Bosonization of fermion operators as linked-cluster expansions

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In order for a boson-expansion theory to be useful for practical purposes, it must satisfy at least two requirements: It must be in the form of a linked-cluster expansion, and the pure (ideal) boson states must be usable as basis states. Previously, we constructed such a boson theory and used it successfully for many realistic calculations. This construction, however, lacked mathematical rigor. In the present paper, we develop an entirely new approach, which results in the same boson expansions obtained earlier, but now in a mathematically rigorous fashion. The achievement of the new formalism goes beyond this. Its framework is much more general and flexible than was that of the earlier formalism, and it allows us to extend the calculations beyond what had been done in the past.

[NUCLEAR STRUCTURE Boson expansion theories, commutator method, Marumori-Yamamura-Tokunaga method, fermion system, norm matrix, linked-cluster expansion, ideal-boson state, physical boson state, even-even nucleus, odd-A nucleus, nuclear collectivity.]

I. INTRODUCTION

About a decade ago, we published a pair of papers,[1,2] which we shall refer to as KT-1 and KT-2, respectively, which presented a detailed formalism and applications of the boson expansion theory (BET) to describe nuclear collective motions. Subsequently, rather extensive calculations were performed to analyze experimental data on Sm,[3] Ru-Pd,[4] Os-Pt,[5] and Ge (Ref. 6) isotopes which obtained good fits to data in most cases. (See also our recent review of these calculations.[7])

In spite of the numerical success[3–7] we have achieved, it appears that some in the nuclear physics community suspect the validity of our formalism on which our previous calculations were based, as seen, e.g., in two recent papers by Marshalek.[8] We ourselves have felt for some time now that it was desirable to review our old formalism, and put it on a firmer, and possibly a much broader, basis. The purpose of this paper is to present results of such renewed formal investigations, and the reader will find that our goal has been accomplished to a large extent. The new formalism is rigorous mathematically, and the obtained results are rather general and flexible. It thus allows us to extend the numerical calculations much beyond what we were able to do before. The reader will also find that the new formalism includes the old formalism as a special limit, thus giving us convincing justification of the latter, and of the ensuing numerical calculations,[3–7] contrary to Marshalek's skepticism.

The new formalism presented here may be called a *linked-cluster expansion* form of BET. It begins along the line set forth sometime ago by Marumori, Yamamura, and Tokunaga (MYT),[9] but departs significantly from it in several aspects. It has been known that the MYT theory, in spite of its mathematical rigor, suffered from a very slow convergence, and it will be seen later that this was the case because the original MYT theory was not formulated so as to get rid of *unlinked-cluster* terms. We shall also show that these unlinked-cluster terms remained because the MYT theory was not formulated (at least not explicitly) so as to use properly normalized states. A reader who is familiar with the Brueckner theory[10] of the nuclear many-body problem will recollect that Goldstone[11] showed that the Brueckner expansion can be brought into a linked-cluster expansion form, when, and only when, correctly normalized states are used, which is what is also done here. Therefore, our new formalism may be regarded as a reformulation of MYT along the lines of Goldstone.

The original MYT theory was formulated using the particle-pair representation (PPR), as was done by many other authors, rather than by using, i.e., the Tamm-Dancoff representation (TDR). To formulate BET by using PPR certainly helps to make the theory transparent. Nevertheless, as was emphasized recently by Tamura, Weeks, and Pedrocchi (TWP),[12] it does not make much sense to bosonize a

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fermion system that is described in terms of PPR, because no fermion state admits a superposition of the same pair, thus leaving no room for introducing boson statistics. A meaningful BET should thus start from a fermion system described in terms of TDR (or its equivalent). Reformulating MYT this way was in fact done by Holzwarth and his coworkers,[13,14] who also used the correct normalization of the states involved. They, nevertheless, formulated their theory by truncating the starting fermion system to purely collective states, which is not the case in our formalism given below. Thus our work may also be regarded as an extension of that of Holzwarth *et al.*

We have stated that the present formalism gives a justification of our previous calculations, and we now wish to explain what is actually meant by this statement. In doing this we first remark that there is an important difference between KT-1 and KT-2 in carrying out the boson expansion (of fermion pair operators). The formulation of KT-1 was done by first examining a similar formalism given earlier by Sorensen,[15] using a method which might be called the *commutator method.* This method leads to a series of sets of equations (which may be called *coefficient equations)* for the unknown coefficients introduced in expanding the fermion pair operators in terms of normal ordered products of boson operators. In KT-1, these equations were solved in an analytic way, by retaining all the TDR components. In spite of the fact that we are able to obtain the BET in TDR in a rather compact form this way, we noticed that it shared with MYT the same shortcoming, i.e., that the convergence was very slow. (Very recently, Pedrocchi and Tamura[16] further clarified the method employed in KT-1, in particular comparing it with that of MYT.)

In KT-2, the method of KT-1, i.e., to solve exactly the coefficient equations, was thus given up, and was replaced by a much simpler, and thus more approximate, method. This was to truncate the commutator equations to purely collective TDR components, and then solve them. We then found that a very fast convergence was achieved, as was stressed again recently.[17] The matrix elements of the fermion pair operators thus bosonized were evaluated in a purely collective ideal boson space. (We use terminology such as *ideal, physical,* and *unphysical* boson states, and/or space in the same way as used by MYT.) We call this the method of KT-2, and note that it is the method used[17] in subsequent calculations.[3-7]

The switch in method from that of KT-1 to that of KT-2, explained above, was stated very explicitly in KT-2 (see p. 345 of KT-2). Very unfortunately, however, this explicit statement appears to have been largely overlooked, resulting in several authors[8,18] having the misunderstanding that we used the method of KT-1, rather than that of KT-2, in our numerical calculations, and casting doubts about our results. (It should not be misunderstood that we gave up all of KT-1. It is *only the second half of Sec. 3 of KT-1* that was given up. The rest of KT-1 remains valid even to date.)

