Improved effective range formula

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A new effective range formula is derived using a conformal mapping variable that incorporates the knowledge of the left-hand cut. Our formula is valid up to higher energies and has good convergence and extrapolation properties. As an illustration, we use the formula for a study of the S and P waves in the n-p scattering.

[NUCLEAR STRUCTURE New effective range formula derived from analyticity using conformal mappings.]

I. INTRODUCTION

The effective range formula[1] contributes significantly towards a phenomenological understanding[2,3] of N-N interaction and the investigation of theoretical concepts such as the isospin invariance,[4] as well as providing a framework for constructing realistic N-N potentials[5,6] and for calculating three[7] and four[8] nucleon dynamics. It was derived by Schwinger[9] using a variational method, although numerous alternate derivations were subsequently made available.[10–14] From the point of view of the analyticity of the partial wave elastic amplitude the formula may be regarded as a one-pole approximation to the left-hand cut,[13,15] or alternatively it may be viewed as a power series expansion that explicitly incorporates elastic unitarity and has the right-hand cut uniformized.[14] From this latter point of view, the formula depends upon the well established analyticity properties[16] and does not subscribe to any specific forms of nuclear forces or potentials. As such it is valid for systems containing tensor forces, nonlocal potentials, or velocity dependent potentials.[12]

The formula is, however, valid only near the threshold up to 10 MeV laboratory energy. This is a severe limitation not only for the phenomenological utility of the formula but also because the higher energy data cannot be used towards the evaluation of its expansion coefficients, with the consequent lack of determination of the shape parameters which in fact contain the finer details of the underlying interaction. An extension of the formula up to 40 MeV laboratory energy has been possible[17] by explicitly computing the one pion exchange (OPE) contribution through a numerical integration. An analytical generalization of the formula up to high energies has been given using multisheet conformal mapping techniques.[14] This generalization has been tested[14] and works very well. In the present paper we give a new formula that works as well but is simpler to use. We expand the usual effective range function $F_l(k)=k^{2l+1}\cot\delta_l$ in a new variable that carries an explicit knowledge of the left-hand cut. The expansion coefficients of our formula are related in a simple way to the expansion coefficients in the conventional effective range formula. Of course, there is a price to pay for this simplification—while our earlier generalization[14] gave both the phase and the modulus of the Jost function, our present formula does not contain information on the modulus.

Our formula offers a precise parametrization of the experimental phase shifts up to higher energies. Besides, the accessibility of the high energy data allows the determination of the higher coefficients, i.e., the shape parameters. Further, we may now also treat the higher partial waves. These features of our formula make it phenomenologically very attractive[15,18] and useful for the construction of realistic N-N interactions to be used as such[6] or as basic input to three- and four-body calculations[7,8,19] such as the triton binding energy.[7,20]

We present our new formula in Sec. II. Some numerical results and their discussion are given in Sec. III. Our conclusions are presented in Sec. IV.

II. NEW EFFECTIVE RANGE FORMULA

Figure 1 shows the analytic structure[16] of a partial wave amplitude $A_l(E)$ describing the elastic scattering of two scalar equal mass particles whose longest range interaction arises from a one-particle

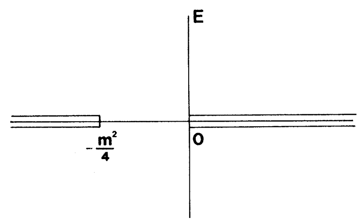

FIG. 1. Analytic structure of equal mass partial wave scattering amplitude with one-particle exchange of mass m.

exchange of mass m, in the $E=k^2$ complex plane where E is the center-of-mass energy and k is the center-of-mass momentum.

The branch cuts of $A_l(E)$ extend along $0\leq E\leq\infty$ and $-\infty\leq E\leq -m^2/4$. The branch point at $E=0$ corresponds to the two-particle threshold and it is a square root type branch point.[15] The branch point at $E=-m^2/4$ arises from a logarithmic term[15,21,22] and it is a branch point of infinite order.[23] In this context the conventional effective range formula arises[14] simply by unfolding the right-hand cut by employing the transformation $k=+\sqrt{E}$ and by using the elastic unitarity which specifies the discontinuity across this cut. In so doing one learns that the effective range function

$$F_l(k)=k^{2l+1}\cot\delta_l(k) \tag{1}$$

is an even function of k and has only the left-hand cut in the complex E plane. This analyticity of $F_l(k)$ is shown in Fig. 2.

