) in Eq. (26) and equate the longitudinal and transverse parts. We change the variable of integration from scattering angles to momentum transfer squared, Δ^2, and integrate between kinematic limits; we then get the expressions

$$N^{Cl}(E_e,\omega)=\frac{\alpha}{2\pi}\frac{\omega^2}{p_1^2}\int_{\Delta_{\min}^2}^{\Delta_{\max}^2}d(\Delta^2)\frac{(E_p^2-\Delta^2)}{\Delta^4}\frac{|F_C^l(\Delta)|^2}{|F_T^{El}(\omega)|^2} , \tag{28a}$$

$$N_T^{\tau l}(E_e,\omega)=\frac{\alpha}{4\pi}\frac{\omega^2}{p_1^2}\int_{\Delta_{\min}^2}^{\Delta_{\max}^2} d(\Delta^2)\left[\frac{(\Delta_{\max}^2-\Delta^2)(\Delta^2-\Delta_{\min}^2)}{\Delta^2(\Delta^2-\omega^2)^2}+\frac{2}{(\Delta^2-\omega^2)}\right]\frac{|F_T^{\tau l}(\Delta)|^2}{|F_T^{\tau l}(\omega)|^2} . \tag{28b}$$

The conventional spectrum is derived from the first order spectrum by taking the long wavelength limit which is equivalent to retaining the lowest order terms in the expansion of the form factors $F_C^l(q)$, $F_T^{El}(q)$, and $F_T^{Ml}(q)$ in powers of q, namely,

$$\frac{|F_C^l(q)|^2}{|F_T^{El}(\omega)|^2}\simeq\frac{l}{l+1}\left[\frac{q}{\omega}\right]^{2l} , \tag{29a}$$

$$\frac{|F_T^{\tau l}(q)|^2}{|F_T^{\tau l}(\omega)|^2}\simeq\left[\frac{q}{\omega}\right]^{2l'} , \tag{29b}$$

where $l'=l-1$ for $\tau=E$ and $l'=l$ for $\tau=M$. Substitution from Eqs. (29) in Eqs. (28) and subsequent integration then yields the results

$$N^{C1}(E_e,\omega)=\frac{\alpha}{\pi}\left[\frac{E_p^2}{2p_1^2}\ln\left[\frac{p_1+p_2}{p_1-p_2}\right]-\frac{p_2}{p_1}\right] ,$$

$$N_T^{E1}(E_e,\omega)=\frac{\alpha}{\pi}\left[\frac{E_p^2}{p_1^2}\ln(\xi)-\frac{E_p^2}{2p_1^2}\ln\left[\frac{p_1+p_2}{p_1-p_2}\right]-\frac{p_2}{p_1}\right] ,$$

$$N^{Cl}(E_e,\omega)=\frac{\alpha}{2\pi}\left[\frac{l}{l+1}\right]\frac{1}{p_1^2}\left[\frac{E_p^2}{l-1}\left[\frac{\Delta_{\max}^{2l-2}-\Delta_{\min}^{2l-2}}{\omega^{2l-2}}\right]-\frac{1}{l}\left[\frac{\Delta_{\max}^{2l}-\Delta_{\min}^{2l}}{\omega^{2l-2}}\right]\right] , \quad l>1$$

$$N_T^{El}(E_e,\omega)=\frac{\alpha}{4\pi}\frac{1}{p_1^2}\left\{2[E_1^2+E_2^2-2(l-1)m_e^2]\left[2\ln(\xi)+\sum_{s=1}^{l-1}\frac{(\Delta_{\max}^{2s}-\Delta_{\min}^{2s})}{s\omega^{2s}}\right]\right.$$
$$\left.+E_p^2\left[\frac{\Delta_{\max}^{2l-4}-\Delta_{\min}^{2l-4}}{\omega^{2l-4}}\right]+\left[\frac{(l-2)}{l-1}E_p^2-2(p_1^2+p_2^2)\right]\left[\frac{\Delta_{\max}^{2l-2}-\Delta_{\min}^{2l-2}}{\omega^{2l-2}}\right]\right\} , \quad l>1$$

$$\xi=\frac{E_1E_2+p_1p_2-m_e^2}{m_e(E_1-E_2)} . \tag{30}$$

In this limit, the expression for magnetic transitions can be obtained from the transverse electric term as a consequence of Eq. (29); the relation is

$$N^{Ml}(E_e,\omega)=N_T^{E(l+1)}(E_e,\omega) . \tag{31}$$

A rough criterion for the validity of the long wavelength limit is $qR_{\text{nuc}}\ll l$, and since it is evidently more appropriate for high l states it is interesting to see how the virtual photon spectrum behaves as a function of l. Figure 1 shows the conventional spectra for electric transitions El for l from 1 through 20, for 100 MeV electrons. In the low photon energy region the major contribution comes from the Coulomb term and the virtual photon spectrum which behaves as $\omega^{-2(l-1)}$ rises sharply (infrared catastrophe). Near the end point the Coulomb contribution becomes negligible in comparison to the transverse contribution and all curves approach zero in the same fashion.

The merit of the conventional spectrum is that it is independent of all nuclear properties and indeed the idea of a spectrum of virtual photons is not useful if its shape is strongly dependent on the model chosen for the transition charge and current densities. We have studied the finite size effects of the nucleus to determine to what extent these results are model specific. One of the models we have used is that introduced by Helm[13] and later elaborated by Rosen *et al.*[14] and the present authors,[15] which takes into account the main features of the nuclear shape and its collective motions and has simple form factors with adjustable parameters. The other model we use is the modified form of the exponential distribution[16] which has both simple form factors and simple charge and current densities. Form factors used for the Helm-type model are from Ref. 15 and Eqs. (3), (8), and (15).

$$F_T^{Ml}(q)=-\frac{1}{2(2l+1)^{1/2}}q\exp(-\tfrac{1}{2}\bar{g}^2q^2)[\sqrt{l}\gamma_{l+1}j_{l+1}(q\bar{R})+\sqrt{l+1}\gamma_{l-1}j_{l-1}(q\bar{R})] \quad (32)$$

for magnetic transitions and

$$F_C^l(q)=\beta_l j_l(qR)\exp(-\tfrac{1}{2}g^2q^2) \quad (33)$$

and

$$F_T^{El}(q)=-\left[\frac{R}{\tilde{R}}\right]^l\left[\frac{l+1}{l}\right]^{1/2}\beta_l\frac{\omega}{q}j_l(q\tilde{R})\exp(-\tfrac{1}{2}\tilde{g}^2q^2) \quad (34)$$

for Coulomb and transverse electric components. For magnetic transitions $\bar{R}$, $\bar{g}$ and the γ's are adjustable parameters. For an electric transition there are two sets of parameters to be derived, R,g and $\tilde{R},\tilde{g}$, and to set these equal is a constraint which also comes from assuming that the flow of nuclear matter is irrotational and incompressible—an assumption which always leads to the relation[15]

$$F_T^{El}(q)=-\left[\frac{l+1}{l}\right]^{1/2}\frac{\omega}{q}F_C^l(q) \quad (35)$$

between Coulomb and transverse electric form factors. For the modified exponential distribution the corresponding form factors are taken to be

$$F_T^{M1}(q)=\frac{N_M q}{(a^2+q^2)^4}\left[1+\frac{q^2}{a^2}(16\epsilon-1)\right] \quad (36)$$

and

$$F_T^{E1}(q)=\frac{N_E}{(a^2+q^2)^4}\left[1+\frac{q^2}{a^2}(16\epsilon-1)\right] \quad (37)$$

for $M1$ and $E1$ transitions. Here ϵ and a are adjustable parameters and N_M,N_E are the normalization constants. We do not need explicit expressions for N_M and N_E since the expressions for the virtual photon spectrum involve ratios of form factors [see Eqs. (28)]. Form factors such as (36) and (37) do not give rise to diffraction structure in the differential cross section and are, therefore, not used for angular distributions. However, we will be interested to see how they fare for integrated cross sections since their simple forms offer many advantages.

To compare different models of nuclear density distributions, we use the transition radius, R_{tr}, which can be obtained from a MacLaurin expansion of the form factor in powers of q^2:

$$\frac{F(q)}{F(\omega)}=\left[\frac{q}{\omega}\right]^{l'}\left[1-\frac{q^2}{2(2l+3)}[R_{tr}^{\tau}]^2+\cdots\right], \quad (38)$$

where $F(q)$ can be $F_C^l(q)$, $F_T^{El}(q)$, or $F_T^{Ml}(q)$, and $l'=l$ for Coulomb and magnetic, and $l'=l-1$ for transverse electric transitions. The corresponding transition radii carry the label τ which can be C, E, or M. We can also write the transition radii as moments of the charge and current distribution[17,24]:

$$[R_{tr}^C]^2=\frac{\int d^3r\, r^{l+2}\rho_n(\vec{r})Y_l^m(\hat{r})}{\int d^3r\, r^l\rho_n(\vec{r})Y_l^m(\hat{r})}, \quad (39)$$

$$[R_{tr}^E]^2=\frac{\int d^3r\, \vec{j}_n(\vec{r})\cdot\nabla\times[r^{l+2}\vec{Y}_{ll}^m(\hat{r})]}{\int d^3r\, \vec{j}_n(\vec{r})\cdot\nabla\times[r^l\vec{Y}_{ll}^m(\hat{r})]}, \quad (40)$$

$$[R_{tr}^M]^2=\frac{\int d^3r\, r^{l+2}\vec{j}_n(\vec{r})\cdot\vec{Y}_{ll}^m(\hat{r})}{\int d^3r\, r^l\vec{j}_n(\vec{r})\cdot\vec{Y}_{ll}^m(\hat{r})}. \quad (41)$$

From Eq. (38) we see that for low values of momentum transfer the behavior of the form factor

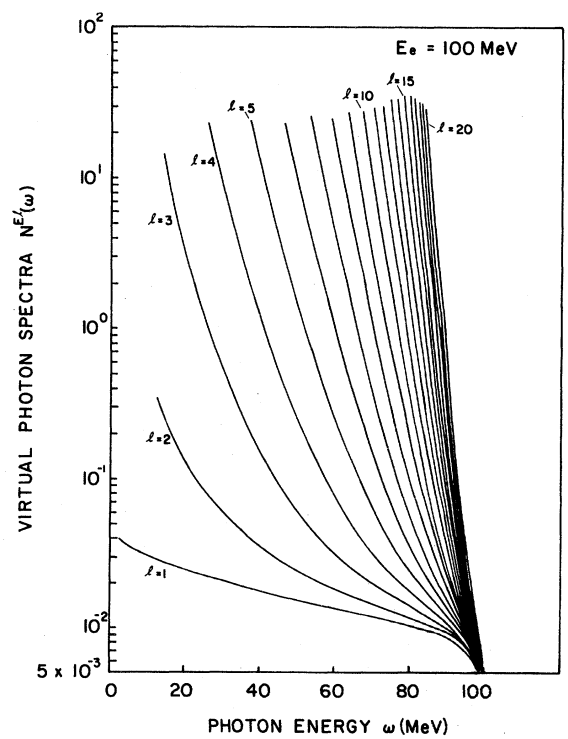

FIG. 1. Spectrum of virtual photons for the first 20 electric multipoles ($l=1$–20), calculated in the plane wave Born approximation and long wavelength limit. The energy of incident electrons is 100 MeV.

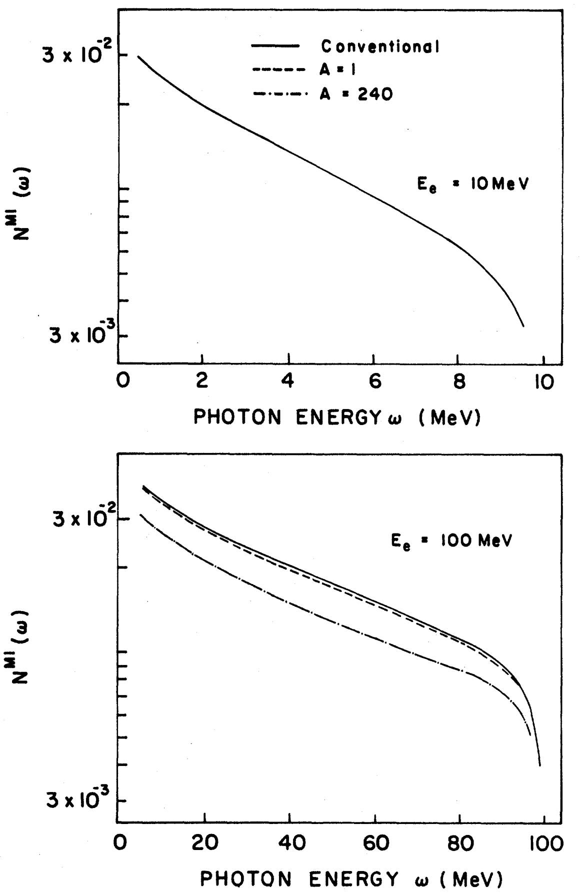

FIG. 2. The $M1$ spectra of virtual photons emitted by 10 and 100 MeV electrons incident upon targets of mass numbers $A=1$ and $A=240$, calculated in PWBA with the finite size effects included and compared with the corresponding conventional spectra. Helm model form factors with $\bar{g}^2=0.3$ fm^2, $\bar{R}=0.87$ fm for $A=1$ and $\bar{R}=6.85$ fm, $\bar{g}^2=0.8$ fm^2 for $A=240$.

is governed by the value of the transition radius and thus we expect its value to govern the behavior of electron cross sections at suitably low energy. For incompressible flow, relation (35) between the form factors also implies $R_{\rm tr}^C=R_{\rm tr}^E$ so that electric transitions can be characterized by a single radius.