In any case, one thing which we shall show is that the formulas that were obtained by the method of KT-2 reemerge as limiting cases of the formulas that are to be derived later. Since the formulas of the present paper are obtained based on a rather firm basis, the above fact may very well be taken as a reconfirmation of the validity of the method of KT-2, and of the ensuing numerical calculations. In this regard, we recall a remark made above that our new method includes that of Lie and Holzwarth[13] (LH) as a special limit. Recall also the fact that both LH and KT-2 introduced a truncation at very early stages of their respective formulations. It would then be a rather natural guess that these two theories are very closely related. In fact, we can show[19] that the KT-2 and LH expansions, and further the generalized Holstein-Primacoff (GHP) expansion (under the same truncation) are all equivalent, none of them containing any unlinked cluster term. This fact is contrary to Marshalek's doubts regarding KT-2, presented in the last sentence of the second paper of Ref. 8.

The equivalence between LH and KT-2 needs a further remark. The results of LH (and of Sorensen[15]) had a problem in that the spacings of the obtained theoretical spectra were too wide, by a factor of 1.5—2, compared with the experimental spacings. This is because the (boson) Hamiltonian was obtained in TDR. With the method of KT-2, explained above, we obtain the same Hamiltonian as that of LH, making these two theories equivalent at this stage. In KT-2, however, we went one step further, before diagonalizing the above Hamiltonian, in switching from the TDR-type bosons to RPA (random-phase-approximation)-type bosons, which helped to remove the above difficulty. In this sense, LH and KT-2 calculations differ significantly. In all the calculations reported,[3-7] we were able to compare our theoretical spectra directly with experiment.

The structure of the present paper is as follows. The material presented in Sec. II is mostly preparatory. In Sec. II A, which is largely a recapitulation of Sec. 2 of KT-1, we first explain basic fermion quantities. In Sec. II B, we recapitulate TWP,[12] in which a few very important concepts were introduced which turned out to be crucial in leading us to the development of the formulation of the present paper. In Sec. II C, we recapitulate MYT, and this is followed by Sec. III, which discusses how to modify MYT. The formulas derived in Sec. III remain relatively abstract. The somewhat lengthy algebra required to give them a more concrete form is presented in Secs. IV and V. As stressed above, it is vital, in order to construct a valid BET, to be able to construct properly normalized states, which in turn requires the calculation of norm(s), or more generally, norm matrices. This task is performed in Sec. IV, with the help of a few mathematical relations derived in Appendices A and B. By using the results of Sec. IV, we then derive in Sec. V the bosonized (boson-expanded) forms of the fermion pair operators. As expected, they are given as linked-cluster expansions. These operators are supposed to satisfy, because of their construction, several requirements, e.g., they satisfy the commutation relations that are satisfied by their original fermion counterparts. That they indeed have these properties is reconfirmed in Sec. VI by going through somewhat lengthy but relatively elementary algebra. The boson expanded forms of the operators obtained in Sec. V are very compact, but the reader might find it somewhat difficult to determine how to use them in practice. We thus take up in Sec. VII a few lower order terms in the above expansion, and give them very explicit forms, including those that correspond to a truncation of the system to purely collective components. As we stressed above, they are found to agree with those obtained with the method of KT-2 (and of LH). All the above formulations were made keeping in mind the application of BET to the analyses of even-even nuclei. We touch very briefly, in Sec. VIII, upon how to extend them to odd-A nuclei. It will be seen that very similar linked-cluster expansions again result. We finally summarize in Sec. IX the results of the present paper and discuss what we intend to do in the future.

II. PRELIMINARIES

This section is subdivided into three subsections. In subsection A, we introduce a few basic fermion quantities, and discuss their properties and the relations they satisfy. It will be seen that the major part of this subsection is a recapitulation of Sec. 2 of KT-1. In subsection B, we discuss from a rather general point of view the basic concepts of BET as a whole. It is thus seen that subsection B essentially summarizes TWP. In subsection C, we survey briefly the original formalism of MYT; we do not add anything of our own. We nevertheless intend to make it clear why it was hard to use the MYT theory for practical purposes. This survey thus prepares us for the presentation that is made in Sec. III, where we discuss the basic idea on how to modify the MYT theory.

A. Fermion descriptions

We denote a shell model orbit in terms of a set of quantum numbers $\{j_1 m_1\}$. The operator $d^\dagger_{j_1 m_1}$ creates a fermion in this orbit, while $d_{j_1 m_1}$ annihilates it.[19(a)] When we say that we use a particle-pair representation (PPR), it means that we use the pair-creation operators defined, e.g., by

$$B^\dagger_{j_1 m_1 j_2 m_2} = d^\dagger_{j_1 m_1} d^\dagger_{j_2 m_2} = \sum_{\lambda\mu} (j_1 m_1 j_2 m_2 \mid \lambda\mu) B^\dagger_{j_1 j_2 \lambda\mu} . \tag{2.1}$$

In obtaining the second equality of (2.1), the angular momenta $\vec{j}_1$ and $\vec{j}_2$ were coupled to result in a new angular momentum λ with projection μ. The factor $(jmj'm' \mid \lambda\mu)$ is the Clebsch-Gordan coefficient. It is clear that the relation inverse to (2.1) is

$$B^\dagger_{j_1 j_2 \lambda\mu} = \sum_{m_1 m_2} (j_1 m_1 j_2 m_2 \mid \lambda\mu) B^\dagger_{j_1 m_1 j_2 m_2} . \tag{2.1'}$$

When the Tamm-Dancoff representation (TDR) is used, we use, instead, the following pair of operators:

$$B^\dagger_a \equiv B^\dagger_{\alpha\lambda\mu} = \sum_{j_1 \le j_2} D^{-1}_{j_1 j_2} \psi^{(\alpha)}_{j_1 j_2} B^\dagger_{j_1 j_2 \lambda\mu} = \frac{1}{2} \sum_{j_1 j_2} D_{j_1 j_2} \psi^{(\alpha)}_{j_1 j_2} B^\dagger_{j_1 j_2 \lambda\mu} , \tag{2.2}$$

$$D_{j_1 j_2} = (1+\delta_{j_1 j_2})^{1/2} .$$

Clearly, Eq. (2.2) defines $a = \{\alpha\lambda\mu\}$, α denoting a TD component for a fixed λ.