The expansion of $F_l(k)$ in the E plane, which is the conventional effective range formula, may now be explicitly written as follows:

$$F_l(k)=\left[\frac{-1}{a_l}\right]+\left[\frac{r_l}{2}\right]E-(P_lr_l\rho_l)E^2+(Q_lr_l\rho_l{}^2)E^3-(R_lr_l\rho_l{}^3)E^4+(S_lr_l\rho_l)E^5\ , \tag{2}$$

where

$$\rho_l=a_lr_l\ . \tag{3}$$

The parameters a_l and r_l are traditionally referred to as the scattering length and the effective range,[24] while the parameters P_l, Q_l, R_l, and S_l are dimensionless numbers[25] which contain the finer details of the underlying interaction. The quantity ρ_l has the dimensions of the square of a length, although it has been the custom in the past to take $\rho_0=r_0{}^2$ for the S wave.

It is obvious from Fig. 2 that the effective range expansion Eq. (2) converges only within a circle of radius $m^2/4$ which roughly corresponds to a laboratory energy of 10 MeV. It is also clear that if we unfold the branch point at $E=-m^2/4$ we will have a larger circle of convergence as well as a faster rate of convergence.[26,27] This can be readily done by the transformation

$$\tau=\ln\left[E+\frac{m^2}{4}\right]\ . \tag{4}$$

However, if we wish to keep our expansion in the τ variable close to the effective range expansion Eq. (2), we should use the combination

$$\omega=\ln\left[\frac{4E}{m^2}+1\right]\ . \tag{5}$$

Our improved effective range formula may now be explicitly written as follows:

$$F_l(k)\equiv F_l(\omega)=\sum_{n=0}^{\infty}A_{\mathrm{l}n}\omega^n\ . \tag{6}$$

In view of Eq. (5) the expansion coefficients in our improved formula [Eq. (6)] are related to those in the conventional formula [Eq. (2)] as follows:

$$-\frac{1}{a_l}=A_{l0}\ , \tag{7}$$

$$\frac{r_l}{2}=\frac{A_{l1}}{\lambda}\ , \tag{8}$$

$$-P_lr_l\rho_l=\frac{1}{2\lambda^2}(2A_{l2}-A_{l1})\ , \tag{9}$$

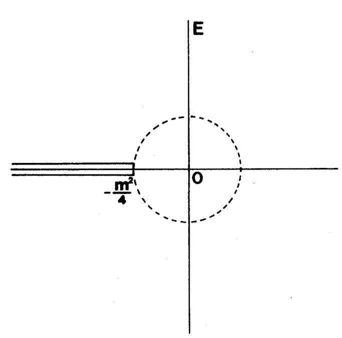

FIG. 2. Analyticity of the effective range function $F_l(k)$ [Eq. (1)] and the circle of convergence of the conventional effective range formula [Eq. (2)].

$$Q_l r_l \rho_l^2 = \frac{1}{\lambda^3}(A_{l3} - A_{l2} + \tfrac{1}{3}A_{l1}) \,, \tag{10}$$

$$-R_l r_l \rho_l^3 = \frac{1}{12\lambda^4}(12A_{l4} - 18A_{l3} + 11A_{l2} - 3A_{l1}) \,, \tag{11}$$

where

$$\lambda = \frac{m^2}{4} \,. \tag{12}$$

The inverse relations are as follows:

$$A_{l2} = \frac{\lambda}{2}\left\{2\lambda(-P_l r_l \rho_l) + \left[\frac{r_l}{2}\right]\right\} \,, \tag{13}$$

$$A_{l3} = \frac{\lambda}{6}\{6\lambda^2(Q_l r_l \rho_l^2) + 6\lambda(-P_l r_l \rho_l) + \tfrac{1}{2} r_l\} \,, \tag{14}$$

$$A_{l4} = \frac{\lambda}{24}\{24\lambda^3(-R_l r_l \rho_l^3) + 36\lambda^2(Q_l r_l \rho_l^2) + 14\lambda(-P_l r_l \rho_l) + \tfrac{1}{2} r_l\} \,. \tag{15}$$

Looked upon as a conformal mapping[26,27] our transformation Eq. (5) maps the E plane of Fig. 2 onto a horizontal strip of width 2π with the origin going to the origin; see Fig. 3. The images of the strip under the translations

$$\omega \to \omega + 2\pi i \tag{16}$$

correspond to the higher Riemann sheets reached in the E plane by crossing the left-hand cut in Fig. 2.