For the modified exponential distributions we find that the transition radii for all three (Coulomb, transverse electric, and magnetic) cases, all have the same expression for a transition of multipolarity $l=1$:

$$R_{\rm tr}^{T}=\left[\frac{10}{a^2}(5-16\epsilon)\right]^{1/2} . \tag{42}$$

For the Helm model, the transition radius for the $M1$ transition is given by

$$R_{\rm tr}^{M}=\left\{\frac{5}{3}\left[\bar{R}^2\left[1-\frac{\sqrt{2}}{5}\frac{\gamma_2}{\gamma_0}\right]+3\bar{g}^2\right]\right\}^{1/2} \tag{43}$$

and the $E1$ transition radii by

$$R_{\rm tr}^{C}=(R^2+5g^2)^{1/2} , \tag{44}$$

$$R_{\rm tr}^{E}=(\tilde{R}^2+5\tilde{g}^2)^{1/2} . \tag{45}$$

We first evaluate the virtual photon spectrum for $M1$ transitions, by substituting the form factor of Eq. (32) in Eq. (28b). Figure 2 shows the effect of finite size on an $M1$ spectrum at different energies. The upper part of this figure shows that at low energies such as 10 MeV nuclear size does not produce an observable effect for the spectra for mass numbers $A=1$ and $A=240$; both coincide with the conventional spectrum. On the other hand, the lower part of Fig. 2 shows that at higher energies such as 100 MeV, inclusion of finite size causes a depression

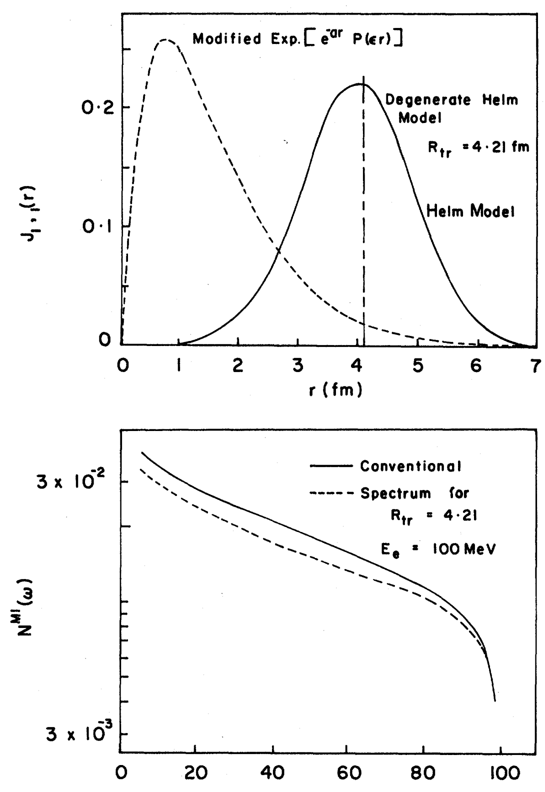

FIG. 3. The upper part of the figure shows current component for three different distributions, the Helm model, a degenerate Helm model (skin thickness $\bar{g}=0$), and a modified exponential shape $[e^{-ar}P(\epsilon r)]$. They all have the same transition radius, $R_{\rm tr}^M=4.21$ fm. The lower part shows the $M1$ virtual photon spectrum corresponding to $R_{\rm tr}^M=4.21$ fm for electrons of kinetic energy $E_e=100$ MeV.

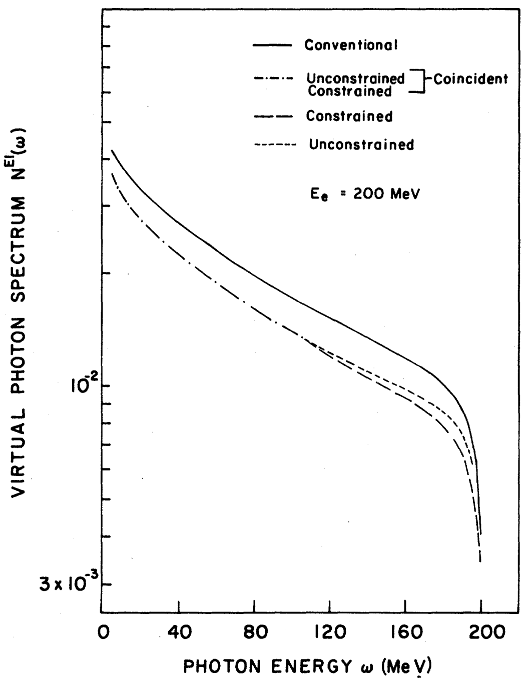

FIG. 4. Comparison of $E1$ virtual photon spectra using Eqs. (33) and (34) with $R=\tilde{R}$, $g=\tilde{g}$ (constrained) and $R\neq\tilde{R}$, $g\neq\tilde{g}$ (unconstrained). The values of the parameters are $R=4.21$ fm, $g^2=1$ fm^2, $\tilde{R}=3.57$ fm, $\tilde{g}^2=0.8$ fm^2, and $E_e=200$ MeV.

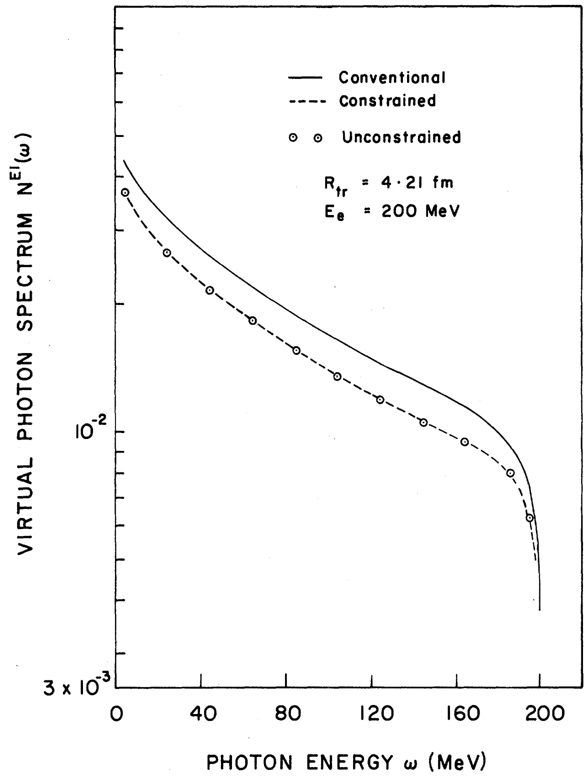

FIG. 5. $E1$ virtual photon spectra using constrained ($R=3.57$ fm, $g^2=0.8$ fm^2) and unconstrained ($R=4.21$ fm, $g^2=0$, $\tilde{R}=3.57$, $\tilde{g}^2=0.8$ fm^2) forms of Eqs. (33) and (34). The transition radii in the two cases are the same. The kinetic energy of incident electrons is $E_e=200$ MeV.

in the virtual photon spectrum in comparison to the conventional value. Figure 3 compares the effect of three different distributions, the Helm model, a degenerate Helm model with zero skin thickness, and a modified exponential distribution. All the three distributions, shown in the upper part of Fig. 3, have the same transition radius. The lower part shows the virtual photon spectra corresponding to these distributions and, interestingly enough, they coincide. From this and many similar examples we conclude that at these energies, irrespective of the details of the form factor, if we keep the transition radius constant, we get the same virtual photon spectrum.

We can make similar observations in the case of electric transitions, but now we have both a charge and a current distribution contributing. One case of particular interest is the effect of tying the two distributions to one another by the constraint of Eq. (35). Taking an $E1$ transition and using form factors given in Eqs. (33) and (34) with $R=4.21$ fm and $g^2=1.0$ fm^2, we compare the effect of setting $\tilde{R},\tilde{g}$ to the same values against making them 20% smaller. Figure 4 shows how the spectra for the two cases are different. However, if we adjust the parameters so that the transition radius in the two cases is the same, then we get coinciding virtual photon spectra as shown in Fig. 5. Hereafter we use relation (35) for simplicity, although a complete analysis should take account of both transverse and Coulomb transition radii as independent parameters.

IV. THE SPECTRUM OF VIRTUAL PHOTONS IN SOBA

In this case we admit that the electron is moving in the Coulomb field of the nucleus and modify the electron wave function in Eq. (23) accordingly. The Green's function for electrons of energy E_j, and momentum p_j, is

$$G(\vec{r},\vec{r}\,')=(i\vec{\alpha}\cdot\nabla-E_j-\beta m_e)\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{(p_j{}^2-k^2-i\kappa)} \tag{46}$$

and the Coulomb potential of the nucleus, also conveniently written as a Fourier transform, is

$$U_C(r) = -4\pi e^2 Z \int \frac{d^3q}{(2\pi)^3} F(q^2) \frac{e^{-i\vec{q}\cdot\vec{r}}}{q^2+\lambda^2} . \tag{47}$$

In Eqs. (46) and (47) κ and λ are small positive quantities which will ultimately be set equal to zero; $F(q)$ is the elastic form factor of the nucleus and Z its charge number. The wave function for the electron, correct to first order in U_C, is now (Ref. 7)

$$\phi_s(\vec{r}) = e^{i\vec{p}_j\cdot\vec{r}} \left[1 + \frac{e^2 Z}{2\pi^2} \int d^3k \frac{(2E_j + \vec{\alpha}\cdot\vec{k}) e^{i\vec{k}\cdot\vec{r}} F(k^2)}{(k^2+\lambda^2)(k^2+2\vec{k}\cdot\vec{p}_j - i\kappa)} \right] \left[\frac{m_e}{E_j V} \right]^{1/2} u_s(\vec{p}_j) , \tag{48}$$

which when inserted in Eqs. (14) and (15) for $R_C(\vec{q})$ and $\vec{R}_T(\vec{q})$ and combined with Eqs. (20), yields the following expressions for the kernels correct to second order Born approximation (SOBA):

$$X_C^l = \frac{(2\pi)^6 m_e^2}{V^2 E_1 E_2} \frac{\delta(\Delta - q')}{\Delta^2} \left[\frac{\delta(\Delta - q)}{\Delta^2} S_C^0 + \frac{\alpha Z}{2\pi^2} \int d\Omega_q \frac{F[(\vec{\Delta}-\vec{q})^2]}{(\vec{\Delta}-\vec{q})^2+\lambda^2} 2\,\mathrm{Re} S_C \right] ,$$

$$X_T^{\tau l} = \frac{(2\pi)^6}{V^2} \frac{m_e^2}{E_1 E_2} \frac{\delta(\Delta - q')}{\Delta^2} \left[\frac{\delta(\Delta - q)}{\Delta^2} S_T^0 + \frac{\alpha Z}{2\pi^2} \int d\Omega_q \frac{F[(\vec{\Delta}-\vec{q})^2]}{(\vec{\Delta}-\vec{q})^2+\lambda^2} 2\,\mathrm{Re} S_{T\tau} \right] ,$$

$$\mathrm{Re}(iX_{EC}^l) = \frac{(2\pi)^6}{V^2} \frac{m_e^2}{E_1 E_2} \frac{\alpha Z}{2\pi^2} \left[\frac{\delta(\Delta - q')}{\Delta^2} \int d\Omega_q \frac{F(\vec{\Delta}-\vec{q})^2}{(\vec{\Delta}-\vec{q})^2+\lambda^2} \mathrm{Re} S_{I1} \right.$$
$$\left. + \frac{\delta(\Delta - q)}{\Delta^2} \int d\Omega_{q'} \frac{F[(\vec{\Delta}-\vec{q}\,')^2]}{(\vec{\Delta}-\vec{q}\,')^2+\lambda^2} \mathrm{Re} S_{I2} \right] , \tag{49}$$

where the superscript "0" indicates the first order contribution and is the same as Eqs. (24), and

$$S_C = \sum_m \sum_{i,f} u_f^\dagger \left[\frac{2E_1 - \vec{\alpha}\cdot(\vec{\Delta}-\vec{q})}{Z_1} \right] u_i u_i^\dagger u_f \frac{(2l+1)}{4\pi} P_l(\hat{q}\cdot\hat{\Delta}) + \mathrm{i.t.} , \tag{50}$$

$$S_{T\tau} = \sum_m \sum_{i,f} u_f^\dagger \vec{\alpha}\cdot\vec{Y}_\tau^*(\hat{q}) \left[\frac{2E_1 - \vec{\alpha}\cdot(\vec{\Delta}-\vec{q})}{Z_1} \right] u_i u_i^\dagger \vec{\alpha}\cdot\vec{Y}_\tau(\hat{\Delta}) u_f + \mathrm{i.t.} , \tag{51}$$

$$S_{I1} = i \sum_m \sum_{i,f} Y_l^m(\hat{\Delta}) u_f^\dagger \vec{\alpha}\cdot\vec{Y}_E^*(\hat{q}) \left[\frac{2E_1 - \vec{\alpha}\cdot(\vec{\Delta}-\vec{q})}{Z_1} \right] u_i u_i^\dagger u_f + \mathrm{i.t.} , \tag{52}$$

$$S_{I2} = -i \sum_m \sum_{i,f} Y_l^{m*}(\hat{q}) u_f^\dagger \left[\frac{2E_1 - \vec{\alpha}\cdot(\vec{\Delta}-\vec{q})}{Z_1} \right] u_i u_i^\dagger \vec{\alpha}\cdot\vec{Y}_E(\hat{\Delta}) u_f + \mathrm{i.t.} , \tag{53}$$

$$Z_1 = q^2 - (p_1^2 - p_2^2) + 2\vec{q}\cdot\vec{p}_2 - i\kappa . \tag{54}$$

In Eqs. (50)—(53) the abbreviation i.t. corresponds to an "interchanged term" obtained from the first term by making the following interchange:

$$\Delta \leftrightarrow -\Delta , \quad q \leftrightarrow -q ,$$
$$p_1 \leftrightarrow p_2 , \quad E_1 \leftrightarrow E_2 . \tag{55}$$