The TD coefficients $\psi^{(\alpha)}_{j_1 j_1 \lambda}$ satisfy the following relations:

$$\psi^{(\alpha)}_{j_1 j_2 \lambda} = \theta_{j_1 j_2 \lambda} \psi^{(\alpha)}_{j_2 j_1 \lambda} ,$$

$$\theta_{j_1 j_2 \lambda} = (-)^{j_1 - j_2 + \lambda} ,$$

$$\sum_\alpha \psi^{(\alpha)}_{j_1 j_2 \lambda} \psi^{(\alpha)}_{j_1' j_2' \lambda} = (1+\theta_{j_1 j_2 \lambda} P_{j_1 j_2}) \times \delta_{j_1 j_1'} \delta_{j_2 j_2'} D_{j_1 j_2}{}^{-2} , \tag{2.3}$$

$$\sum_{j_1 j_2} D_{j_1 j_2}{}^2 \psi^{(\alpha)}_{j_1 j_2 \lambda} \psi^{(\alpha')}_{j_1' j_2' \lambda} = 2\delta_{\alpha\alpha'} .$$

In (2.3), $P_{j_1 j_2}$ exchanges j_1 and j_2 in the expression that follows it. An important consequence of (2.3) is that

$$B^{\dagger}_{j_1m_1j_2m_2}=\sum_{\alpha\lambda\mu}(j_1m_1j_2m_2\mid\lambda\mu)\times D_{j_1j_2}\psi^{(\alpha)}_{j_1j_2\lambda}B^{\dagger}_{\alpha\lambda\mu}\ . \tag{2.4}$$

It is obvious that, if we choose

$$\psi^{(\alpha)}_{j_1j_2\lambda}=D_{j_1j_2}{}^{-1}\delta_{\alpha,(j_1j_2)}\ ,$$

Equation (2.4) reduces to Eq. (2.1), and thus TDR reduces to PPR. In this sense TDR is more general, and in the following, we use TDR almost exclusively.

In addition to the pair creation operators (and their conjugates), we also need the so-called scattering operator defined by

$$C^{\dagger}_{j_1m_1j_2m_2}=d^{\dagger}_{j_1m_1}d_{j_2m_2}=\sum_{\lambda\mu}(j_1m_1\widetilde{j_2m_2}\mid\lambda\mu)C^{\dagger}_{j_1j_2\lambda\mu}\ , \tag{2.5}$$

where

$$(j_1m_1\widetilde{j_2m_2}\mid\lambda\mu)=(-)^{j_2-m_2}(j_1m_1j_2-m_2\mid\lambda\mu)\ .$$

If we abbreivate $B^{\dagger}_{j_1m_1j_2m_2}$ by $B^{\dagger}_{12}$, and $C^{\dagger}_{j_1m_1j_2m_2}$ by $C^{\dagger}_{12}$, we have the following commutation relations (in the PPR)

$$\begin{aligned}[][B_{12},B^{\dagger}_{34}]&=(\delta_{13}\delta_{24}-\delta_{14}\delta_{23})-(1-P_{12})(1-P_{34})\delta_{13}C^{\dagger}_{42}\ ,\\ [C^{\dagger}_{12},B^{\dagger}_{34}]&=\delta_{23}B^{\dagger}_{14}-\delta_{24}B^{\dagger}_{13}\ ,\\ [C^{\dagger}_{12},C^{\dagger}_{34}]&=\delta_{23}C^{\dagger}_{14}-\delta_{14}C^{\dagger}_{32}\ .\end{aligned} \tag{2.6}$$

The commutation relations for the TDR operators are given by

$$[B_a,B^{\dagger}_b]=\delta_{ab}-\sum_p P^{(p)}_{a,b}C^{\dagger}_p\ , \tag{2.7a}$$

$$[C^{\dagger}_p,B^{\dagger}_a]=\sum_b P^{(p)}_{a,b}B^{\dagger}_b\ , \tag{2.7b}$$

where

$$p=\{j_1j_2kq\}\ ,$$

with

$$\vec{k}=\vec{j}_1+\vec{j}_2$$

and

$$q=m_1+m_2\ .$$

Further,

$$\begin{aligned}P^{(j_1j_2kq)}_{\alpha\lambda\mu;\alpha'\lambda'\mu'}&=(\lambda\mu\lambda'\mu'\mid kq)\hat{P}^{(j_1j_2k)}_{\alpha\lambda;\alpha'\lambda'}\ ,\\ \hat{P}^{(j_1j_2k)}_{\alpha\lambda;\alpha'\lambda'}&=\hat{\lambda}\hat{\lambda}'\sum_j\psi^{(\alpha)}_{jj_2\lambda}\psi^{(\alpha')}_{j_1j\lambda}\times W(j_2j_1\lambda\lambda';kj)D_{jj_1}D_{jj_2}\ .\end{aligned} \tag{2.8}$$

Note that a quantity called $Y(abcd)$ was very important in KT-1, and will also be used below. It is defined by

$$Y(abcd)=\sum_p P^{(p)}_{ab}P^{(p)}_{cd}\ . \tag{2.9}$$

B. Recapitulation of TWP

The reason we introduce bosons to describe nuclei which consist entirely of fermions should be obvious. This may be done when it is (believed to be) guaranteed that a boson calculation can copy faithfully (or reasonably well) a fermion calculation, and the former is practicable while the latter is not. It should be stressed that it is rather awkward to introduce bosons for problems for which a fermion description is feasible. As a consequence, when introducing bosons, one should clearly keep in mind what the corresponding fermion calculations would be like. These considerations constituted the starting point of TWP.

