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Quasipotential approach to scattering theory: Spectral analysis of the vertex function

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The vertex function of the quasipotential in scattering equations is spectral analyzed in terms of the strength eigenstates, and a criterion for its effectiveness established. A simple variational procedure is given to construct the strength eigenstates, by which vertex functions may be evaluated.

[NUCLEAR REACTIONS Quasipotential method, convergence of the iteration series.]

In a recent paper,[1] we have derived a quasipotential (QP) approach which can be used to solve a class of scattering equations with divergent iteration kernels. It is a direct extension of the earlier work by Sasakawa[2] and Austern,[3] combined with the quasiparticle approach of Weinberg.[4] The vertex function φ_t, which appears in the QP of Ref. 1, was chosen rather arbitrarily, and its effectiveness was examined numerically for a simple form. A more systematic way of constructing φ_t is desirable and, in this paper, we report both on a spectral analysis of φ_t in terms of the strength eigenfunctions and a minimum criterion for the generation of a convergent iterative solution.

(1) Following Ref. 1, we consider a simple potential scattering of the form

$$D_0|u\rangle = V|u\rangle \ , \tag{1}$$

where $D_0=E-H_0$, and H_0 may contain some distortion potentials, especially in the case of multichannel scattering. The corresponding integral equation is given by

$$|u\rangle = |u_0\rangle + G_0V|u\rangle \ , \tag{2}$$

where

$$D_0|u_0\rangle = 0, \quad D_0G_0=1 \ . \tag{3}$$

The quasipotential approach of Ref. 1 is obtained by a replacement of V in (1) and (2) by V_s defined by

$$V_s(u_t,\varphi_t) = V|u_t\rangle \frac{1}{\langle \varphi_t^\dagger|V|u_t\rangle}\langle \varphi_t^\dagger|V \ . \tag{4}$$

In particular, an iterative procedure is obtained when $|u_t\rangle$ is chosen to be the ith iterated solution of the equation

$$[D_0 - V_s(u_i,\varphi_t)]|u_{i+1}\rangle = 0 \ . \tag{5}$$

The vertex function φ_t is still arbitrary, but the convergence property of (5) depends *critically* on the choice of φ_t. A simple choice $|\varphi_t\rangle = |u_0\rangle$ was seen to result in the Sasakawa-Austern procedure,[2,3] while the form $|\varphi_t\rangle = f|u_0\rangle$, with $f=e^{-ar}$ and a a nonlinear adjustable parameter, gave a strongly convergent result in the case of $p+{}^{16}O$ scattering.

As is well known,[4,5] the convergence property of an iteration series can be studied conveniently in terms of the strength eigenfunctions $\{\varphi_n\}$ generated by the iteration kernel $K_0 \equiv G_0V$, as

$$K_0|\varphi_n\rangle = \lambda_n|\varphi_n\rangle \ \text{ with } \langle \varphi_n^\dagger|V|\varphi_m\rangle = \delta_{nm} \ , \tag{6}$$

where φ_n are regular at $r=0$ and satisfy a purely outgoing-wave boundary condition beyond some $r>r_c$. Thus K_0 may be expanded as

$$K_0 = \sum_m |\varphi_n\rangle \lambda_n \langle \varphi_n^\dagger|V \ . \tag{7}$$

The scattering function $|u\rangle$ is given by

$$|u\rangle = |u_0\rangle + \sum_n a_n|\varphi_n\rangle \ , \tag{8}$$

with $a_n=\lambda_n b_n/(1-\lambda_n)$ and $b_n=\langle\varphi_n^\dagger|V|u_0\rangle$. The usual Born series results when $(1-\lambda_n)^{-1}$ is expanded in powers of λ_n; obviously, the series would diverge if any λ_n in (8) would be outside the unit circle, i.e., $|\lambda_n|\geq 1$.

The iteration series obtained from (5) with (4) may be summarized for the ith iteration as follows[1]:

$$|\chi_i\rangle = K_0|u_{i-1}\rangle \ , \tag{9a}$$

$$y_i = \frac{\langle\varphi_t^\dagger|V|U_0\rangle}{\langle\varphi_t|V|u_{i-1}\rangle - \langle\varphi_t^\dagger|V|\chi_i\rangle} \ , \tag{9b}$$

and thus the wave function is given by

$$|u_i\rangle = |u_0\rangle + y_i|\chi_i\rangle \ . \tag{9c}$$

If the series converges, we expect that

$$|u_i\rangle \to |u\rangle, \quad y_i \to 1 \ , \tag{10}$$

and, after the ith iteration,

$$|u_i\rangle = |u_0\rangle + \sum_{n=1}^{\infty}\lambda_n b_n|\varphi_n\rangle\left(1+\lambda_n y_{i-1}+\lambda_n{}^2 y_{i-1}y_{i-2}+\cdots+\lambda_n^{i-1}\prod_{j-1}^{i-1}y_j\right)y_i \ , \tag{11}$$