The entire dependence on electron spins and magnetic quantum number "m" is contained in the functions S_s. The sums over electron spins are straightforward but require some special techniques.[11] The m sums take different forms for electric as compared with magnetic transitions, so we need to assign the "value" E or M to τ before carrying out these sums. In doing these sums we take the unit vector $\hat{\Delta}$ to be along with z direction and $\vec{p}_1, \vec{p}_2$ to be in the x-z plane; the various sums are tedious but straightforward and all details are given in Ref. 6. The result is

$$S_C = \frac{(2l+1)}{8\pi m_e^2 Z_1} \{ E_1(E_p^2 - \Delta^2) - (E_1\vec{p}_2 + E_2\vec{p}_1)\cdot(\vec{\Delta}-\vec{q}) \} P_l(\hat{q}\cdot\hat{\Delta}) + \mathrm{i.t.} , \tag{56}$$

$$S_{TM}=\frac{(2l+1)}{8\pi m_e{}^2 Z_1}\left[\left\{\frac{E_m}{2}(p_1{}^2-p_2{}^2+\Delta^2)-E_1E_m{}^2-E_m\vec{p}_2\cdot\vec{q}+E_2\vec{\Delta}\cdot\vec{q}+2E_1p_{2x}{}^2\right\}P_l(\hat{q}\cdot\hat{\Delta})\right.$$
$$+\frac{1}{l(l+1)}\{(\vec{p}_2\cdot\hat{q}-p_{2z}\hat{q}\cdot\hat{\Delta})(-4E_1p_{2z}-E_m\Delta+E_mq\hat{q}\cdot\hat{\Delta})$$
$$+(E_2q\Delta-E_mq\,p_{2z})(1-(\hat{q}\cdot\hat{\Delta})^2)\}\frac{P_l{}^1(\hat{q}\cdot\hat{\Delta})}{[1-(\hat{q}\cdot\hat{\Delta})^2]^{1/2}}$$
$$\left.+\frac{2E_1}{l(l+1)}\{2(\vec{p}_2\cdot\hat{q}-p_{2z}\hat{q}\cdot\hat{\Delta})^2-p_{2x}{}^2(1-(\hat{q}\cdot\hat{\Delta})^2)\}\frac{P_l{}^2(q\cdot\Delta)}{[1-(\hat{q}\cdot\hat{\Delta})^2]}\right]+\text{i.t.}\ , \tag{57}$$

$$S_{TE}=\frac{(2l+1)}{8\pi m_e{}^2 Z_1}\left[\left\{\frac{q}{2}(E_2\Delta-E_mp_{2z})+(2E_1p_2{}^2+E_m\Delta^2+\tfrac{3}{2}E_m\Delta p_{2z}-E_1E_m{}^2)\hat{q}\cdot\hat{\Delta}\right.\right.$$
$$\left.-\left[2E_1p_{2z}+\frac{E_m\Delta}{2}\right]\vec{p}_2\cdot\hat{q}+\frac{q\Delta E_2}{2}(\hat{q}\cdot\hat{\Delta})^2-\frac{E_m}{2}qp_2(\hat{q}\cdot\hat{p}_2)(\hat{q}\cdot\hat{\Delta}\right\}P_l(\hat{q}\cdot\hat{\Delta})$$
$$+\frac{1}{l(l+1)}\{(4E_1p_{2x}{}^2+E_m\Delta^2+E_m\Delta p_{2z}-E_1E_m{}^2-E_m\vec{p}_2\cdot\vec{q}+E_2\vec{\Delta}\cdot\vec{q})(1-(\hat{q}\cdot\hat{\Delta})^2)$$
$$-4E_1(\vec{p}_2\cdot\hat{q}-p_{2z}\hat{q}\cdot\hat{\Delta})^2\}\frac{P_l{}^1(\hat{q}\cdot\hat{\Delta})}{(1-(\hat{q}\cdot\hat{\Delta})^2)^{1/2}}$$
$$+\frac{1}{2l(l+1)}\{(q(E_mp_{2z}-E_2\Delta)+[4E_1(p_{2x}{}^2-p_{2z}{}^2)-E_m\Delta p_{2z}](\hat{q}\cdot\hat{\Delta})\ +(4E_1p_{2z}+E_m\Delta)\vec{p}_2\cdot\hat{q}$$
$$+q\Delta E_2(\hat{q}\cdot\hat{\Delta})^2-E_mp_2q(\hat{q}\cdot\hat{\Delta})(\hat{q}\cdot\hat{p}_2))(1-(\hat{q}\cdot\hat{\Delta})^2)$$
$$\left.-\,8E_1\hat{q}\cdot\hat{\Delta}(\vec{p}_2\cdot\hat{q}-p_{2z}\hat{q}\cdot\hat{\Delta})^2\}\frac{P_l{}^2(\hat{q}\cdot\hat{\Delta})}{1-(\hat{q}\cdot\hat{\Delta})^2}\right]$$
$$+\text{i.t.}\ , \tag{58}$$

$$S_{I1}=\frac{-(2l+1)}{8\pi m_e{}^2 Z_1}\frac{1}{[l(l+1)]^{1/2}}\left[(1-(\hat{q}\cdot\hat{\Delta})^2)\left\{2E_1E_pP_{2z}+\frac{\Delta}{2}(\Delta^2-E_m{}^2)+p_{2z}\vec{q}\cdot\vec{\Delta}-\Delta\vec{p}_2\cdot\vec{q}\right\}\right.$$
$$\left.-\,\hat{q}\cdot\hat{\Delta}(\vec{p}_2\cdot\hat{q}-p_{2z}\hat{q}\cdot\hat{\Delta})(2E_1E_p-\Delta^2+\vec{q}\cdot\vec{\Delta})\right]\ \frac{P_l{}^1(\hat{q}\cdot\hat{\Delta})}{[1-(\hat{q}\cdot\hat{\Delta})^2]^{1/2}}+\text{i.t.}\ , \tag{59}$$

$$S_{I2}=-\frac{(2l+1)}{8\pi m_e{}^2 Z_1}\frac{1}{[l(l+1)]^{1/2}}\left[\frac{q}{2}(\Delta^2-E_m{}^2)(1-(\hat{q}\cdot\hat{\Delta})^2)+\{2E_1E_p-2(\Delta p_{2z}-\vec{q}\cdot\vec{p}_2)-\Delta^2+\vec{\Delta}\cdot\vec{q}\}\right.$$
$$\left.\times(\vec{P}_2\cdot\hat{q}-p_{2z}(\hat{q}\cdot\hat{\Delta}))\right]\frac{P_l{}^1(\hat{q}\cdot\hat{\Delta})}{[1-(\hat{q}\cdot\hat{\Delta})^2]^{1/2}}+\text{i.t.}\ , \tag{60}$$

where $E_m = E_1 - E_2$.

These relations together with Eqs. (49) yield expressions for the kernels for Coulomb, transverse, and interference terms expressed as integrals over the solid angle $d\Omega_q$. In the above equations, we use series forms[18] for the Legendre and associated Legendre functions $P_l(x)$ and $P_l^m(x)$:

$$P_l^m(x) = (1-x^2)^{m/2}\left[\frac{d}{dx}\right]^m P_l(x) ,\qquad P_l(x) = \sum_{k=0}^{E(l/2)} a_k x^{l-2k} , \tag{61}$$

where

$$a_k = \frac{1}{2^l}\frac{(-1)^k(2l-2k)!}{k!(l-k)!(l-2k)!}$$

and $E(n)$ means "integral part of n."

For the choice of elastic form factor we find the Yukawa form factor

$$F_\gamma(q^2) = \frac{\gamma^2}{q^2+\gamma^2} ,\quad \gamma = \text{Yukawa parameter} \tag{62}$$

is most convenient for analytic purposes. We could handle any form factor with singularities restricted to poles in the q plane but find the choice of $F(q)$ not especially critical in this problem. With this form factor we can write the combination

$$\frac{F_\gamma[(\vec{\Delta}-\vec{q})^2]}{(\vec{\Delta}-\vec{q})^2+\lambda^2} = \left[\frac{1}{(\vec{\Delta}-\vec{q})^2+\lambda^2} - \frac{1}{(\vec{\Delta}-\vec{q})^2+\gamma^2}\right] + O(\lambda^2) \tag{63}$$

and the term $O(\lambda^2)$ can be dropped since we take the limit $\lambda \to 0$. On inserting the set of relations (56)–(63) in Eqs. (49), we find that any of the integrals over solid angle of the form

$$I_s = \int d\Omega_q \frac{F[(\vec{\Delta}-\vec{q})^2]}{(\vec{\Delta}-\vec{q})^2+\lambda^2}\,\mathrm{Re}(S_s) , \tag{64}$$

where S_s can be S_C, S_{TM}, S_{TE}, S_{I1}, or S_{I2}, can be expressed as a difference of two similar integrals:

$$I_s = \frac{(2l+1)}{8\pi m_e^2}\,\mathrm{Re}[I_l(\lambda) - I_l(\gamma)] . \tag{65}$$

The integral $I_l(\beta)$, $\beta = \lambda$ or γ, when expressed in its simplest form, looks like

$$I_l(\beta) = \frac{1}{n(\beta)V_1}\left[\sum_{n=0}^{l+2}(A_{nl}\mathscr{g}_n + B_{nl}\mathscr{h}_n) + C_l J + D_l\right] + \text{i.t.} , \tag{66}$$

$$V_1 = q^2 - p_1^2 - p_2^2 ,\quad n(\beta) = q^2 + \Delta^2 + \beta^2 , \tag{67}$$

where A_{nl}, B_{nl}, C_l, and D_l, besides being functions of the kinematic parameters E_1, E_2, p_1, p_2, q, and Δ, are also dependent on β and the coefficients a_k of Eq. (61). The quantities $\mathscr{g}_n$, $\mathscr{h}_n$, and J are the solid angle integrals:

$$\mathscr{g}_n = \int d\Omega_q \frac{(\hat{q}\cdot\hat{\Delta})^n}{1+k_1\hat{q}\cdot\hat{p}_2} , \tag{68}$$

$$\mathscr{h}_n(\beta) = \int d\Omega_q \frac{(\hat{q}\cdot\hat{\Delta})^n}{1+k_2\hat{q}\cdot\hat{\Delta}} , \tag{69}$$

$$J(\beta) = \int d\Omega_q \frac{1}{(1+k_1\hat{q}\cdot\hat{p}_2)(1+k_2\hat{q}\cdot\hat{\Delta})} , \tag{70}$$

where

$$k_1 = \frac{2qp_2}{V_1 - i\kappa} \text{ and } k_2(\beta) = -\frac{2q\Delta}{n(\beta)} . \tag{71}$$

All these integrals can be evaluated analytically. The integral $\mathscr{g}_n$ can be expressed as the sum

$$\mathscr{g}_n = 2\pi\sum_{m=0}^{E(n/2)}\frac{1}{2^{2m}}\frac{n!(\hat{\Delta}\cdot\hat{p}_2)^{n-2m}}{(n-2m)!m!}[1-(\hat{\Delta}\cdot\hat{p}_2)^2]^m\sum_{k=0}^{m}\frac{(-1)^{m-k}}{k!(m-k)!}\mathscr{f}(n-2k) , \tag{72}$$

where

$$\mathscr{I}_n=\frac{1}{k_1}\left[\frac{1}{n}\{1-(-1)^n\}-\mathscr{I}_{(n-1)}\right] \tag{73}$$

and

$$\mathscr{I}_0=\frac{V_1}{2qp_2}\ln\left|\frac{V_1+2qp_2}{V_1-2qp_2}\right| . \tag{74}$$

The integrals $\mathscr{f}_n$ are found to obey the recurrence relation

$$\mathscr{f}_n=\frac{1}{k_2}\left[\frac{2\pi}{n}\{1-(-1)^n\}-\mathscr{f}_{n-1}\right] \tag{75}$$

with

$$\mathscr{f}_0(\beta)=\frac{2\pi}{k_2}\ln\left[\frac{1+k_2}{1-k_2}\right] . \tag{76}$$

The expression for the integral J is simply

$$J=\frac{\pi V_1 n(\beta)}{q(p_1{}^2q^4+uq^2+v)^{1/2}}\ln\left|\frac{(q^2-\Delta^2)V_2+\gamma^2V_1+2q(p_1{}^2q^4+uq^2+v)^{1/2}}{(q^2-\Delta^2)V_2+\gamma^2V_1-2q(p_1{}^2q^4+uq^2+v)^{1/2}}\right| , \tag{77}$$

$$\begin{aligned}
&V_2=q^2+p_1{}^2-p_2{}^2 ,\\
&u(\beta)=-2\Delta^2p_1{}^2+\beta^2(p_1{}^2+p_2{}^2)-\beta^2\Delta^2 ,\\
&v(\beta)=p_1{}^2\Delta^4+\beta^2[\Delta^2(p_1{}^2+p_2{}^2)-(p_1{}^2-p_2{}^2)^2]+p_2{}^2\beta^4 .
\end{aligned} \tag{78}$$

With an arbitrary value of the angular momentum l, the expressions at this point start to get rather complicated and our present investigation is limited to $E1$ and $M1$ transitions.

A. Spectrum for $M1$ transitions

For magnetic dipole transitions, the result of evaluating integral I_s of Eq. (64) is

$$I_M=\frac{3}{64m_e{}^2q^2}I_{\Omega M} , \tag{79}$$

where

$$\begin{aligned}
I_{\Omega M}=\frac{\gamma^2}{\Delta}&\left[\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\ln\left|\frac{q-\Delta_{\min}}{q+\Delta_{\min}}\right|-\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\ln\left|\frac{q-\Delta_{\max}}{q+\Delta_{\max}}\right|\right]\\
&+\frac{4}{\Delta^2}[E_m(p_1{}^2-p_2{}^2)-E_p\Delta^2]\left[2\ln\left|\frac{q-\Delta}{q+\Delta}\right|-\ln\left[\frac{(q-\Delta)^2+\gamma^2}{(q+\Delta)^2+\gamma^2}\right]\right]\\
&+\frac{(q^4+K_1q^2+K_2)}{\Delta(q^2-\Delta^2)}\left[\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\ln\left|\frac{q-\Delta_{\min}}{q+\Delta_{\min}}\right|-\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\ln\left|\frac{q-\Delta_{\max}}{q+\Delta_{\max}}\right|\right]\\
&-\frac{(E_1q^4+K_3q^2+K_4)}{\Delta(p_1{}^2q^4+uq^2+v)^{1/2}}\ln\left|\frac{(q^2-\Delta^2)V_2+\gamma^2V_1+2q(p_1{}^2q^4+uq^2+v)^{1/2}}{(q^2-\Delta^2)V_2+\gamma^2V_1-2q(p_1{}^2q^4+uq^2+v)^{1/2}}\right|\\
&-\frac{(E_2q^4+\bar{K}_3q^2+\bar{K}_4)}{\Delta\sqrt{p_2{}^2q^4+\bar{u}q^2+\bar{v}}}\ln\left|\frac{(q^2-\Delta^2)V_1+\gamma^2V_2+2q(p_2{}^2q^4+\bar{u}q^2+\bar{v})^{1/2}}{(q^2-\Delta^2)V_1+\gamma^2V_2-2q(p_2{}^2q^4+\bar{u}q^2+\bar{v})^{1/2}}\right| ,
\end{aligned} \tag{80}$$

with

$$K_1=2[\Delta^2+2(p_1{}^2+p_2{}^2-E_m{}^2)]\ ,$$
$$K_2=\Delta^2[\Delta^2+4(p_1{}^2+p_2{}^2-E_m{}^2)]-4(p_1{}^2-p_2{}^2)^2\ ,$$
$$K_3=E_1K_1+E_p\gamma^2\ ,$$
$$K_4=E_1K_2+\gamma^2[E_p\Delta^2+2E_m(p_1{}^2-p_2{}^2)+4E_1(2p_2{}^2-E_m{}^2)]+E_2\gamma^4\ . \tag{81}$$

In Eq. (80), the quantities u,v are the same as those defined by Eq. (78) but with $\beta=\gamma$, and the quantities with an overbar, $\bar{u}$, $\bar{v}$, $\bar{K}_1$, $\bar{K}_2$, etc., are interchanged terms, i.e., they are given by the expressions for the corresponding u, v, K_1, K_2, etc., subject to the interchange of Eq. (55). Hereafter we shall use the overbar exclusively to indicate interchanged terms.