When performing shell-model-like calculations, one would begin by first constructing basis states. If PPR is used, these basis states may be written as

$$\mid n\rangle=B^{\dagger}_{12}B^{\dagger}_{34}\cdots B^{\dagger}_{2n-1,2n}\mid 0\rangle\ . \tag{2.10}$$

In (2.10), we have in mind a state which has n pairs, and we have denoted such a state by $\mid n\rangle$, the state $\mid 0\rangle$ standing for the fermion vacuum. It is obvious that, for (2.10) to be nonvanishing, all the indices $1,2,\ldots,2n$ must be different. The state $\mid n\rangle$ of (2.10), which is made nonvanishing this way, is normalized; no additional normalization factor is needed.

If TDR is used, on the other hand, an n fermion-pair state may be written as

$$\mid n\rangle=N_F{}^{-1}\{(B^{\dagger}_a)^{n_a}(B^{\dagger}_b)^{n_b}\cdots\}\mid 0\rangle\ ;\quad (n_a+n_b+\cdots=n)\ . \tag{2.11}$$

An important difference between (2.10) and (2.11) is that in the latter it is permissible to take, e.g., $n_a\geq 2$, i.e., to operate with the same operators repeatedly. Obviously, (2.11) is a complicated linear combination of states of the type of (2.10), which requires us to introduce a normalization factor $N_F{}^{-1}$ explicitly in (2.11).

Having thus constructed the fermion basis states, we may now consider the bosonization of the fermion system, which is often done by introducing a concept called *mapping*, being denoted by using an arrow in the following way:

$$O_F\rightarrow O_B\ , \tag{2.12}$$

$$\mid n\rangle\rightarrow\mid n)\ . \tag{2.13}$$

Obviously, Eq. (2.12) means a mapping of a fermion operator O_F onto a corresponding boson operator O_B, while (2.13) does the same of a fermion state $| n \rangle$ on to a corresponding boson state $| n)$.

One of the most important remarks made in TWP is that, if one intends to construct an *exact* BET, it must be designed so that an equality

$$\langle m | O_F | n \rangle = (m | O_B | n) \tag{2.14}$$

holds. In fact, if (2.14) holds for all the matrix elements, it is guaranteed that the same numerical results are obtained for any physical quantity (observable), irrespective of whether the calculation is done in a fermion way or in a boson way. In other words, for a bosonization to make sense, the two mappings of (2.12) and (2.13) must be done simultaneously and consistently, so that (2.14) is guaranteed.

Let us take three basic fermion pair operators B_a, $B_a^\dagger$, and $C_p^\dagger$ introduced in subsection A, and rewrite them as B_F, $B_F^\dagger$, and $C_F^\dagger$ to signify that they are fermion operators. We now assume that their mapping onto B_B, $B_B^\dagger$, and $C_B^\dagger$ has been done in such a way that these bosonized pair operators satisfy the commutation relations which are the same as (2.7), except that these bosonized pairs replace the fermion pairs there. Since any fermion operator is constructed out of these three basic operators, the above procedure completes the mapping of (2.12). The mapping of (2.13) may be rewritten, somewhat more explicitly, as

$$| n \rangle = N_F{}^{-1} \{ B_F^\dagger \}^n | 0 \rangle \rightarrow | n) = N_F{}^{-1} \{ B_B^\dagger \}^n | 0) \; . \tag{2.15}$$

A very obvious abbreviation of (2.11) was made in describing the state $| n \rangle$ that appears in (2.15).

The above procedure clearly makes the two mappings (2.12) and (2.13) consistent, because the same set of pair operators are used. It will also be obvious that Eq. (2.14) is satisfied, because in evaluating both sides of this equation, the same set of commutation relations is used.

It is rather reassuring that we found in this way a method to carry out a bosonization exactly, but actually we have not achieved anything, because the evaluation of the rhs of (2.14) is at least equally involved as is that of the lhs of (2.14). The bosonization has not yet introduced any simplification of the numerical task involved.

In the terminology of MYT, the bosonized state $| n)$ given by (2.15) is a physical state, and its appearance in (2.14) is the reason why the evaluation of $(m | O_B | n)$ is so much involved. Suppose now that we find a method to replace this matrix element by $(\bar{m} | \bar{O}_B | \bar{n})$, $| \bar{n})$ being an ideal boson state, so that (2.14) is replaced by

$$\langle m | O_F | n \rangle = (\bar{m} | \bar{O}_B | \bar{n}) \; . \tag{2.16}$$

Then the matter changes drastically. The construction of $| \bar{n})$ is orders of magnitude easier than is that of $| n)$. Thus, unless $\bar{O}_B$ becomes excessively complicated, the evaluation of the rhs of (2.16) becomes feasible, even when that of the lhs of (2.16) is not. In other words, what we said at the beginning of the present subsection can be rephrased by saying that meaningful BET's are those that permit the use of (2.16), in place of that of (2.14). As we remarked in Sec. I, the use of the method of KT-2 meant the use of (2.16). Nevertheless, we used (2.16) without a sufficient proof that (2.14) can indeed be replaced by (2.16), and to fill this gap is what we intend to do in the following sections.