The branch point at $E = -m^2/4$ is mapped onto $\omega = \infty$ and our improved formula [Eq. (6)] converges[28] in the whole ω plane. However, if there are any poles of $F_l(k)$ such as that corresponding to the change of sign in the S-wave phase shifts, such poles will have to be taken care of before our expan-

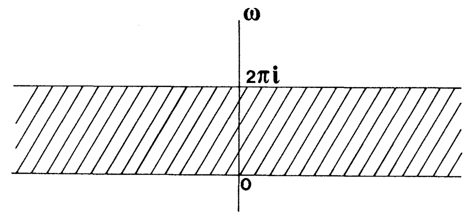

FIG. 3. The horizontal strip is the image of the entire E plane of Fig. 2 under the transformation Eq. (5).

TABLE I. Comparison of our improved effective range formula [Eq. (6)] with the experimental data.

Laboratory energy (MeV)	$^1S_0$ phases (deg) Experimental	$^1S_0$ phases (deg) Theoretical Eq. (17)	$^3S_1$ phases (deg) Experimental	$^3S_1$ phases (deg) Theoretical Eq. (18)
1	62.43±0.01	62.430	147.85±0.01	147.850
2	65.03±0.03	65.020	136.55±0.02	136.552
3	65.35±0.06	65.363	128.83±0.03	128.830
4	65.06±0.08	65.095	122.92±0.05	122.916
5	64.53±0.11	64.601	118.12±0.07	118.114
6	63.91±0.14	64.013	114.07±0.09	114.068
8	63.57±0.20[a]	62.744	107.49±0.13	107.488
10	61.23±0.26	61.459	102.22±0.16	102.237
12	59.95±0.32	60.198	97.84±0.20	97.857
14	58.73±0.37	58.973	94.07±0.23	94.099
16	57.56±0.42	57.786	90.78±0.26	90.783
18	56.46±0.47	56.637	87.84±0.29	87.831
20	55.41±0.52	55.524	85.18±0.32	85.164
25	52.96±0.62	52.898	79.48±0.38	79.442
30	50.73±0.71	50.484	74.76±0.41	74.715
40	46.72±0.86	46.243	67.19±0.46	67.246
50	43.16±0.98	42.689	61.20±0.47	61.549
60	39.90±1.10	39.702		
70	36.89±1.21	37.178		
80	34.08±1.33	35.026		

[a]This data point seems to be in error.

TABLE II. Values of the effective range parameters obtained from our analysis.

Parameters		Values for $^1S_0$		Values for $^3S_1$
Scattering length		-23.679 fm		5.395 fm
Effective range		2.505 fm		1.752 fm
P	(i)	-0.012	(i)	-0.127
	(ii)	0.0013	(ii)	-0.041
Q	(i)	-0.216	(i)	-0.636
	(ii)	-0.0024	(ii)	-0.067

sion [Eq. (6)] can be expected to converge in the whole ω plane. This matter will be discussed in detail elsewhere. For the moment we remark that our Eq. (6) will converge considerably slowly in the vicinity of such a pole if the pole has not been explicitly accounted for, as is the case of our present study.

The expansion coefficients A_{1n} in our improved formula [Eq. (6)] can be evaluated by using data up to higher energies. The corresponding coefficients in the conventional formula [Eq. (2)] are then determined using Eqs. (7)–(12). Thus the higher coefficients P_l, Q_l, R_l which specify the details of N-N interaction may now be determined. These terms could not be evaluated by the conventional formula owing to its circle of convergence being limited (see Fig. 2) so that the data beyond 10 MeV could not be used. For the same reasons we can now evaluate the effective range expansion parameters for higher partial waves, which could not be done with Eq. (2) because the higher partial waves are too small below 10 MeV and the data above 10 MeV lie outside the circle of convergence of Eq. (2).