The expression for the square of the matrix element then reduces to

$$|H^{M1}|^2=\frac{4\alpha}{3\pi^2}\int dq\,dq'q^2q'^2\frac{F_T^{M1}(q)F_T^{M1}(q')}{(\omega^2-q^2)(\omega^2-q'^2)}X_T^{M1}\ , \tag{82}$$

where the kernel can be written as a sum of two terms

$$X_T^{M1}=X_{BA}^{M1}+X_{\rm corr}^{M1}\ ,$$

the first of which

$$X_{BA}^{M1}=\frac{(2\pi)^6}{V^2}\frac{m_e{}^2}{E_1E_2}\frac{\delta(\Delta-q')\delta(\Delta-q)}{\Delta^4}S_T^0$$

yields the expression in first order. The second term

$$X_{\rm corr}^{M1}=\frac{(2\pi)^6}{V^2}\frac{3}{64E_1E_2}\frac{\delta(\Delta-q')}{\Delta^2}\frac{\alpha Z}{\pi^2}\cdot\frac{1}{q^2}I_{\Omega M}$$

yields the second order correction to the matrix element which we now write as

$$|H^{M1}|^2=|H_{BA}^{M1}|^2+\hbar_{TM}\ , \tag{83}$$

where

$$\hbar_{TM}=\frac{\alpha}{4E_1E_2}\frac{(2\pi)^4}{V^2}\frac{\alpha Z}{\pi^2}\frac{F_T^{M1}(\Delta)}{(\omega^2-\Delta^2)}\mathscr{I}_M\ . \tag{84}$$

The q dependence of the correction part $\hbar_{TM}$ is required explicitly for the integral

$$\mathscr{I}_M=\int_0^\infty dq\frac{F_T^{M1}(q)}{\omega^2-q^2}I_{\Omega M} \tag{85}$$

with $I_{\Omega M}$ given by Eq. (80). We notice that at the origin of the complex q plane, the argument of each of the logarithmic terms becomes unity so that $(I_{\Omega M})_{q=0}=0$, and the integrand of the q integral in Eq. (85) vanishes. All q integrals are to be evaluated in the complex q plane shown in Fig. 6. There are simple poles at $q=\pm\omega,\pm\Delta$ and four branches in the positions shown, to which must be added any singularities in the factor F_T.

Before deciding the mode of integration for the integral $\mathscr{I}_M$ we recall that the evaluation of the virtual photon spectrum involves yet another integration of the matrix element over the physical momentum transfer Δ which in general must be performed numerically. Thus, if we perform the q integral of Eq. (85) also numerically, then our final expression for the virtual photon spectrum will involve a double numerical integration, one of them over an infinite range. Although in principle this is possible, things get somewhat clumsy. Therefore, it is a decided advantage to evaluate the integral $\mathscr{I}_M$ analytically. This can be achieved only by making an appropriate choice for the inelastic form factor.

In Sec. III we observed that the virtual photon spectrum is insensitive to the details of the inelastic

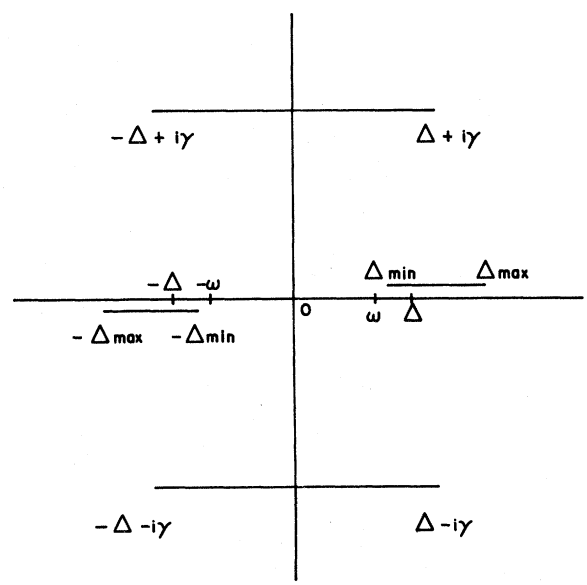

FIG. 6. Complex q plane.

form factor and depends only on the transition radius for a large range of energies. This observation gives us the freedom of choosing an inelastic distribution for which the inelastic form factor also has only poles. Such a form factor is given by Eq. (36), which we rewrite as

$$F_T^{M1}(q)=\frac{N_m q}{(a^2+q^2)^4}(1+C_2q^2)\ , \tag{86}$$

where

$$C_2=(16\epsilon-1)/a^2\ .$$

This form factor has only fourth order poles at $q=\pm ia$ and together with the expression $I_{\Omega M}$ of Eq. (80), then has a suitable analytic structure for the q integrals of $\mathscr{I}_M$ [Eq. (85)] to be evaluated by contour integration. Let us define the quantities

$$\mathscr{D}(k)=\int_0^\infty \frac{dq\, q^{2k-1}}{(a^2+q^2)^4(\omega^2-q^2)}\ln\left[\frac{(q-\Delta)^2+\gamma^2}{(q+\Delta)^2+\gamma^2}\right]\ , \tag{87}$$

$$\mathscr{F}(k,y,g)=\int_0^\infty \frac{dq\, q^{2k-1}}{(a^2+q^2)^4(g^2-q^2)}\ln\left|\frac{q-y}{q+y}\right|\ , \tag{88}$$

$$\mathscr{L}(k)=\int_0^\infty \frac{dq\, q^{2k-1}}{(a^2+q^2)^4(\omega^2-q^2)(p_1{}^2q^4+uq^2+v)^{1/2}}\ln\left|\frac{(q^2-\Delta^2)V_2+\gamma^2V_1+2q(p_1{}^2q^4+uq^2+v)^{1/2}}{(q^2-\Delta^2)V_2+\gamma^2V_1-2q(p_1{}^2q^4+uq^2+v)^{1/2}}\right|\ , \tag{89}$$

$$\mathscr{P}(k,y)=\frac{1}{\omega^2-\Delta^2}[\mathscr{F}(k,y,\omega)-\mathscr{F}(k,y,\Delta)]\ , \tag{90}$$

all of which can be evaluated in closed form, which are unfortunately rather long and are given in the Appendix. If we now introduce the notation

$$\begin{aligned}
&\mathscr{D}_M=\mathscr{D}(1)+C_2\mathscr{D}(2)\ ,\\
&\mathscr{T}_M(y)=\mathscr{F}(1,y,\omega)+C_2\mathscr{F}(2,y,\omega)\ ,\\
&\mathscr{P}_M(y)=\frac{1}{\Delta}[C_2\mathscr{P}(4,y)+(1+C_2K_1)\mathscr{P}(3,y)+(K_1+C_2K_2)\mathscr{P}(2,y)+K_2\mathscr{P}(1,y)]\ ,\\
&\mathscr{S}_M=E_1C_2\mathscr{L}(4)+(E_1+C_2K_3)\mathscr{L}(3)+(K_3+C_2K_4)\mathscr{L}(2)+K_4\mathscr{L}(1)\ ,
\end{aligned} \tag{91}$$

then the expression for the integral $\mathscr{I}_M$ can be written as

$$\begin{aligned}
\mathscr{I}_M=N_M\Bigg\{\frac{\gamma^2}{\Delta}\Bigg[&\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\mathscr{T}_M(\Delta_{\min})-\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\mathscr{T}_M(\Delta_{\max})\\
&+\frac{4}{\Delta^2}[E_M(p_1{}^2-p_2{}^2)-E_p\Delta^2][2\mathscr{T}_M(\Delta)-\mathscr{D}_M]\\
&+\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\mathscr{P}_M(\Delta_{\min})-\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\mathscr{P}_M(\Delta_{\max})-\frac{1}{\Delta}(\mathscr{S}_M+\mathscr{T}_M)\Bigg]\Bigg\}\ .
\end{aligned} \tag{92}$$

The virtual spectrum for $M1$ radiation is then obtained by following the same procedure as in Sec. III, and the result is

$$\begin{aligned}
N^{M1}(E_e,\omega)=\frac{\alpha}{2\pi}\frac{\omega^2}{p_1{}^2}\frac{1}{|F_T^{M1}(\omega)|^2}\int_{\Delta_{\min}}^{\Delta_{\max}}\Bigg[&\left\{\frac{(\Delta_{\max}{}^2-\Delta^2)(\Delta^2-\Delta_{\min}{}^2)}{\Delta^2}+2(\Delta^2-E_M{}^2)\right\}\\
&\times\frac{|F_T^{M1}(\Delta)|^2}{(\omega^2-\Delta^2)^2}+\frac{\alpha Z}{2\pi}\frac{F_T^{M1}(\Delta)}{(\omega^2-\Delta^2)}\mathscr{I}_M\Bigg]\Delta\, d\Delta\ .
\end{aligned} \tag{93}$$

B. Spectrum for $E1$ transitions

The expression for the $E1$ virtual photon spectrum can be obtained by the same sequence of steps as used for the case of $M1$ transitions, but is more complicated because we have contributions from transverse electric, Coulomb, and interference terms. The transverse electric form factor is given by

$$F_T^{E1}(q)=\frac{N_E}{(a^2+q^2)^4}(1+C_2q^2) \tag{94}$$

and the Coulomb form factor is obtained using relation (35). The square of the matrix element for electric dipole transitions can be expressed as

$$|H^{E1}|^2=|H_{BA}^{E1}|^2+\mathscr{h}_{\rm TE}+\mathscr{h}_C+\mathscr{h}_I\ , \tag{95}$$

where the first term can be identified with the first order matrix element, and $\mathscr{h}_{\rm TE}$, $\mathscr{h}_C$, and $\mathscr{h}_I$ give the contributions of the transverse electric, Coulomb, and interference terms to the second order correction and are given by expressions similar to Eq. (84), namely:

$$\mathscr{h}_{\rm TE}=\frac{\alpha}{4E_1E_2}\frac{(2\pi)^4}{V^2}\frac{\alpha Z}{\pi^2}\frac{F_T^{E1}(\Delta)}{\Delta^2(\omega^2-\Delta^2)}\mathscr{I}_E\ , \tag{96}$$

$$\mathscr{h}_C=\frac{\alpha}{4E_1E_2}\frac{(2\pi)^4}{V^2}\frac{\alpha Z}{\pi^2}\frac{F_T^{E1}(\Delta)}{2\omega^2\Delta^2}\mathscr{I}_C\ , \tag{97}$$

$$\mathscr{h}_I=\frac{\alpha}{4E_1E_2}\frac{(2\pi)^4}{V^2}\frac{\alpha Z}{\pi^2}\frac{F_T^{E1}(\Delta)}{\omega\Delta^2}\left[\mathscr{I}_{IE}+\frac{1}{(\Delta^2-\omega^2)}\mathscr{I}_{IC}\right]\ , \tag{98}$$

where for the transverse contribution, $\mathscr{I}_E$ can be expressed as

$$\begin{aligned}\mathscr{I}_E=N_E\Bigg\{&\gamma^2\omega^2\left[\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\mathscr{T}_E(\Delta_{\min})-\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\mathscr{T}_E(\Delta_{\max})\right]\\&+2(p_1{}^2-p_2{}^2)\left[\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\mathscr{C}_E(\Delta_{\min})-\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\mathscr{C}_E(\Delta_{\max})\right]\\&+\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\mathscr{P}_E(\Delta_{\min})-\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\mathscr{P}_E(\Delta_{\max})+4\Delta(2\mathscr{G}_E-\mathscr{D}_E)-(\mathscr{S}_E+\overline{\mathscr{P}}_E)\Bigg\}\ ,\end{aligned} \tag{99}$$

where $\mathscr{T}_E$, $\mathscr{C}_E$, $\mathscr{P}_E$, etc., can be written in terms of the integrals defined by the set of Eqs. (87)–(90):

$$\mathscr{G}_E=C_2E_p\mathscr{F}(2,\Delta,\omega)+[E_p-C_2E_M(p_1{}^2-p_2{}^2)]\mathscr{F}(1,\Delta,\omega)-E_M(p_1{}^2-p_2{}^2)\mathscr{F}(0,\Delta,\omega)\ ,$$

$$\mathscr{C}_E(y)=-\Delta^2\mathscr{F}(0,y,\omega)+(1-C_2\Delta^2)\mathscr{F}(1,y,\omega)+C_2\mathscr{F}(2,y,\omega)\ ,$$

$$\mathscr{D}_E=C_2E_p\mathscr{D}(2)+[E_p-C_2E_M(p_1{}^2-p_2{}^2)]\mathscr{D}(1)-E_M(p_1{}^2-p_2{}^2)\mathscr{D}(0)$$

$$\mathscr{T}_E(y)=\mathscr{F}(0,y,\omega)+C_2\mathscr{F}(1,y,\omega)\ ,$$

$$\mathscr{P}_E(y)=l_1C_2\mathscr{P}(3,y)+(l_1+C_2l_2)\mathscr{P}(2,y)+(l_2+C_2l_3)\mathscr{P}(1,y)+l_3\mathscr{P}(0,y)\ ,$$

$$\mathscr{S}_E=l_6\mathscr{L}(0)+(l_5+C_2l_6)\mathscr{L}(1)+(l_4+C_2l_5)\mathscr{L}(2)+C_2l_4\mathscr{L}(3)\ ,$$

and the quantities l_1, l_2, etc., depend on kinematic parameters and are given by

$$l_1=2\Delta^2-E_M{}^2\ ,$$

$$l_2=2[\Delta^4-3\Delta^2E_M{}^2+4\Delta^2(p_1{}^2+p_2{}^2)-(p_1{}^2-p_2{}^2)^2]\ ,$$

$$l_3=-\Delta^2[\Delta^2E_M{}^2+2(p_1{}^2-p_2{}^2)^2]\ ,$$

$$l_4=E_1[l_1-2(p_1{}^2-p_2{}^2)]\ ,$$

$$l_5=E_1[l_2+4\Delta^2(p_1{}^2-p_2{}^2)]+\gamma^2[2\Delta^2(E_1+E_p)-E_p(p_1{}^2-p_2{}^2)-2E_1E_M{}^2]\ ,$$
$$l_6=E_1[l_3-2\Delta^4(p_1{}^2-p_2{}^2)]-\gamma^2[2E_1E_M{}^2\Delta^2+E_p\Delta^2(p_1{}^2-p_2{}^2)-2E_1(p_1{}^2-p_2{}^2)^2]$$
$$-\gamma^4[E_1E_M{}^2-E_M(p_1{}^2-p_2{}^2)]\ .$$