At this stage, it is worthwhile to remark that TWP introduced a method called *step-by-step operation* (SSO), which is very useful in obtaining $| n)$ explicitly and understanding its properties. As is well known[1,9] (see below), a standard form we obtain for $B_B^\dagger$ is given, in a somewhat abbreviated notation, as (A and $A^\dagger$ are pure boson operators)

$$B_B^\dagger = \sum_{n=1} X_{2n-1} (A^\dagger)^n A^{n-1} \; . \tag{2.17}$$

In constructing $| n)$, we first operate $B_B^\dagger$ of (2.17) upon $| 0)$. Clearly, only the X_1 term of (2.17) contributes to the result. In the second-step operation, however, X_1 and X_3 terms contribute; and so forth. In obtaining $| n)$ exactly, it is thus clear that it is sufficient to have (2.17) terminated at the nth term. It is also clear that $| n)$ is written as a linear combination of the terms of the form $(A^\dagger)^n | 0)$, i.e., of $| \bar{n})$.

An important use of SSO was made in TWP. It was to apply it to a system in which the TDR space was limited to a single collective component of monopole nature. We then found that $| n) = | \bar{n})$, i.e., that (2.14) and (2.16) were equivalent, even with the choice of $O_B = \bar{O}_B$. In TWP, we gave this result as a (partial) proof of the validity of the KT-2 method. A generalization of this proof will be given in Sec. VI B.

C. The MYT Theory

As is well known, the bosonization of MYT is done by first introducing the so-called *modified Usui operator* U. If the PPR is used, as was the case in MYT, the explicit form of U is given as

$$U=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\times\sum_{\mu_1\nu_1\cdots\mu_n\nu_n}[(2n-1)!!]^{-1/2}\times\sum_P(-)^P PA^\dagger_{\mu_1\nu_1}\cdots A^\dagger_{\mu_n\nu_n}|0)\times\langle 0|d_{\mu_1}d_{\nu_1}\cdots d_{\mu_n}d_{\nu_n}\ . \tag{2.18}$$

In (2.18), μ's and ν's denote fermion orbits, and the operator P exchanges any number of these indices that are attached to different $A^\dagger_{\mu\nu}$ operators. It is clear that the number of such exchanges equals $(2n-1)!!$, and thus that

$$|n)\equiv[(2n-1)!!]^{-1/2}\times\sum_P(-)^P PA^\dagger_{\mu_1\nu_1}\cdots A^\dagger_{\mu_n\nu_n}|0) \tag{2.19a}$$

is a normalized n-boson state, making it quite appropriate to denote it as $|n)$.

The state (2.19a) was called a *physical boson state* in MYT; it satisfies the Pauli principle, i.e., it is antisymmetric with respect to an exchange of any pair of the fermion indices μ and ν. Note that $A^\dagger_{\mu\nu}=-A^\dagger_{\nu\mu}$. [The state (2.15), constructed via the commutator method, is also physical[12].] For later convenience we also define, as done by MYT, the (normalized) *ideal boson states*

$$|\bar{n})=A^\dagger_{\mu_1\nu_1}\cdots A^\dagger_{\mu_n\nu_n}|0)\ . \tag{2.19b}$$

Clearly they do not satisfy the Pauli principle. Comparison of (2.19a) and (2.19b) shows that $|n)$ is written as a linear combination of $|\bar{n})$.

The last factor of (2.18), i.e.,

$$\langle 0|d_{\mu_1}d_{\nu_1}\cdots d_{\mu_n}d_{\nu_n}$$

can be regarded as the Hermitian conjugate of (2.10), and may thus be abbreviated as $\langle n|$. Then

$$U=\sum_n|n)\langle n|\ , \tag{2.20}$$

and

$$|n)=U|n\rangle\ . \tag{2.21}$$

This replaces (2.13), which means that the mapping of the states, which was defined in (2.13) only by using an arrow, is now described by an equality, making the mathematical concept of the mapping much clearer. As shown in MYT, the mapping of the operator (2.12) is also replaced by an equality

$$O_M=UO_FU^\dagger\ . \tag{2.22}$$

Note that we denoted the lhs of (2.22) by O_M, rather than by O_B. This is because we want to reserve the notation O_B to mean something else, as will be seen in Sec. III.

In spite of having the capability to define the mapping very clearly, as shown above, which also made the explicit construction of O_M rather straightforward, the MYT theory had not been used very much in the past in practice. This was partly caused by the fact that, as seen from (2.18) and (2.22), the operator O_M contains a factor

$$\Gamma\equiv|0)(0|=\sum_k[(-)^k/(2^k k!)](A^\dagger)^k A^k\ , \tag{2.23}$$

which converges extremely slowly. As will be clarified shortly, all the terms in (2.23) (excepting the $k=0$ term, which equals 1) are unlinked-clustered, and make O_M contain many unlinked-cluster terms. [See Eq. (5.9a).]

As regards the question on which of the states, physical or ideal, are to be used as basis states, the answer is clear. Equations (2.21) and (2.22) result in the following algebra:

$$\langle m|O_F|n\rangle=\langle m|\vec{1}_F O_F\vec{1}_F|n\rangle=\langle m|U^\dagger UO_FU^\dagger U|n\rangle=(m|O_M|n)\ , \tag{2.24}$$

which is nothing but (2.14), appropriate for the MYT theory. It shows that the physical states of (2.19a) must be taken.