which is to be compared with the ith iterated Born series obtained when all the y's are set equal to unity. Before discussing the role of the vertex function φ_t, we note the following properties: (a) From (8) and (11), the φ_n content of $|u\rangle$ is determined essentially by the overlap $b_n=\langle\varphi_n^\dagger|V|u_0\rangle$, so that, for those φ_m's with $b_m=0$, no convergence problem arises; (b) the convergence (or divergence) of the iteration series depends on the factor in (11), inside the square bracket. Although eventually $y_i\to 1$ as $i\to\infty$, all the y's conspire in such a way that (11) is fundamentally different from the Born series. The precise way in which this occurs will be illustrated below. [The Born series is obtained by setting all $y_i=1$ in Eq. (11).]

(2) Now, we consider the crucial question of the choice of φ_t, and start with a simple case in which only one $|\lambda_1|$ lies outside the unit circle, i.e., $|\lambda_1|>1$ and $|\lambda_n|<1$ for all $n>1$. For an arbitrary constant c_1, we let

$$|\varphi_t\rangle=c_1|\varphi_1\rangle \quad . \tag{12}$$

[Of course, the QP form (4) is independent of the overall normalization of φ_t.] Then, it is a simple matter to show that

$$y_1=\frac{1}{1-\lambda_1},\quad y_n=1,\quad n>1 \quad , \tag{13}$$

and $|u_i\rangle$ after the ith iteration are explicitly given by

$$\begin{aligned}
|u_1\rangle&=|u_0\rangle+\sum_{n\neq 1}^{\infty}{}' \lambda_n b_n|\varphi_n\rangle\frac{1}{1-\lambda_1}+a_1|\varphi_1\rangle \quad ,\\
|u_2\rangle&=|u_0\rangle+\sum_{n\neq 1}^{\infty}{}' \lambda_n b_n|\varphi_n\rangle\left[1+\frac{\lambda_n}{1-\lambda_1}\right]+a_1|\varphi_1\rangle \quad ,\\
|u_3\rangle&=|u_0\rangle+\sum_{n\neq 1}^{\infty}{}' \lambda_n b_n|\varphi_n\rangle\left[1+\lambda_n+\lambda_n^2\frac{1}{1-\lambda_1}\right]+a_1|\varphi_1\rangle,\ \text{etc.}
\end{aligned} \tag{14}$$

The above result reveals many salient features of the QP approach: (i) After the *first* iteration, the φ_1 part of u_i has converged completely, and the subsequent iterations do not modify this part of u; (ii) the φ_n parts ($n>1$) of u_i are obtained exactly as in the Born series, *except* for the $(i-1)$th term which is multiplied by y_1. (This may be improved by allowing φ_t to contain φ_n's other than φ_1.) (iii) $y_i\to 1$ immediately after the first iteration, although the overall u_i has still not converged. Therefore $y_i\to 1$ is *not* a reliable indicator of the convergence.

(3) Next, consider the case in which $|\lambda_1|$, $|\lambda_2|>1$, while $|\lambda_n|<1$ for $n\geqslant 3$, and try for φ_t a form

$$|\varphi_t\rangle=c_1|\varphi_1\rangle+c_2|\varphi_2\rangle \quad , \tag{15}$$

with c_1 and c_2 to be determined below.

This case is of special interest, because our QP is still a *single* term of rank 1, while the conventional quasipotential should be of rank 2. In fact, we have shown[6] that, when φ_t is chosen properly, the weaker form of rank 1 is *sufficient* to guarantee the convergence of the u_i series. The form (15) may also be used, even when $|\lambda_2|<1$, to improve the overall convergence. A simple algebraic manipulation yields for the first iteration

$$y_1=\frac{c_1b_1+c_2b_2}{c_1b_1(1-\lambda_1)+c_2ba_2(1-\lambda_2)}$$

and

$$\begin{aligned}|u_1\rangle=|u_0\rangle&+\sum_{n\neq 1,2}^{\infty}{}' \lambda_n b_n|\varphi_n\rangle y_1\\
&+\{\lambda_1b_1|\varphi_1\rangle y_1+\lambda_2b_2|\varphi_2\rangle y_2\} \quad .\end{aligned} \tag{16}$$

On the other hand, the result (14) with one φ_1 suggests that we would like to have, in (16),

$$\{\ \}\to a_1|\varphi_1\rangle+a_2|\varphi_2\rangle \quad , \tag{17}$$

where $a_n=\lambda_nb_n/(1-\lambda_n)$. (This choice is, of course, not unique but is good enough for our purpose.) Obviously, this is not possible no matter how c_1 and c_2 in y_1 are adjusted. Therefore, for the present case, we need *at least* two iterations. We have, for $i=2$ and from (11),

$$\begin{aligned}|u_2\rangle=|u_0\rangle&+\sum_{n\neq 1,2}^{\infty}{}' \lambda_n b_n|\varphi_n\rangle(1+\lambda_ny_1)y_2\\
&+\{\lambda_1b_1|\varphi_1\rangle(1+\lambda_1y_1)y_2\\
&+\lambda_2b_2|\varphi_2\rangle(1+\lambda_2y_1)y_2\} \quad .\end{aligned} \tag{18}$$