In order to write an explicit expression for the integral $\mathscr{I}_C$ we need to define additional integrals:

$$D(k)=\int_0^\infty dq\frac{q^{2k-1}}{(a^2+q^2)^4}\ln\left[\frac{(q-\Delta)^2+\gamma^2}{(q+\Delta)^2+\gamma^2}\right]\ , \tag{100}$$

$$\mathscr{F}_C(k,y)=\int_0^\infty dq\frac{q^{2k-1}}{(a^2+q^2)^4}\ln\left|\frac{q-y}{q+y}\right|\ , \tag{101}$$

$$\mathscr{L}_C(k)=\int_0^\infty \frac{dq\,q^{2k-1}}{(a^2+q^2)^4(p_1{}^2q^4+uq^2+v)^{1/2}}\ln\left[\frac{(q^2-\Delta^2)V_2+\gamma^2V_1+2q(p_1{}^2q^4+uq^2+v)^{1/2}}{(q^2-\Delta^2)V_2+\gamma^2V_1-2q(p_1{}^2q^4+uq^2+v)^{1/2}}\right]\ . \tag{102}$$

The results of integration for various values of the integer k are given in the Appendix. In terms of these integrals, if we introduce the notation

$$\mathscr{T}_C(y)=\mathscr{F}_C(0,y)+C_2\mathscr{F}_C(1,y)\ ,$$
$$\mathscr{G}_C=\Delta^2\mathscr{F}_C(0,\Delta)+(1+C_2\Delta^2)\mathscr{F}_C(1,\Delta)+C_2\mathscr{F}_C(2,\Delta)\ ,$$
$$\mathscr{D}_C=(\Delta^2+\gamma^2)D(0)+\{1+C_2(\Delta^2+\gamma^2)\}D(1)+C_2D(2)\ ,$$
$$\mathscr{E}_C(y)=-l_8\mathscr{F}(0,y,\Delta)-(l_7+C_2l_8)\mathscr{F}(1,y,\Delta)+(1-C_2l_7)\mathscr{F}(2,y,\Delta)+C_2\mathscr{F}(3,y,\Delta)\ ,$$
$$\mathscr{S}_C=l_0\mathscr{L}_C(0)+(l_9+C_2l_0)\mathscr{L}_C(1)+(-E_1+C_2l_9)\mathscr{L}_C(2)-C_2E_1\mathscr{L}_C(3)\ ,$$

with

$$l_7=2(E_p{}^2-\Delta^2)\ ,$$
$$l_8=\Delta^2(2E_p{}^2-\Delta^2)\ ,$$
$$l_9=E_1l_7-E_m\gamma^2\ ,$$
$$l_0=E_1l_8+\gamma^2(2E_1E_p{}^2-E_M\Delta^2)+E_2\gamma^4\ ,$$

then the expression for $\mathscr{I}_C$ is given by

$$\mathscr{I}_C=N_E\left[\gamma^2\left\{\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\mathscr{T}_C(\Delta_{\min})-\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\mathscr{T}_C(\Delta_{\max})\right\}+\frac{2E_p}{\Delta}(-2\mathscr{G}_C+\mathscr{D}_C)\right.$$
$$\left.-\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\mathscr{E}_C(\Delta_{\min})+\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\mathscr{E}_C(\Delta_{\max})-(\mathscr{S}_C+\overline{\mathscr{T}}_C)\right]\ . \tag{103}$$

The contribution of the interference terms, $\hbar_I$, Eq. (98), contains the integrals $\mathscr{I}_{IE}$ and $\mathscr{I}_{IC}$. Of these, the constituent integrals of $\mathscr{I}_{IE}$ are similar in nature to those of $\mathscr{I}_M$ and $\mathscr{I}_E$. Thus if we define

$$\mathscr{D}_{IE}=(\Delta^2+\gamma^2)\mathscr{D}(0)+[1+C_2(\Delta^2+\gamma^2)]\mathscr{D}(1)+C_2\mathscr{D}(2)\ ,$$
$$\mathscr{G}_{IE}=\Delta^2\mathscr{F}(0,\Delta,\omega)+(1+C_2\Delta^2)\mathscr{F}(1,\Delta,\omega)+C_2\mathscr{F}(2,\Delta,\omega)\ ,$$
$$\mathscr{E}_{IE}(y)=[2E_p(p_1{}^2-p_2{}^2)-E_M(\Delta^2+\gamma^2)]\mathscr{F}(0,y,\omega)$$
$$+[E_M+C_2\{2E_p(p_1{}^2-p_2{}^2)-E_M(\Delta^2+\gamma^2)\}]\mathscr{F}(1,y,\omega)+C_2E_M\mathscr{F}(2,y,\omega)\ ,$$
$$\mathscr{S}_{IE}=i_3\mathscr{L}(0)+(i_2+C_2i_3)\mathscr{L}(1)+(i_1+C_2i_2)\mathscr{L}(2)+i_1C_2\mathscr{L}(3)\ ,$$

with

$$i_1=E_1(E_p+2E_1)\ ,$$
$$i_2=-[2E_1\{2E_1\Delta^2-E_p(p_1{}^2-p_2{}^2)\}+\gamma^2(2\Delta^2-E_M{}^2-2E_1E_p)]\ ,$$
$$i_3=E_1\Delta^2[E_m\Delta^2-2E_p(p_1{}^2-p_2{}^2)]+\gamma^2[\Delta^2E_M{}^2-2E_1E_p(p_1{}^2-p_2{}^2)]-E_2E_M\gamma^4\ ,$$

then the integral $\mathscr{I}_{IE}$ can be expressed as

$$\mathscr{I}_{IE}=N_E\left[\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\mathscr{E}_{IE}(\Delta_{\min})-\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\mathscr{E}_{IE}(\Delta_{\max})\right.$$
$$\left.-2E_p\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\mathscr{T}_M(\Delta_{\min})+\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\mathscr{T}_M(\Delta_{\max})+\frac{2E_ME_p}{\Delta}(2\mathscr{G}_{IE}-\mathscr{D}_{IE})-\mathscr{S}_{IE}+\overline{\mathscr{T}}_{IE}\right]\ . \tag{104}$$

On the other hand, the constituent integrals of $\mathscr{I}_{IC}$ are similar in nature to those of $\mathscr{I}_C$, Eqs. (100)—(102). In terms of the integrals of Eqs. (100)—(102), if we introduce notation

$$\mathscr{D}_{IC}=D(0)+C_2D(1)\ ,$$
$$\mathscr{E}_{IC}(y)=[2E_p(p_1{}^2-p_2{}^2)+E_M(\Delta^2+\gamma^2)]\mathscr{F}_C(0,y)$$
$$+[-E_M+C_2\{2E_p(p_1{}^2-p_2{}^2)+E_M(\Delta^2+\gamma^2)\}]\mathscr{F}_C(1,y)-C_2E_M\mathscr{F}_C(2,y)\ ,$$
$$\mathscr{S}_{IC}=i_5\mathscr{L}_C(0)+(i_4+C_2i_5)\mathscr{L}_C(1)+(E_1E_M+C_2i_4)\mathscr{L}_C(2)+C_2E_1E_M\mathscr{L}_C\ ,(3)$$

with

$$i_4=-[2E_1\{2E_1\Delta^2+E_p(p_1{}^2-p_2{}^2)\}+\gamma^2(2\Delta^2-E_M{}^2)]\ ,$$
$$i_5=E_1\Delta^4(2E_1+E_p)+2E_1E_p\Delta^2(p_1{}^2-p_2{}^2)+\gamma^2[2E_1E_p(\Delta^2-p_1{}^2+p_2{}^2)+\Delta^2E_M{}^2]-E_2E_M\gamma^4\ ,$$

then

$$\mathscr{I}_{IC}=N_E\left[\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\mathscr{E}_{IC}(\Delta_{\min})-\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\mathscr{E}_{IC}(\Delta_{\max})\right.$$
$$-2E_p\Delta^2\left\{\left[\frac{E_2}{p_2}+\frac{E_1}{p_1}\right]\mathscr{T}_C(\Delta_{\min})-\left[\frac{E_2}{p_2}-\frac{E_1}{p_1}\right]\mathscr{T}_C(\Delta_{\max})\right\}$$
$$\left.+4E_ME_p\Delta(2\mathscr{T}_C(\Delta)-\mathscr{D}_{IC})+\mathscr{S}_{IC}-\overline{\mathscr{T}}_{IC}\right]\ . \tag{105}$$

In terms of the integrals $\mathscr{I}_E$, $\mathscr{I}_C$, $\mathscr{I}_{IE}$, and $\mathscr{I}_{IC}$ the expression for the virtual photon spectrum in second order Born approximation for electric dipole transitions can be written as

$$N^{E1}(E_e,\omega)=\frac{\alpha}{2\pi}\frac{\omega^2}{p_1{}^2}\frac{1}{|F_T^{E1}(\omega)|^2}\int_{\Delta_{\min}}^{\Delta_{\max}}\left[\left\{\frac{(\Delta_{\max}{}^2-\Delta^2)(\Delta^2-\Delta_{\min}{}^2)}{\Delta^2}+2(\Delta^2-E_M{}^2)\right\}\right.$$
$$\times\frac{|F_T^{E1}(\Delta)|^2}{(\omega^2-\Delta^2)^2}+\frac{|F_T^{E1}(\Delta)|^2}{\omega^2\Delta^2}(E_p{}^2-\Delta^2)+\frac{\alpha Z}{2\pi}\frac{F_T^{E1}(\Delta)}{\Delta^2}$$
$$\left.\times\left\{\frac{1}{\omega^2-\Delta^2}\mathscr{I}_E+\frac{1}{2\omega^2}\mathscr{I}_C+\frac{1}{\omega}\left[\mathscr{I}_{IE}+\frac{1}{\Delta^2-\omega^2}\mathscr{I}_{IC}\right]\right\}\right]\Delta\,d\Delta\ . \tag{106}$$

V. RESULTS AND CONCLUSIONS

The evaluation of the integrals $\mathscr{I}_M$, $\mathscr{I}_E$, etc., has involved rather extensive manipulations, and in order to check these expressions we also performed the same integrations by numerical integration for selected values of the parameters. The two modes of integration have provided a vital check on one another.

Since our calculation takes account of charge effects of the nucleus to order αZ only, we have compared our second order calculations with three other approximations, the distorted wave Born (DWBA), the plane wave Born approximation (PWBA), and the conventional long wavelength method. In DWBA (Ref. 19) the size and charge corrections are taken into account, in principle, to all orders in αZ, but the results are restricted by computational difficulties. By PWBA we mean that produced by the first order term only and accounting only for the finite size of the nucleus. The conventional method produces results independent of both size and charge of the nucleus.

In the results that follow we are using primarily the form factors of Eqs. (36) and (37) with the value of $\epsilon=0.1$ so that the value of the parameter a in Eq. (42) is

$$a=\frac{\sqrt{34}}{R_{tr}} . \tag{107}$$

We have related the transition radius to the root-mean-square radius by

$$R_{tr}=1.27R_{rms} , \tag{108}$$

where R_{rms} itself is determined from the mass number A of the nucleus[20]:

$$R_{rms}=(\tfrac{3}{5})^{1/2}\left[1.12A^{1/3}+\frac{2.15}{A^{1/3}}-\frac{1.75}{A}\right] . \tag{109}$$

The Yukawa parameter of Eq. (62) is related to R_{rms} by

$$\gamma=\frac{\sqrt{6}}{R_{rms}} . \tag{110}$$

Instead of the differential cross section $d\sigma^{ML}/d\Omega$ we display the relative cross section, defined for magnetic transitions by

$$\sigma_{rel}(\Delta)=\frac{d\sigma^{ML}}{d\Omega}\Bigg/\left\{4\pi e^2\frac{p_2}{p_1}\left[\frac{(\Delta_{max}{}^2-\Delta^2)(\Delta^2-\Delta_{min}{}^2)}{2\Delta^2(\Delta^2-\omega^2)^2}+\frac{1}{(\Delta^2-\omega^2)}\right]\right\} ,$$

so that in first order this should reduce to $|F_T^{ML}(\Delta)|^2$ according to Eq. (25b). Figure 7 shows the $M1$ relative cross section evaluated using second order and distorted wave calculations (DUELS) (Ref. 21), and compared with the form factor squared. We observe that there is good agreement between the second order and distorted wave calculations.