It has often been remarked (see, e.g., Ref. 8), that a unique advantage of the MYT operators is that, when they are operated upon an ideal state, a physical state results, thus preventing one from straying into unphysical space. To put too much emphasis on this aspect can be misleading, however. As remarked above, a correct use of MYT results in (2.14), and since in general

$$(m|O_M|n)\neq(\bar{m}|O_M|\bar{n})\ ,$$

MYT does not allow for taking advantage of (2.16). (The matter is different if $\bar{O}_M$ can be constructed, but it means going beyond the framework of MYT.) When we stay with (2.14), however, the MYT and the commutator-method operators behave in exactly the same way.[8,12] Therefore, the advantage of MYT mentioned above disappears.

The O_M is full of unlinked-cluster terms, as we remarked above, and so is the operator $B^\dagger_M=UB^\dagger_F U^\dagger$. Let us now consider constructing $|n)$ by using the $B^\dagger_M$ and employing the SSO method of the preceding subsection. It is interesting to note that one finds in this procedure that all the unlinked-cluster-term contributions cancel out (which is not unexpected, because $|n)$ is properly normalized). Thus, one of the disadvantages of MYT is removed this way. (Exactly the same is true for the KT-1 operators.[12]) This cannot be taken, however, as a relief. To carry out the SSO in practical problems is prohibitively involved. One cannot avoid this so long as one is forced to stay with (2.14).

We should look at the matter somewhat differently, and begin to realize that a unique advantage of the MYT method lies in its introduction of the operator U to start with. The above-mentioned problems with the original MYT remain if one retains the form of U as given in (2.18). The picture changes drastically, however, if one chooses a somewhat different form for U.

III. REFORMULATION OF THE MYT THEORY

The operator U, as defined by (2.18), was constructed in such a way that, when it is operated upon a normalized fermion (pair) state, it results in a normalized physical boson state. An unsatisfactory feature of this form of U is that it is defined in PPR. As we remarked in the beginning of Sec. II B, the fermion states with which we start differ very significantly when they are written in PPR and in TDR. In the former, no pair is repeated, and thus there is no room to introduce boson statistics in its boson image. This indicates that a procedure in which one starts from PPR, completes bosonization, and then finally transforms to TDR, is rather unlikely to allow us to construct a transparent BET. We will be much better off if we start by first constructing a new U which is described in terms of quantities in TDR. This is what we intend to do now. This was also the spirit of LH.[13]

Since, as we remarked at the end of Sec. II C, to achieve (2.16) is our goal, it may be appropriate to construct the new U in such a way that, when it is operated upon a normalized fermion state in TDR, it results in an ideal boson state which is also normalized. We may thus write it as

$$U=\sum_{n,(a)} |n;\bar{a})\langle n;a| \ , \tag{3.1}$$

where

$$(a)\equiv\{a_1,a_2,\cdots,a_n\} \ ,$$

a_i denoting a TD component, and

$$\begin{aligned}|n;\bar{a})&=(1/\sqrt{n!})(A_a^\dagger)^n|0)\\ &\equiv(1/\sqrt{n!})A_{a_1}^\dagger\cdots A_{a_n}^\dagger|0) \ .\end{aligned} \tag{3.2}$$

The fermion state $|n;a\rangle$, of which the Hermitian conjugate $\langle n;a|$ appeared in (3.2), is defined by

$$|n;a\rangle=\sum_{(b)}(Z_n{}^{-1})_{a,b}|n;b\rangle\rangle \ , \tag{3.3}$$

where

$$\begin{aligned}|n;a\rangle\rangle&=(1/\sqrt{n!})(B_a^\dagger)^n|0\rangle\\ &\equiv(1/\sqrt{n!})B_{a_1}^\dagger\cdots B_{a_n}^\dagger|0\rangle \ ,\end{aligned} \tag{3.4}$$

very similarly as in (3.2), while $(Z_n{}^{-1})_{b,a}$ is defined as the inverse of the square root of the norm matrix

$$(Z_n{}^2)_{a;b}=(Z_n{}^2)_{b;a}$$

defined by

$$\langle\langle n;a|m;b\rangle\rangle=(Z_n{}^2)_{a;b}\delta_{mn} \ . \tag{3.5}$$

Obviously, the states $|n;a\rangle\rangle$ are not orthonormal in general, but the states $|n;a\rangle$ defined by (3.3) are. Therefore, the U defined by (3.1) does have all the required properties.

It is easy to see that (3.1) results in two expressions given as

$$|n;a\rangle=U^\dagger|n;\bar{a})$$

and

$$\langle m;b|=(m;\bar{b}|U \ . \tag{3.6}$$

The lhs of (2.16), which should now be generalized as $\langle m;b|O_F|n;a\rangle$, can then be rewritten as

$$\begin{aligned}\langle m;b|O_F|n;a\rangle&=(m;\bar{b}|UO_FU^\dagger|n;\bar{a})\\ &\equiv(m;\bar{b}|O_B|n;\bar{a}) \ ,\end{aligned} \tag{3.7}$$

defining the boson image of O_F [which we write as O_B, rather than as $\bar{O}_B$ as in (2.16)] as

$$\begin{aligned}O_B&=UO_FU^\dagger\\ &=\sum_{n,m}\sum_{(a,b)}|n;\bar{a})\langle n;a|O_F|m;b\rangle(m;\bar{b}| \ .\end{aligned} \tag{3.8}$$

The O_B can also be written as

$$O_B=\sum_{n,m}\sum_{(a,b,c,d)}[1/\sqrt{m!n!}]\,|n;\bar{a})(Z_b{}^{-1})_{ac}\langle 0|(B_c)^nO_F(B_d^\dagger)^m|0\rangle(z_m{}^{-1})_{db}(m;\bar{b}| \ . \tag{3.9}$$

The derivation of (3.9) completes the bosonization of the fermion operator O_F, and this procedure has turned out to be extremely simple. This simplicity is, however, only apparent. To construct explicitly the rhs of (3.8) is rather involved, as we shall soon see.