It has been shown[29] that low energy phase shifts are influenced by contributions from momentum components up to 6 fm^{-1}. Our improved effective range formula, besides providing an accurate description of high energy data, also provides a tool to study how the low energy parameters are influenced when high energy data are incorporated.

III. NUMERICAL RESULTS AND DISCUSSION

We illustrate the theory given in the previous section using the phase shifts reported in Ref. 30. Although a more recent phase shift analysis[31] exists, the data reported in Refs. 30 and 31 are in agreement over the energy region used by us, while there are more data in the lower energy region in Ref. 30.

Parametrization of the n-p data in the S wave by our improved effective range formula [Eq. (6)] is given by Eqs. (17) and (18) below. Equation (17) describes the $^1S_0$ phases while Eq. (18) gives $^3S_1$ phases.

$$\begin{aligned} k\cot\delta_{os} &= 0.042\,2324+0.150\,2916\omega \\ &\quad +0.077\,8852\omega^2-0.008\,9737\omega^3 \\ &\quad +0.031\,781\omega^4\,, \end{aligned} \tag{17}$$

$$\begin{aligned} k\cot\delta_{ot} &= -0.185\,3477+0.105\,125\omega \\ &\quad +0.062\,3624\omega^2+0.009\,163\omega^3 \\ &\quad +0.015\,6891\omega^4\,. \end{aligned} \tag{18}$$

In Table I we display the experimental results and those reproduced by our Eqs. (17) and (18). The values of the effective range parameters corre-

TABLE III. Comparison of our values of the scattering length and the effective range with those from the literature.

Laboratory energy (MeV)	Experimental $^1S_0$ phases (deg)	Shape independent effective range approximation Ref. 1 $a=-23.719$ fm $r=2.76$ fm	Present analysis $a=-23.679$ $r=2.505$
1	62.43 ± 0.01	61.831	62.429
2	65.03 ± 0.03	64.090	64.993
3	65.35 ± 0.06	64.167	65.285
4	65.06 ± 0.08	63.663	64.950
5	64.53 ± 0.11	62.954	64.381
6	63.91 ± 0.14	62.169	63.716
8	63.57 ± 0.20[a]	60.562	62.302
10	61.23 ± 0.26	59.011	60.903

[a]This data point seems to be in error.

TABLE IV. Comparison of our improved effective range formula [Eq. (6)] with the experimental data.

Laboratory energy (MeV)	Experimental $^1P_1$ phases	Theoretical [Eq. (19)]
1	-0.17 ± 0.00	-0.1700
2	-0.43 ± 0.00	-0.4299
3	-0.71 ± 0.01	-0.714
4	-1.00 ± 0.01	-1.000
5	-1.28 ± 0.01	-1.279
10	-2.46 ± 0.03	-2.464
14	-3.18 ± 0.04	-3.177
20	-4.03 ± 0.06	-4.018
25	-4.61 ± 0.08	-4.610
30	-5.14 ± 0.10	-5.157
40	-6.16 ± 0.17	-6.199
50	-7.19 ± 0.24	-7.231
60	-8.27 ± 0.32	-8.277
70	-9.40 ± 0.40	-9.345
80	-10.55 ± 0.47	-10.433
90	-11.72 ± 0.54	-11.537
100	-12.90 ± 0.59	-12.652
120	-15.23 ± 0.69	-14.894
140	-17.47 ± 0.76	-17.111
160	-19.59 ± 0.82	-19.262
180	-21.57 ± 0.88	-21.310
200	-23.41 ± 0.95	-23.231
220	-25.10 ± 1.03	-25.008
240	-26.65 ± 1.13	-26.634
260	-28.07 ± 1.26	-28.106
280	-29.36 ± 1.41	-29.430
300	-30.54 ± 1.58	-30.611
320	-31.60 ± 1.77	-31.661
340	-32.56 ± 1.97	-32.590
360	-33.43 ± 2.19	-33.410
380	-34.21 ± 2.41	-34.131
Values below are extrapolations of Eq. (19)		
400	-34.91 ± 2.64	-34.764
420	-35.53 ± 2.87	-35.319
440	-36.09 ± 3.11	-35.805
460	-36.58 ± 3.35	-36.231
480		-36.603
500		-36.929
600		-38.037
700		-38.606
800		-38.901
900		-39.061
1000		-39.158

sponding to Eqs. (17) and (18) as obtained via Eqs. (7)–(12) are shown in Table II.