It is now possible to adjust the parameters c_1 and c_2 in φ_t of (15) such that the term $\{\ \}$ of (18) satisfies (17). That is,

$$\frac{c_1}{c_2}=-\frac{\lambda_2b_2}{\lambda_1b_1} \quad . \tag{19}$$

Thus, aside from the overall constant and with $b_n\equiv\langle\varphi_n^\dagger|V|u_0\rangle$,

$$|\varphi_t\rangle=\frac{\lambda_1}{b_1}|\varphi_1\rangle-\frac{\lambda_2}{b_2}|\varphi_2\rangle \quad . \tag{20}$$

The choice (20) gives the following iteration series:

$$\begin{aligned}
y_1&=\frac{1}{1-(\lambda_1+\lambda_2)} \quad ,\\
y_2&=\frac{1-(\lambda_1+\lambda_2)}{(1-\lambda_1)(1-\lambda_2)} \quad ,\\
y_n&=1,\quad n\geqslant 3 \quad ,
\end{aligned} \tag{21}$$

and

$$|u_1\rangle = |u_0\rangle + \sum_{n\neq 1,2}^{\infty}{}' \lambda_n b_n |\varphi_n\rangle y_1 + (\lambda_1 b_1 |\varphi_1\rangle y_1 + \lambda_2 b_2 |\varphi_2\rangle y_1) \ ,$$

$$|u_2\rangle = |u_0\rangle + \sum_{n\neq 1,2}^{\infty}{}' \lambda_n b_n |\varphi_n\rangle (1+\lambda_n y_1) y_2 + (a_1|\varphi_1\rangle + a_2|\varphi_2\rangle) \ , \qquad (22)$$

$$|u_3\rangle = |u_0\rangle + \sum_{n\neq 1,2}^{\infty}{}' \lambda_n b_n |\varphi_n\rangle (1+\lambda_n y_2 + \lambda_n^2 y_1 y_2) + (a_1|\varphi_1\rangle + a_2|\varphi_2\rangle), \text{etc.} \ ,$$

so that, after the second iteration, $|u_i\rangle$ will contain the correct φ_1 and φ_2 components, as we expected. The other components will be just as in the Born series, except for the last two terms, which are multiplied by y_2 and $y_1 y_2$, respectively.

The above procedure could be carried further for the φ_t with more than two φ_n's. This proves that (5) with (4) will converge for all cases if φ_t is chosen *properly*. Even if some $|\lambda_n| < 1$ for which $c_n \neq 0$, the iteration series will be made to converge faster. On the other hand, φ_t need not contain all the troublesome φ_n's in precisely the form (20), but should include "enough" of them so that the series converges, in the sense of (20), either by cancellations or by reduction in strength.

In summary, we have shown that the rank 1 form (4) for V_s and the resulting iteration by (5) will converge with a proper choice of φ_t even when a finite number of φ's occur with $|\lambda_n| > 1$. What is not yet clear is whether a simple and systematic way to generate a useful form of φ_t can be found. We have not been able to derive such a method, although a variational principle for the φ_n's can be formulated.[7] From a practical point of view, it is highly desirable to avoid, if possible, the use of exact φ_n's in the construction of V_s, especially when the scattering system is complex and many channels are open. Since the main requirement on the role of V_s is to *reduce* the strength of those λ_n's with $|\lambda_n| \geq 1$, such that the remaining part of these states in the iteration series is convergent,[4] often a crude approximation to φ_n may be sufficient. For example, from (6), we have

$$VG_0V|\varphi_n\rangle = \lambda_n V|\varphi_n\rangle \ , \qquad (23)$$

in which case[1] a simultaneous diagonalization of the matrices $\langle\varphi_{mt}|VG_0V|\varphi_{nt}\rangle$ and $\langle\varphi_{mt}|V|\varphi_{nt}\rangle$ will provide a set $\{\varphi_{nt}\}$ with $\{\lambda_{nt}\}$. Alternatively,[7]

$$D_0|\varphi_n\rangle = \frac{1}{\lambda_n} V|\varphi_n\rangle \ , \qquad (24)$$

in which case the matrices to be diagonalized are $\langle\varphi_{mt}|D_0|\varphi_{nt}\rangle$ and $\langle\varphi_{mt}|V|\varphi_{nt}\rangle$. This provides an approximate set $\{\varphi_{nt}\}$.

Other iterative procedures in which a rank 1 QP of the form (4) is introduced will be discussed elsewhere.[6]

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