Great care must be exercised in defining the space in which the $Z_n{}^2$ and $Z_n{}^{-1}$ matrices are constructed. In writing several of the formulas given above, it has tacitly been assumed that the inverse of Z_n, i.e., $Z_n{}^{-1}$, exists. However, under certain circumstances, $Z_n{}^{-1}$ does not exist, making the formulas given above invalid.

Let us first consider the case with $n=2$. The case with $n=1$ is trivial; by construction of the TD states, we simply have

$$(Z_1{}^2)_{a_1;b_1}=(Z_1{}^{-1})_{a_1;b_1}=\delta_{a_1b_1} .$$

Let us denote by s the total number of (fermion) single particles orbits (jm) which we decided to include in our calculations. The total number of one-pair states, i.e., the total number of TD states $a=\{\alpha\lambda\mu\}$ [cf. below Eq. (2.2)], which we denote by t, is given as $t={}_sC_2$, where ${}_sC_2$ is a binomial coefficient. For $n=2$, the state $|2;a_1a_2\rangle\rangle$, defined by (3.4), is of the form $B_{a_1}^\dagger B_{a_2}^\dagger|0\rangle$, whose number is t^2, making $Z_2{}^2$ a $t^2\times t^2$ matrix. The number of two-pair (four-fermion) states that can be constructed without violating the Pauli principle is given, however, as ${}_sC_4$, which is smaller than t^2. This means that if we diagonalize the $t^2\times t^2$ matrix $Z_2{}^2$, we will have zero eigenvalues with the degeneracy equal to $t^2-{}_sC_4>0$. Therefore,

$$\det|Z_2{}^2|=0 ,$$

and thus $Z_2{}^{-1}$ cannot exist. (Very similar arguments can be repeated for $n>2$.)

From what we showed above, it is clear that we have to truncate the $|n,a\rangle\rangle$ space one way or another. One possibility is to use, e.g., the Schmidt orthogonalization procedure for each n, retaining all the possible ${}_sC_{2n}$ basis states. This certainly keeps the theory exact, but makes it all but intractable. Since we are going to truncate the TDR state at any rate, at least at the stage of numerical calculations, it would make more sense to introduce the truncation at an early stage of the formulation. Probably the easiest and the most transparent way would be to divide the TDR components into two groups, T and $(1-T)$, the space T including components that are retained, and $(1-T)$ those that are not. More precisely, all the indices that are attached to $(Z_n{}^{-1})$ matrices, for any n, should be picked up from the space T. However, the indices attached, e.g., to $(Z_n{}^2)$ may belong either to T or $(1-T)$. We will encounter a variety of examples of this mixed use of the indices as the formalism is developed below.

An extreme of T is the case in which only the collective, quadrupole component is retained. Our theory is then reduced to that of LH (and of KT-2). Our formalism allows, however, for a much wider choice of T; e.g., we may not encounter any singular $(Z_n{}^{-1})$, unless we make n excessively large by retaining in T all the TD components with $\lambda=2$.

It is convenient to show at this stage how the boson images of the two basic fermion pair operators $B_e^\dagger$ and $C_p^\dagger$ look. They are, of course, obtained as special examples of (3.9), and are found to be

$$(B_e^\dagger)_B=\sum_n\sqrt{n}\sum_{(a,b)}\left[\sum_{(c)}(Z_n)_{(a);e(c)}(Z_{n-1}{}^{-1})_{cb}\right]|n;a)(n-1;b| , \tag{3.10}$$

and

$$(C_p^\dagger)_B=\sum_n(1/n!)\sum_{(a,b)}\left[\sum_{(c,d)}(Z_n{}^{-1})_{ac}\langle 0|(B_c)^nC_p^\dagger(B_d^\dagger)^n|0\rangle(Z_n{}^{-1})_{db}\right]|n;a)(n;b| . \tag{3.11}$$

We also remark here that the vacuum projector Γ, appropriate for the TDR, is given as

$$\Gamma=|0)(0|=\sum_{k=0}[(-)^k/k!]\sum_{(a)}(A_a^\dagger)^k(A_a)^k . \tag{3.12}$$

Note that we wrote, e.g., $|n;a)$ in place of $|n;\bar{a})$ in (3.10) and (3.11). Since we exclusively use ideal boson states in the rest of the present paper, this simplification will not cause any confusion. In other words, from now on, $|n;a)$ means an ideal boson state, not the physical boson state, unless it is explicitly stated otherwise.

We may make an additional remark here. Let us define a projection operator T by

$$T=\sum_n\sum_{(a\in T)}|n;a)(n;a|$$

and a norm operator Λ by

$$\Lambda=\sum_n\sum_{(a,b\in T)}|n;a)[n!(Z_n{}^2)_{ab}](n;b| .$$

One then sees that the operator O_B given in (3.8) may be cast into the following form:

$$O_B=(T\Lambda T)^{-1/2}(TO_MT)(T\Lambda T)^{-1/2} , \tag{3.13}$$

where we denoted by O_M an operator which we might have used, had we decided to work in the physical boson space. The relation (3.13) then means that we have derived an effective operator O_B out of an original operator O_M, so that we can now work in the ideal boson space. Equations like (3.13) have been used in the past, in both particle physics[20] and nuclear physics.[21]

IV. PROPERTIES OF THE Z_n MATRIX

The $Z_n{}^2$ matrix was first introduced in Eq. (3.5), which we shall reproduce here in a slightly different form:

$$(Z_n{}^2)_{ab} \equiv (1-Y_n)_{ab} = (n!)^{-1}\langle 0|(B_a)^n(B_b^\dagger)^n|0\rangle \; . \qquad (4.1)$$

Clearly, the second version of (4.1) defines a new matrix called Y_n, which is related to Z_n by the relation

$$Z_n = [1-Y_n]^{1/2} \; .$$

Because of the appearance of the square root, the manipulation of the matrix Z_n is somewhat more involved than is that of Y_n. We shall thus first discuss the properties of Y_n.