We have chosen two nuclei, ${}^{40}_{20}$Ca and ${}^{114}_{48}$Cd, to illustrate the second order results. The choice of nuclei is largely arbitrary; we avoid the very light nuclei where the effects we are discussing are small and the heaviest nuclei where we are not confident the second order approximation applies (a point which will be discussed later). Figure 8 presents the effective $M1$ virtual photon spectra for a range of electron energies for ${}^{40}_{20}$Ca and ${}^{114}_{48}$Cd. Around 10 MeV, the finite size does not contribute and the correction is mainly due to the charge; consequently the corrected spectrum lies well above the conventional spectrum. Around 40–50 MeV the charge and the size corrections neutralize each other and the second order results almost coincide with the conventional spectrum. At 100 MeV the size effect

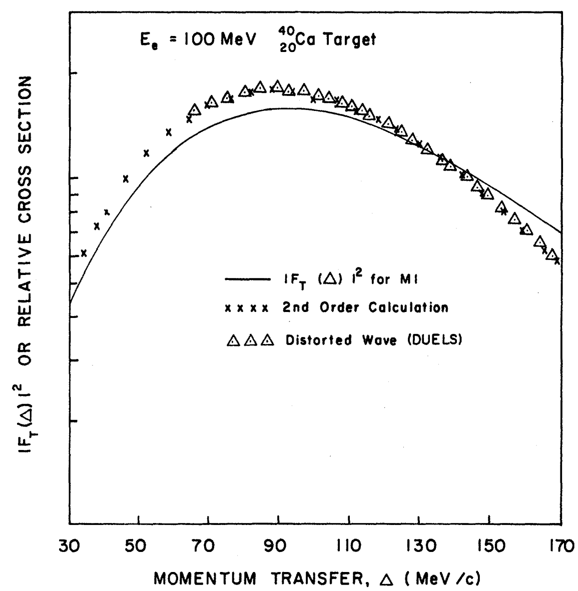

FIG. 7. Comparison of PWBA form factor with SOBA and DWBA relative cross sections for $M1$ transitions. The target nucleus is ${}^{40}_{20}$Ca and incident electron energy $E_e=100$ MeV. The values of the parameters are $a=264$ MeV, $\gamma=141$ MeV.

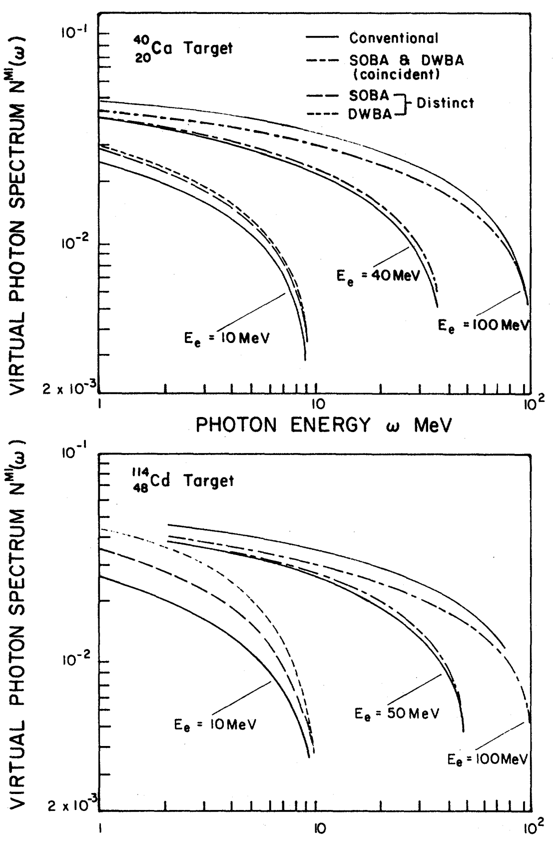

FIG. 8. $M1$ virtual photon spectra evaluated in SOBA compared with the corresponding DWBA results. For $^{40}_{20}$Ca the incident electron kinetic energies are 10, 40, and 100 MeV and for $^{114}_{48}$Cd the kinetic energies are 10, 50, and 100 MeV.

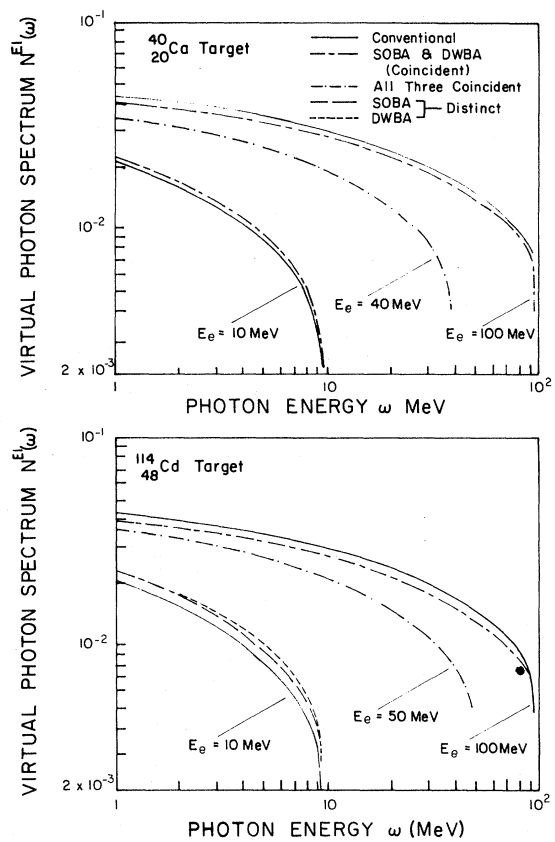

FIG. 9. Same as Fig. 8 but for $E1$ virtual photon spectra.

has taken over and the spectrum is below the conventional value. In all three cases we see that the second order and the distorted wave results compare very well for $^{40}_{20}$Ca. However, this agreement is not quite as good at higher Z and low energies, as can be seen from the spectrum for $^{114}_{48}$Ca at 10 MeV. There is still fairly good agreement between second order and distorted wave results at 50 and 100 MeV. We note that where the distorted wave calculation gives different results it is larger than the second order contribution. By contrast, for $E1$ spectra shown in Fig. 9, not only do SOBA and DWBA spectra coincide for $^{40}_{20}$Ca and $^{114}_{48}$Cd, but the difference between SOBA and DWBA results at 10 MeV is not quite as drastic as it is for the $M1$ spectrum. In either case we observe that with increasing electron energy the agreement between SOBA and DWBA results becomes better.

The charge correction tends to zero with an increase in electron energy and we are left with an increasingly important finite size correction. At appropriately high electron energies (depending on Z) the first order spectrum is entirely adequate; neither second order nor distorted wave calculations produce any appreciable corrections.

In the spirit of the observations made in Sec. III, we expect to find that the corrected virtual photon spectrum is insensitive to the shapes of the charge and current distributions over a wide range of energies and depends only on the root-mean-square and transition radii. For this we have used the distorted wave calculations for the same nuclei and for the same energies as in Figs. 8 and 9, but using charge and current densities derived from a Fermi distribution, we find the same agreement between second order and distorted wave results. This is true even for heavy nuclei; Fig. 10 illustrates the model independence of the $E1$ virtual photon spectrum for electrons of kinetic energy 100 MeV scattered from a $^{238}_{92}$U target.

We have carried out these comparisons for a wide variety of electron energies using several nuclei and have summarized our results in a range of validity

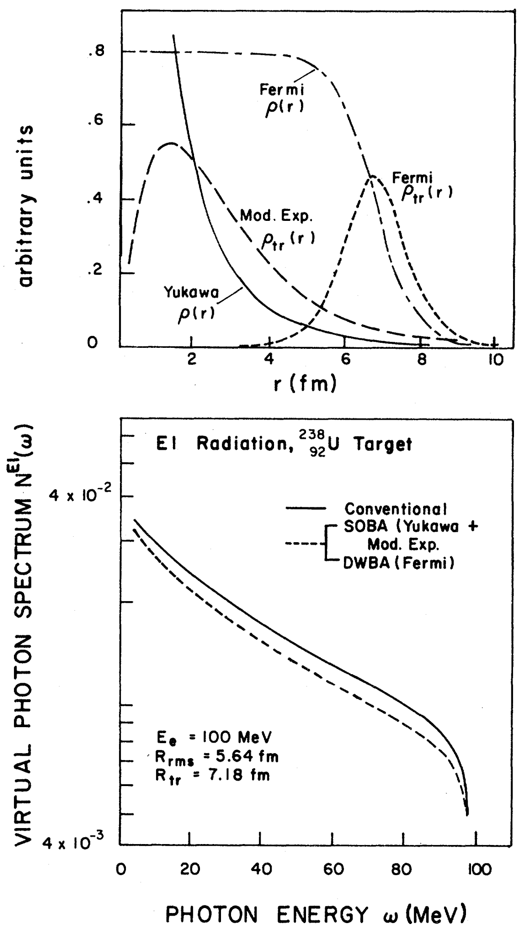

FIG. 10. The upper part shows two different ground state charge densities, both corresponding to the same $R_{rms}=5.64$ fm, and two transition charge densities, both with $R_{tr}=7.18$ fm. The lower part shows that with the different distributions we still get the same spectrum. The parameters correspond to a $^{238}_{92}$U target.

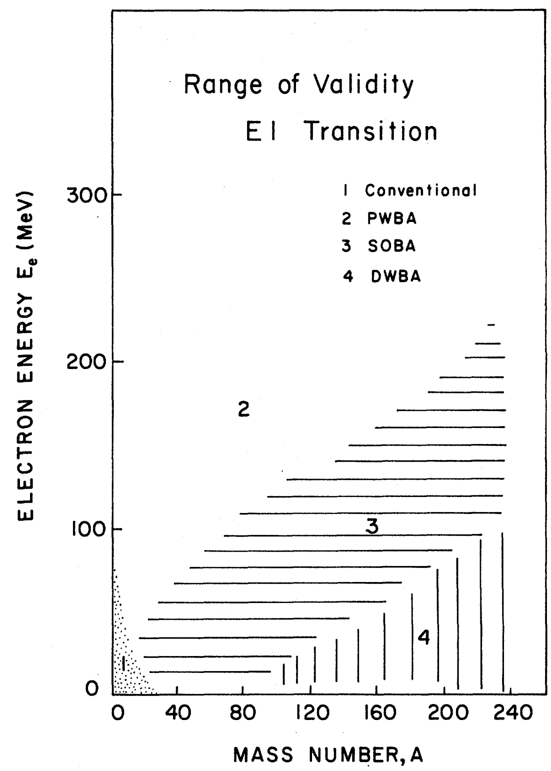

FIG. 11. Range of validity for $E1$ virtual photon spectra evaluated using conventional method, PWBA, SOBA, and DWBA.

plot. Figure 11 presents such a plot for $E1$ spectra. For very light nuclei at low energies the conventional spectra evaluated using plane wave and point nucleus approximations yields reasonably good results. This is a consequence of the fact that for very light nuclei at low energies there is virtually no overlap of electron and nuclear wave functions. This region is depicted by dots in Fig. 11 (region 1). For light nuclei at moderately high energies and for heavier nuclei at very high energies, it is sufficient to consider only the first order Born approximation (region 2 of Fig. 11). In the remaining region we need to include both the size and charge of the nucleus. It can be further divided into two regions; the first, depicted by horizontal lines in Fig. 11, gives the region in which the second order results agree with the distorted wave results to an accuracy of 2%, and the second, marked by vertical lines, gives the region where we cannot do without distorted waves. In Fig. 12 we show the same for $M1$ spectra. Although the second order Born approximation has some limitations where the Coulomb distortion is very strong, where it is valid it has several advantages.

We can pursue second order calculations for arbitrarily high energies and extreme values of ω, both high and low, and still get consistent results. The distorted wave program involves several double infinite sums; consequently it suffers from convergence problems in some regions. At high energies it is further limited by the number of partial waves that must be used. Also in comparison to the distorted wave program very little computer time is consumed (from $\frac{1}{50}$ to $\frac{1}{30}$ of the CPU time for the distorted wave program). The second order results can be also used for other leptons such as the muons,[22] by simply replacing the electron mass by the lepton mass.

Owing to the difficulties encountered by the distorted wave program at high energies, it is sometimes difficult to separate the validity range. In Fig. 12 we show, by leaving a cross-shaded region, where

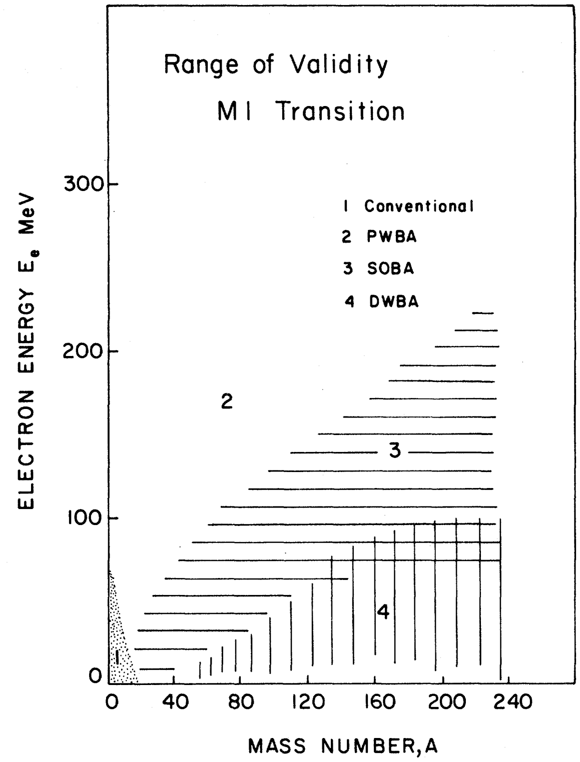

FIG. 12. Range of validity for $M1$ virtual photon spectra evaluated using conventional method, PWBA, SOBA, and DWBA.