Two values are given for each of the parameters P and Q. The first value corresponds to the traditional choice $\rho_0=r_0{}^2$ in Eq. (2), while the second value conforms to our Eq. (3). The values of the scattering length and the effective range parameters shown in Table II are rather different from those generally quoted for these parameters in the literature[1,32] At this stage it is useful to recall[4] that the values quoted in the literature are not determined from a direct use of the phase shift analysis; rather they are determined from the measurements of the coherent and the incoherent scattering cross sections in the 1–500 eV energy region.[33,34,3] On the other hand, we have directly employed the results of the phase shift analysis and have based our calculations upon data above 1 MeV. In Table III we show the experimental results for the $^1S_0$ phase shifts versus the values given by the shape independent effective range approximation with the values of the scattering length and the effective range parameters[35] taken from the literature[1] and the same values taken from our analysis as given in Table II.

Finally, we present a study of $^1P_1$ phases in n-p scattering. The data[30] are parametrized by our formula as follows:

$$\begin{aligned} k^3\cot\delta_1 = &-0.392\,0596-0.549\,3702\omega \\ &-0.119\,9949\omega^2-0.346\,6586\omega^3 \\ &-0.208\,7987\omega^4+0.172\,3316\omega^5 \\ &-0.028\,727\omega^6\ . \end{aligned} \tag{19}$$

Equation (19) gives an accurate description of data as is shown in Table IV. Data up to 380 MeV were used in Eq. (19). It extrapolates up to 460 MeV, still remaining well within the error bars. This remarkable extrapolation is possible because there is no singularity of $F_1(k)$ in the neighborhood so that our Eq. (6) converges notably faster compared to the case of the S wave where there is a pole singularity at about 300 MeV. Incidentally, this is another demonstration for the need to preserve the correct analytic structure of the amplitude when seeking approximations.[24,25]

The effective range parameters for the $^1P_1$ state corresponding to the parametrization (19) are as follows:

$$a_1=2.551\ \mathrm{fm}^3\ ,$$

$$r_1=-9.156\ \mathrm{fm}^{-1}\ ,$$

$$P_1=0.050\ ,$$

$$Q_1=0.047\ ,$$

$$R_1=-0.014\ .$$

IV. CONCLUSIONS

Using the analyticity of the partial wave elastic scattering amplitude we have obtained a much improved version of the effective range formula by expanding the effective range function in terms of a new variable that has been constructed to incorporate the knowledge about the left-hand cut. This formula affords a precise description of phase shift data up to high energies; the values given by the formula being identical to the data well within the error bars (see Table IV). We recommend the use of our formula instead of the conventional one. Further, there are uses of our formula in situations where the conventional formula does not help owing to its validity being restricted to data below 10 MeV. These include (i) extrapolation of the higher energy data to threshold in order to determine the threshold parameters; (ii) parametrization of the high energy data as well as their interpolation and extrapolation; (iii) the investigation of the higher partial waves; and (iv) determination of the higher expansion coefficients (the shape parameters) which in turn can be employed to construct realistic N-N interactions to be used in two or many nucleon problems.

In regard to the values of the S-wave scattering length and the effective range, it is a matter of how much memory of the data in the $1-500$ eV energy region is still retained by the data in the MeV energy region. In our view the matter deserves further attention.

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[1]G. E. Brown and A. D. Jackson, *The Nucleon-Nucleon Interaction* (North-Holland, Amsterdam, 1976).

[2]M. J. Moravcsik, *The Two-Nucleon Interaction* (Clarendon, Oxford, 1963).

[3]R. Wilson, *The Nucleon-Nucleon Interaction* (Interscience, New York, 1963).

[4]H. P. Noyes, Annu. Rev. Nucl. Sci. 22, 465 (1972).

[5]K. Chadan and P. C. Sabatier, *Inverse Problem in Quantum Scattering Theory* (Springer, New York, 1977).

[6]S. S. Ahmad, Nuovo Cimento 65A, 77 (1981).

[7]J. S. Levinger, in *Springer Tracts in Modern Physics,* edited by G. Hohler (Springer, New York, 1974), Vol. 71.

[8]R. Perne and W. Sandhas, Phys. Rev. Lett. 39, 788 (1977).

[9]J. S. Schwinger, Phys. Rev. 72, 742A (1947).

[10]G. F. Chew and M. L. Goldberger, Phys. Rev. 75, 1637 (1949).

[11]H. A. Bethe, Phys. Rev. 76, 38 (1949).

[12]N. Austern, Nucl. Phys. 7, 195 (1958).

[13]G. F. Chew, *S-matrix Theory of Strong Interaction* (Benjamin, New York, 1962).

[14]A. R. Choudhary, Nuovo Cimento 24A, 161 (1974).

[15]M. J. Moravcsik and H. P. Noyes, Annu. Rev. Nucl. Sci. 11, 95 (1961).

[16]S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966), p. 386.

[17]H. P. Noyes and D. Y. Wong, Phys. Rev. Lett. 3, 191 (1959).

[18]E. Lomon and R. Wilson, Phys. Rev. C 9, 1329 (1974).

[19]*Proceedings of the VIII International Conference on Few Body Systems and Nuclear Forces,* edited by H. Zingle, M. Hoftel, and H. Zankel (Springer, New York, 1978).

[20]A. C. Phillips, Nucl. Phys. A107, 209 (1968).

[21]A. R. Choudhary and R. B. Jones, Nucl. Phys. B22, 343 (1970).

[22]V. de Alfaro and T. Regge, *Potential Scattering* (North-Holland, Amsterdam, 1965).

[23]A. R. Choudhary and R. B. Jones, J. Phys. A 5, 981 (1972).

[24]If k is measured in fm^{-1} then a_l and r_l have the dimensions fm^{2l+1} and fm^{-2l+1}, respectively. Thus only in the S wave do they have the dimensions of length. The quantity $\rho_l = a_l r_l$ has the dimensions fm^2.

[25]The parameters R_l and S_l introduced here are pure numbers and have nothing to do with the R matrix and the S matrix.

[26]A. R. Choudhary, Lett. Nuovo Cimento 16, 307 (1976).

[27]A. R. Choudhary, Lett. Nuovo Cimento 12, 33 (1975).

[28]This statement is true in the model situation where the higher exchanges are neglected. The branch cuts corre-

sponding to the higher exchanges get mapped onto the horizontal line through $\omega = i\pi$ in Fig. 3. In the event that these higher exchanges become important, our formula [Eq. (6)] will converge within the circle $|\omega| < \pi$ which corresponds to a laboratory energy of 220 MeV. However, a proper handling of these higher singularities would be desirable for further improvement of the formalism. This improvement is in the spirit of the uniformization variable approximation (Refs. 14, 21, 23, 26, and 27), and in the sense that the region of validity of the formula in the energy plane gets considerably extended. An improved rate of convergence, although very much expected and actually achieved in all test cases, is not rigorously established through a mathematical proof.

[29]G. E. Brown, A. D. Jackson, and T. T. S. Kuo, Nucl. Phys. A133, 481 (1969).

[30]M. G. McGregor, R. A. Arndt, and R. M. Wright, Phys. Rev. 182, 1714 (1969).

[31]R. A. Arndt, R. H. Hackman, and L. D. Roper, Phys. Rev. C 15, 1002 (1977).

[32]A. Bohr and B. R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1969), Vol. I.

[33]E. Melkonian, Phys. Rev. 76, 1744 (1949).

[34]L. Koester and W. Nistler, Phys. Rev. Lett. 27, 956 (1971).

[35]We have used $a_s = -23.719$ fm and $r_{os} = 2.76$ fm as given in Ref. 1. Corresponding values from our analysis are $a_s = -23.679$ fm and $r_{os} = 2.505$ fm.