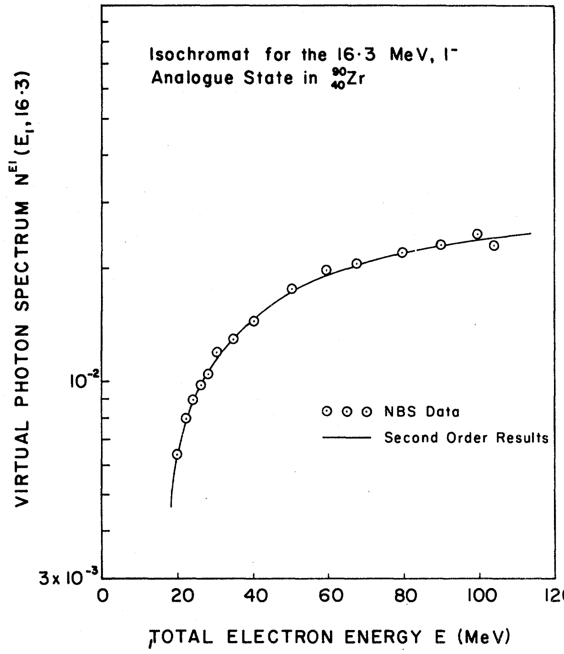

FIG. 13. Isochromat for the excitation of the 16.3 MeV, 1^- isobaric analog state in ${}^{90}_{40}Zr$. The size of the circles is indicative of the statistical error.

we had difficulty marking a boundary for the region of applicability of the second order results.

In a recent experiment Dodge *et al.*[23] measured the cross section for the excitation of the 16.3 MeV 1^- isobaric analog state (IAS) in ${}^{90}_{40}Zr$ with both real and virtual photons in the energy range 17 to 105 MeV. Since this is a very narrow state it is an opportunity to study the virtual photon spectrum at fixed energy ω but variable electron energy E_1. Such a plot is called an isochromat. The open circles in Fig. 13 show the virtual photon spectrum as an isochromat, extracted from their data. The solid line in Fig. 13 represents the same spectrum evaluated from our second order results, and we can say that the agreement could not be better.

The second order calculations presented in this work take the finite size and charge of the nucleus into account simultaneously, thus providing an improvement over all published calculations for the virtual photon spectra. Distorted wave calculations which include the size and charge effects simultaneously have been done in a preliminary form by Onley[19] and are used to help check and establish the range of validity of the second order calculation. To some extent the second order calculations have helped in tuning the distorted wave program (for example, in determining the conditions necessary to ensure convergence). The second order expression is free of singularities and does not involve infinite sums or infinite integrals; consequently it yields very consistent results. In the range of validity graphs (Figs. 11 and 12), we see that the existence of second order results removes the critical need for extending the distorted wave programs to higher energies where it plunges into computational problems.

We would welcome experiments measuring the excitation of narrow states of known spin and parity as a function of electron energy. These effectively are a measurement of the virtual photon spectrum. Similar experiments comparing excitation by electrons and positrons or muons of both signs of charge would be particularly helpful in establishing the charge dependence of the spectrum.

Calculations reported here are incorporated in a FORTRAN program named SOVPS and run the Ohio University IBM 4341 computer. Copies are available from the authors, although full documentation of the program and extension to higher multipoles will be, we anticipate, the subject of a later communication.

ACKNOWLEDGMENTS

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APPENDIX

The integrals $\mathscr{D}(k)$, $\mathscr{F}(k,y,g)$, $D(k)$, $\mathscr{F}_C(k,y)$, $\mathscr{L}(k)$, and $\mathscr{L}_C(k)$ defined in Eqs. (87)—(89) and (100)—(102) have been evaluated by contour integration. We give only the results here. For details see Ref. 6.

$$\begin{aligned}\mathscr{D}(0)=&-\frac{\pi}{\omega^2(a^2+\omega^2)^4}\left[\arctan\left[\frac{\gamma}{\Delta+\omega}\right]+\arctan\left[\frac{\gamma}{\Delta-\omega}\right]\right]+\frac{2\pi}{a^8\omega^2}\arctan\left[\frac{\gamma}{\Delta}\right]\\&-\frac{2\pi(4a^6+6a^4\omega^2+4a^2\omega^4+\omega^6)}{a^8(a^2+\omega^2)^4}\arctan\left[\frac{a+\gamma}{\Delta}\right]\\&+\frac{\pi\Delta}{12a^5(a^2+\omega^2)[\Delta^2+(a+\gamma)^2]}\left[\frac{33}{2a^2}+\frac{9(a+\gamma)}{a[\Delta^2+(a+\gamma)^2]}+\frac{3(a+\gamma)^2-\Delta^2}{[\Delta^2+(a+\gamma)^2]^2}\right.\\&\left.\qquad+\frac{6}{a^2+\omega^2}\left[3+\frac{a(a+\gamma)}{\Delta^2+(a+\gamma)^2}\right]+\frac{3(3a^2-\omega^2)}{(a^2+\omega^2)^2}\right],\end{aligned}\tag{A1}$$

$$\begin{aligned}\mathscr{D}(1)=&-\frac{\pi}{(a^2+\omega^2)^4}\left[\arctan\left[\frac{\gamma}{\Delta+\omega}\right]+\arctan\left[\frac{\gamma}{\Delta-\omega}\right]\right]\\&+\frac{\pi}{12a^3}\left[\frac{24a^3}{(a^2+\omega^2)^4}\arctan\left[\frac{a+\gamma}{\Delta}\right]-\frac{\Delta}{(a^2+\omega^2)[\Delta^2+(a+\gamma)^2]}\right.\\&\times\left\{\frac{3}{2a^2}+\frac{3(a+\gamma)}{a[\Delta^2+(a+\gamma)^2]}+\frac{3(a+\gamma)^2-\Delta^2}{[\Delta^2+(a+\gamma)^2]^2}+\frac{3}{(a^2+\omega^2)}\right.\\&\left.\left.\times\left[1+\frac{4a^2}{(a^2+\omega^2)}+\frac{2a(a+\gamma)}{\Delta^2+(a+\gamma)^2}\right]\right\}\right],\end{aligned}\tag{A2}$$

$$\begin{aligned}\mathscr{D}(2)=&-\frac{\pi\omega^2}{(a^2+\omega^2)^4}\left[\arctan\left[\frac{\gamma}{\Delta+\omega}\right]+\arctan\left[\frac{\gamma}{\Delta-\omega}\right]\right]\\&+\frac{\pi}{12a}\left[\frac{24a\omega^2}{(a^2+\omega^2)^4}\arctan\left[\frac{a+\gamma}{\Delta}\right]+\frac{\Delta}{(a^2+\omega^2)[\Delta^2+(a+\gamma)^2]}\right.\\&\times\left\{-\frac{3}{2a^2}-\frac{3(a+\gamma)}{a[\Delta^2+(a+\gamma)^2]}+\frac{3(a+\gamma)^2-\Delta^2}{[\Delta^2+(a+\gamma)^2]^2}+\frac{3}{(a^2+\omega^2)}\right.\\&\left.\left.\times\left[-3+\frac{4a^2}{(a^2+\omega^2)}+\frac{2a(a+\gamma)}{[\Delta^2+(a+\gamma)^2]}\right]\right\}\right],\end{aligned}\tag{A3}$$

$$\begin{aligned}\mathscr{F}(0,y,g)=&-\frac{\pi(4a^6+6a^4g^2+4a^2g^4+g^6)}{a^8(a^2+g^2)^4}\left[\arctan\left[\frac{a}{y}\right]-\frac{\theta\pi}{2}\right]+\frac{\pi y}{24a^5(a^2+g^2)(a^2+y^2)}\\&\times\left[\frac{33}{2a^2}+\frac{9}{a^2+y^2}+\frac{3a^2-y^2}{(a^2+y^2)^2}+\frac{6}{a^2+g^2}\left[3+\frac{a^2}{a^2+y^2}\right]+\frac{3(3a^2-g^2)}{(a^2+g^2)^2}\right],\end{aligned}\tag{A4}$$

where

$$\theta=\begin{cases}1 & g>y\\ 0 & \text{otherwise}\,,\end{cases}$$

$$\mathscr{F}(1,y,g)=\frac{\pi}{24a^3}\left[\frac{24a^3}{(a^2+g^2)^4}\left\{\arctan\left[\frac{a}{y}\right]-\frac{\theta\pi}{2}\right\}-\frac{y(15a^4+10a^2y^2+3y^4)}{2a^2(a^2+y^2)^3(a^2+g^2)}\right.$$
$$\left.-\frac{6y(2a^2+y^2)}{(a^2+y^2)^2(a^2+g^2)^2}-\frac{3y(3a^2-g^2)}{(a^2+y^2)(a^2+g^2)^3}\right]\,, \tag{A5}$$

$$\mathscr{F}(2,y,g)=\frac{\pi}{24a}\left[\frac{24ag^2}{(a^2+g^2)^4}\left\{\arctan\left[\frac{a}{y}\right]-\frac{\theta\pi}{2}\right\}-\frac{y(3a^4+14a^2y^2+3y^4)}{2a^2(a^2+y^2)^3(a^2+g^2)}\right.$$
$$\left.-\frac{3y(a^2+3y^2)}{(a^2+y^2)^2(a^2+g^2)^2}+\frac{12ya^2}{(a^2+y^2)(a^2+g^2)^3}\right]\,, \tag{A6}$$

$$\mathscr{F}(3,y,g)=\frac{\pi a}{24}\left[\frac{24g^4}{a(a^2+g^2)^4}\left\{\arctan\left[\frac{a}{y}\right]-\frac{\theta\pi}{2}\right\}-\frac{y(3a^4+10a^2y^2+15y^4)}{2a^2(a^2+y^2)^3(a^2+g^2)}\right.$$
$$\left.+\frac{3y(5a^2+7y^2)}{(a^2+y^2)^2(a^2+g^2)^2}-\frac{12ya^2}{(a^2+y^2)(a^2+g^2)^3}\right]\,, \tag{A7}$$

$$\mathscr{F}(4,y,g)=\frac{\pi}{6}\left[\frac{6g^6}{(a^2+g^2)^4}\left\{\arctan\left[\frac{a}{y}\right]-\frac{\theta\pi}{2}\right\}+\frac{ay(33a^4+82a^2y^2+57y^4)}{8(a^2+y^2)^3(a^2+g^2)}\right.$$
$$\left.-\frac{3a^3y(9a^2+11y^2)}{4(a^2+y^2)^2(a^2+g^2)^2}+\frac{3a^5g}{(a^2+y^2)(a^2+g^2)^3}\right]\,, \tag{A8}$$

$$D(0)=\frac{2\pi}{a^8}\left[\arctan\left[\frac{\gamma}{\Delta}\right]-\arctan\left[\frac{a+\gamma}{\Delta}\right]\right]$$
$$+\frac{\pi\Delta}{12a^5[\Delta^2+(a+\gamma)^2]}\left[\frac{33}{2a^2}+\frac{9(a+\gamma)}{a[\Delta^2+(a+\gamma)^2]}+\frac{[3(a+\gamma)^2-\Delta^2]}{[\Delta^2+(a+\gamma)^2]^2}\right]\,, \tag{A9}$$

$$D(1)=-\frac{\pi\Delta}{12a^3[\Delta^2+(a+\gamma)^2]}\left\{\frac{3}{2a^2}+\frac{3(a+\gamma)}{a[\Delta^2+(a+\gamma)^2]}+\frac{3(a+\gamma)^2-\Delta^2}{[\Delta^2+(a+\gamma)^2]^2}\right\}\,, \tag{A10}$$

$$D(2)=-\frac{\pi\Delta}{12a[\Delta^2+(a+\gamma)^2]}\left\{\frac{3}{2a^2}+\frac{3(a+\gamma)}{a[\Delta^2+(a+\gamma)^2]}-\frac{[3(a+\gamma)^2-\Delta^2]}{[\Delta^2+(a+\gamma)^2]^2}\right\}\,, \tag{A11}$$

$$\mathscr{F}_C(0,y)=-\frac{\pi}{a^8}\arctan\left[\frac{a}{y}\right]+\frac{\pi y}{24a^5(a^2+y^2)}\left[\frac{33}{2a^2}+\frac{9}{a^2+y^2}+\frac{3a^2-y^2}{(a^2+y^2)^2}\right]\,, \tag{A12}$$

$$\mathscr{F}_C(1,y)=-\frac{\pi y}{24a^3(a^2+y^2)}\left[\frac{3}{2a^2}+\frac{3}{a^2+y^2}+\frac{3a^2-y^2}{(a^2+y^2)^2}\right]\,, \tag{A13}$$

$$\mathscr{F}_C(2,y)=-\frac{\pi y}{24a(a^2+y^2)}\left[\frac{3}{2a^2}+\frac{3}{a^2+y^2}-\frac{(3a^2-y^2)}{(a^2+y^2)^2}\right]\,, \tag{A14}$$

$$\mathscr{F}_C(3,y)=-\frac{\pi ya}{24(a^2+y^2)}\left[\frac{15}{2a^2}-\frac{9}{a^2+y^2}+\frac{3a^2-y^2}{(a^2+y^2)^2}\right]\,. \tag{A15}$$

The expressions for the integral $\mathscr{L}(k)$ for different values of k can be expressed in the form

$$\mathscr{L}(k)=\pi[\mathscr{R}(k)+\mathscr{B}(k)]\,, \tag{A16}$$

where

$$\mathscr{R}(k)=\frac{1}{6}\left[-\frac{6}{\sqrt{\mathscr{D}}}\left[\mathscr{A}_3+\frac{t_1}{\mathscr{D}}+\frac{t_2}{\mathscr{D}^2}+\frac{t_3}{\mathscr{D}^3}\right]\arctan\left[\frac{2a\sqrt{\mathscr{D}}}{\varkappa}\right]\right.$$
$$-\frac{4}{\eta\xi}\left\{S_1+\frac{S_2}{\mathscr{D}}+\frac{S_3}{\mathscr{D}^2}+\frac{\mathscr{A}_0\mathscr{O}_4}{\mathscr{D}^3}+\frac{1}{\xi}\left[S_4+\frac{S_5}{\mathscr{D}}+\frac{\mathscr{A}_0\mathscr{O}_7}{\mathscr{D}^2}\right]\right.$$
$$\left.\left.-\frac{2\mathscr{A}_0a^2d_1^2}{\xi^2}\left[\varkappa-2a^2\mathscr{G}+\frac{\varkappa a^2d_2}{\mathscr{D}}\right]\right\}\right], \tag{A17}$$

where $\mathscr{A}_0(k)$, $\mathscr{A}_1(k)$, $\mathscr{A}_2(k)$, and $\mathscr{A}_3(k)$ are different for different values of k:

$$t_1=-a\mathscr{A}_2d_2+\mathscr{A}_1(4p_1^2a^2+d_2)+4\mathscr{A}_0ap_1^2,$$
$$t_2=-3\mathscr{A}_1a^2d_2^2-3\mathscr{A}_0ad_2(4p_1^2a^2+d_2),$$
$$t_3=5\mathscr{A}_0a^3d_2^3,$$
$$S_1=3\mathscr{A}_2(\varkappa-2a^2\mathscr{G})+3\mathscr{A}_1a[2(4a^2+\mathscr{G})+K(\varkappa-2a^2\mathscr{G})]+\mathscr{A}_0\mathscr{O}_1,$$
$$S_2=3\mathscr{A}_2\varkappa a^2d_2-3\mathscr{A}_1a[4a^2\mathscr{G}d_2+\varkappa(4p_1^2a^2+d_2)-K\varkappa a^2d_2)]+\mathscr{A}_0\mathscr{O}_2,$$
$$S_3=9\mathscr{A}_1\varkappa a^3d_2^2+\mathscr{A}_0\mathscr{O}_3,$$
$$S_4=3\mathscr{A}_1ad_1(\varkappa-2a^2\mathscr{G})+\mathscr{A}_0\mathscr{O}_5,$$
$$S_5=3\mathscr{A}_1d_1\varkappa a^3d_2+\mathscr{A}_0\mathscr{O}_6,$$
$$\mathscr{O}_1=2(16a^2+\mathscr{G})+K[\varkappa-2a^2(8a^2+3\mathscr{G})]-2a^2\left[K^2-\frac{4}{\eta}\right](\varkappa-2a^2\mathscr{G}), \tag{A18}$$
$$\mathscr{O}_2=-4a^2[3\varkappa p_1^2+4\mathscr{G}(2p_1^2a^2+d_2)+4a^2d_2]+2Ka^2[2\varkappa(2p_1^2a^2+d_2)+3a^2\mathscr{G}d_2]-2\varkappa a^4d_2\left[K^2-\frac{4}{\eta}\right],$$
$$\mathscr{O}_3=a^2d_2[9\varkappa(4p_1^2a^2+d_2)+20a^2\mathscr{G}d_2-5K\varkappa a^2d_2],$$
$$\mathscr{O}_4=-15\varkappa a^4d_2^3,$$
$$\mathscr{O}_5=d_1[\varkappa-2a^2(8a^2+3\mathscr{G})]-2a^2(Kd_1-4)(\varkappa-2a^2\mathscr{G}),$$
$$\mathscr{O}_6=2a^2d_1[2\varkappa(2p_1^2a^2+d_2)+3a^2\mathscr{G}d_2]-2\varkappa a^4d_2(Kd_1-4),$$
$$\mathscr{O}_7=-5d_1\varkappa a^4d_2^2,$$
$$\eta=a^4+2a^2(p_1^2+p_2^2)+(p_1^2-p_2^2)^2,$$
$$K=4(a^2+p_1^2+p_2^2)/\eta,$$
$$\xi=\Delta^4+2\Delta^2(a^2+\gamma^2)+(a^2-\gamma^2)^2,$$
$$\varkappa=\Delta^2[a^2-(p_1^2-p_2^2)]-(\gamma^2+a^2)(p_1^2-p_2^2)-a^2\gamma^2+a^4,$$
$$\mathscr{D}=p_1^2\Delta^4+\Delta^2[2p_1^2a^2+\gamma^2(p_1^2+p_2^2+a^2)]+p_1^2a^4-\gamma^2[a^2(p_1^2+p_2^2)+(p_1^2-p_2^2)^2]+p_2^2\gamma^4,$$
$$\mathscr{G}=2a^2-\gamma^2+\Delta^2-(p_1^2-p_2^2),$$
$$d_1=4(a^2-\gamma^2+\Delta^2),$$
$$d_2=\Delta^2(2p_1^2+\gamma^2)+2p_1^2a^2-\gamma^2(p_1^2+p_2^2);$$

for $k=0$:

$$\mathscr{A}_0=-\frac{1}{16a^5(a^2+\omega^2)}\,,$$

$$\mathscr{A}_1=\frac{5a^2+3\omega^2}{16a^6(a^2+\omega^2)^2}\,,$$

$$\mathscr{A}_2=\frac{29a^4+32a^2\omega^2+11\omega^4}{16a^7(a^2+\omega^2)^3}\,,$$

$$\mathscr{A}_3=-\frac{(4a^6+6a^4\omega^2+4a^2\omega^4+\omega^6)}{a^8(a^2+\omega^2)^4}\,;$$

for $k=1$:

$$\mathscr{A}_0=\frac{1}{16a^3(a^2+\omega^2)}\,,\quad \mathscr{A}_1=\frac{-(3a^2+\omega^2)}{16a^4(a^2+\omega^2)^2}\,,$$

$$\mathscr{A}_2=\frac{-(11a^4+4a^2\omega^2+\omega^4)}{16a^5(a^2+\omega^2)^3}\,,\quad \mathscr{A}_3=\frac{1}{(a^2+\omega^2)^4}\,;$$

for $k=2$;

$$\mathscr{A}_0=\frac{-1}{16a(a^2+\omega^2)}\,,\quad \mathscr{A}_1=\frac{a^2-\omega^2}{16a^2(a^2+\omega^2)^2}\,,$$

$$\mathscr{A}_2=\frac{a^4-8a^2\omega^2-\omega^4}{16a^3(a^2+\omega^2)^3}\,,\quad \mathscr{A}_3=\frac{\omega^2}{(a^2+\omega^2)^4}\,;$$

for $k=3$:

$$\mathscr{A}_0=\frac{a}{16(a^2+\omega^2)}\,,\quad \mathscr{A}_1=\frac{a^2+3\omega^2}{16(a^2+\omega^2)^2}\,,$$

$$\mathscr{A}_2=\frac{a^4+4a^2\omega^2-5\omega^4}{16a(a^2+\omega^2)^3}\,,\quad \mathscr{A}_3=\frac{\omega^4}{(a^2+\omega^2)^4}\,;$$

and for $k=4$:

$$\mathscr{A}_0=\frac{-a^3}{16(a^2+\omega^2)}\,,\quad \mathscr{A}_1=\frac{-a^2(3a^2+5\omega^2)}{16(a^2+\omega^2)^2}\,,$$

$$\mathscr{A}_2=\frac{a(5a^4+16a^2\omega^2+19\omega^4)}{16(a^2+\omega^2)^3}\,,\quad \mathscr{A}_3=\frac{\omega^6}{(a^2+\omega^2)^4}\,.$$

The values of $\mathscr{B}(k)$ for different values of k are the following:

$$\mathscr{B}(0)=\frac{1}{a^2}\left[-\frac{1}{a^2+\omega^2}b_4-\frac{(2a^2+\omega^2)}{a^2(a^2+\omega^2)^2}b_3-\frac{[(a^2+\omega^2)^3-a^6]}{\omega^2a^4(a^2+\omega^2)^3}b_2\right.$$
$$\left.-\frac{[(a^2+\omega^2)^4-a^8]}{\omega^2a^6(a^2+\omega^2)^4}b_1-\frac{a^2}{\omega^2(a^2+\omega^2)^4}b_\omega+\frac{1}{a^6\omega^2}b_0\right]\,,$$

$$\mathscr{B}(1)=\frac{1}{(a^2+\omega^2)}\left[b_4+\frac{1}{a^2+\omega^2}b_3+\frac{1}{(a^2+\omega^2)^2}b_2+\frac{1}{(a^2+\omega^2)^3}(b_1-b_\omega)\right]\,,$$

$$\mathscr{B}(2)=\frac{1}{a^2+\omega^2}\left[-a^2b_4+\frac{\omega^2}{a^2+\omega^2}\left\{b_3+\frac{1}{a^2+\omega^2}b_2+\frac{1}{(a^2+\omega^2)^2}(b_1-b_\omega)\right\}\right]\,,$$

$$\mathscr{B}(3)=\frac{1}{a^2+\omega^2}\left[a^4b_4-\frac{a^2(a^2+2\omega^2)}{(a^2+\omega^2)}b_3+\frac{\omega^4}{(a^2+\omega^2)^2}\left\{b_2+\frac{1}{a^2+\omega^2}(b_1-b_\omega)\right\}\right]\,,$$

$$\mathscr{B}(4)=\frac{1}{a^2+\omega^2}\left[-a^6b_4+\frac{a^4(2a^2+3\omega^2)}{a^2+\omega^2}b_3-\frac{a^2}{a^2+\omega^2}\left[3\omega^2+\frac{a^4}{a^2+\omega^2}\right]b_2+\frac{\omega^6}{(a^2+\omega^2)^3}(b_1-b_\omega)\right],$$

where

$$b_1=\frac{1}{\sqrt{\mathscr{D}}}\arctan\left[\frac{2\sqrt{\mathscr{D}}}{\gamma(a^2-\Delta^2-\gamma^2-2(p_1{}^2-p_2{}^2))}\right],$$

$$b_2=\frac{\gamma[n(\Delta^2+p_1{}^2-p_2{}^2)+2\gamma^2\Delta^2]}{w_2\mathscr{D}}+\frac{1}{2\mathscr{D}}[2p_1{}^2n+\gamma^2(\Delta^2+p_1{}^2-p_2{}^2)]b_1,$$

$$b_3=\frac{\Delta^2\gamma^3}{w_2{}^2\mathscr{D}}\left[n-2u_2-\frac{3(nu_2+2\Delta^2\gamma^2)}{4\Delta^2\mathscr{D}}(2nw_1+u_2w_3)\right]$$

$$-\frac{\gamma^2}{8w_2{}^2\mathscr{D}^2}[-8\gamma^2n^2w_1{}^2+4p_1{}^2w_2{}^2(w_1+4p_1{}^2\Delta^2)-\gamma^2u_2{}^2(3w_3{}^2+16\Delta^2\gamma^2n^2)-8\gamma^2nu_2w_1w_3]b_1$$

$$+\frac{n}{w_2}\left[2b_2-\frac{n}{w_2}b_1\right],$$

$$b_4=\frac{-2\Delta^2\gamma^2}{w_2{}^3\mathscr{D}}\left[2\gamma\left[\frac{1}{3}-\frac{5\gamma^2w_4{}^2}{32\Delta^2\mathscr{D}^2}+\frac{v}{6\Delta^2\gamma^2\mathscr{D}}\right](nu_2+2\gamma^2\Delta^2)\right.$$

$$\left.+\frac{5\gamma^3}{12\mathscr{D}}(n-2u_2)(2nw_1+u_2w_3)+\frac{(2nw_1+u_2w_3)}{8\Delta^2\mathscr{D}}\left[\frac{5\gamma^4w_4{}^2}{4\mathscr{D}}-3v\right]b_1\right]$$

$$+\frac{n}{w_2}\left[3b_3-\frac{3n}{w_2}b_2+\frac{n^2}{w_2{}^2}b_1\right],$$

$$b_\omega=\lim_{a^2\to-\omega^2}(b_1),\quad b_0=\lim_{a^2\to 0}(b_1),$$

$$n=\Delta^2-\gamma^2+a^2,$$

$$u_2=\Delta^2+(p_1{}^2-p_2{}^2),$$

$$w_1=\gamma^2\Delta^2-u_2{}^2,\quad w_2=n^2+4\Delta^2\gamma^2,\quad w_3=n^2-4\Delta^2\gamma^2,\quad w_4=2nw_1+u_2w_3,$$

$$v=\gamma^2[4\Delta^2\gamma^2(4\Delta^2p_1{}^2-nu_2)+(w_1+4p_1{}^2\Delta^2)n^2].$$

The expressions for the integrals $\mathscr{L}_C(k)$ can also be written in the same form as Eq. (A16), i.e.,

$$\mathscr{L}_C(k)=\pi[\mathscr{R}(k)+\mathscr{B}(k)]$$

with $\mathscr{R}(k)$ given by Eq. (A17); the only quantities that are different are the $\mathscr{A}_0(k)$, $\mathscr{A}_1(k)$, $\mathscr{A}_2(k)$, $\mathscr{A}_3(k)$, and the $\mathscr{B}(k)$; these are now, for $k=0$,

$$\mathscr{A}_0=-\frac{1}{16a^5},\quad \mathscr{A}_1=\frac{3}{16a^6},\quad \mathscr{A}_2=\frac{11}{16a^7},\quad \mathscr{A}_3=-\frac{1}{a^8};$$

for $k=1$,

$$\mathscr{A}_0=\frac{1}{16a^3},\quad \mathscr{A}_1=-\frac{1}{16a^4},\quad \mathscr{A}_2=-\frac{1}{16a^5},\quad \mathscr{A}_3=0;$$

for $k=2$,

$$\mathscr{A}_0=-\frac{1}{16a}\;,\;\;\mathscr{A}_1=-\frac{1}{16a^2}\;,\;\;\mathscr{A}_2=-\frac{1}{16a^3}\;,\;\;\mathscr{A}_3=0\;;$$

for $k=3$,

$$\mathscr{A}_0=\frac{a}{16}\;,\;\;\mathscr{A}_1=\frac{3}{16}\;,\;\;\mathscr{A}_2=-\frac{5}{16a}\;,\;\;\mathscr{A}_3=0\;;$$

and

$$\mathscr{B}(0)=\frac{1}{a^2}\left[\frac{1}{a^6}(b_0-b_1)-\frac{b_2}{a^4}-\frac{b_3}{a^2}-b_4\right],$$

$$\mathscr{B}(1)=b_4\;,\;\;\mathscr{B}(2)=b_3-a^2b_4\;,\;\;\mathscr{B}(3)=b_2-2a^2b_3+a^4b_4\;.$$

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$$R_{\rm tr}^C=R_C\;,$$

$$R_{\rm tr}^E=\left[\frac{l+3}{l+1}\right]^{1/2}R_E\;,$$

$$R_{\rm tr}^E=\left[\frac{l+3}{l+1}\right]^{1/2}R_M\;.$$

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