A. Properties of Y_n

In Sec. III, we called $Z_n{}^2$ a *norm matrix,* and we shall retain this name for simplicity, although it may be more appropriate to call $Z_n{}^2$ a *reduced norm matrix,* because it differs by a factor of $n!$ from the usual norm matrix. As is clear from the definition made through Eq. (4.1), the matrix Y_n measures the deviation of this reduced matrix from 1. If $Y_n=0$, it means that the operators $(B_a)^n$ and $(B_b^\dagger)^n$ are behaving as if they were pure boson operators $(A_a)^n$ and $(A_b^\dagger)^n$. To the extent which Y_n deviates from 1, the former set of operators behaves differently from the pure boson operators. It is obvious that $Y_0=0$. We also see that $Y_1=0$, because

$$\langle 0|B_aB_b^\dagger|0\rangle = \delta_{ab}$$

by construction of the TD operators. For $n\geq 2$, however, $Y_n\neq 0$ in general.

Instead of writing the matrix element of Y_n as $(Y_n)_{ab}$, as done in (4.1), we prefer to write it as $(Y_n)_{aa'}$. The suffix a here stands for a set of n indices $\{a_1,a_2,\ldots,a_n\}$, and similarly for a'. Thus, we may simply write

$$a=\{1,2,\ldots,n\} \; ,$$

and

$$a'=\{1',2',\ldots,n'\}$$

without causing any confusion. We understand that $(Y_n)_{aa'}$ and $(Y_n)_{12\cdots n;1'2'\cdots n'}$ mean the same matrix element.

In the following, we very often encounter products of Kronecker deltas, and we find it convenient to introduce a quantity denoted by $\Delta_{12\cdots i;1'2'\cdots i'}$ which is constructed as follows. Construct a product of i Kronecker deltas, each Kronecker delta taking an element from the set $\{1,2,\ldots,i\}$ and another from the set $\{1',2',\ldots i'\}$. We can construct $i!$ different products of this form, and we sum them together and divide the sum by $i!$. This results in $\Delta_{12\cdots i;1',2'\cdots i'}$. Thus, e.g.,

$$\Delta_{12;1'2'} = (\delta_{11'}\delta_{22'}+\delta_{12'}\delta_{21'})/2! \; , \qquad (4.2a)$$

$$\Delta_{123;1'2'3'} = (\delta_{11'}\delta_{22'}\delta_{33'}+\delta_{11'}\delta_{23'}\delta_{32'}+\cdots)/3! \; . \qquad (4.2b)$$

At this stage, it is worthwhile to note that $\Delta_{12\cdots i;1'2'\cdots i'}$ is totally symmetric with respect to any interchange of the indices $\{12\cdots i\}$, and of the indices $\{1'2'\cdots i'\}$. We further remark that this *doubly-symmetric* nature is also shared by the $(Y_n)_{12\cdots n;1'2'\cdots n'}$ matrix, as is easily seen from its definition, (4.1).

We shall now consider a product written as

$$P_{a'}^{(i)}[\Delta_{12\cdots i;1'2'\cdots i'}(Y_{n-i})_{(i+1)\cdots n;(i+1)'\cdots n'}] \; . \qquad (4.3)$$

The operand, i.e., the quantity in the square bracket of (4.3), is symmetric with respect to the exchange of the first i indices $\{1'2'\cdots i'\}$ and of the last $(n-i)$ indices $\{(i+1)',\ldots,n'\}$. The symmetrizer $P_{a'}^{(i)}$ then exchanges indices in the above two sets, so that expression (4.3) as a whole becomes totally symmetric with respect to any exchange of indices from the set $\{1'2'\cdots n'\}$. In other words, expression (4.3) is a sum of ${}_nC_i$ terms, where ${}_nC_i$ is a binomial coefficient.

In the following, we shall often abbreviate the set $\{1,2,\ldots,i\}$ of indices as (i). Similarly, we write $\{n-i\}$ for the set $\{i+1,\ldots,n\}$ of indices. We may thus write

$$P_{a'}^{(i)}[\Delta_{(i);(i)'}(Y_{n-i})_{\{n-i\};\{n-i\}'}] \; , \qquad (4.4)$$

which has the same meaning as the expression given in (4.3).

It is easy to calculate $(Y_2)_{aa'}$ from (4.1). One sees that

$$(Y_2)_{12;1'2'} = \tfrac{1}{2}Y(11'2'2) \; , \qquad (4.5)$$

where $Y(abcd)$ was defined in (2.9), and in obtaining (4.5), the commutation relations in (2.7) were used.

The calculation of $(Y_3)_{a,a'}$ gets somewhat lengthier, but not yet very much involved. We find that it can be written as

$$(Y_3)_{123;1'2'3'}=P_{a'}^{(1)}\Delta_{1,1'}(Y_2)_{23;2'3'}+(Y_3)_{123;1'2'3'}^{(L)} \tag{4.6}$$

with

$$(Y_3)_{123;1'2'3'}^{(L)}=-\tfrac{2}{3}P_{a'}^{(1)} \times\left[\sum_g (Y_2)_{23;1'g}(Y_2)_{1g;2'3'}\right] . \tag{4.7}$$

The first term of (4.6) is in the form of (4.3), and its meaning is clear. In the second term of (4.6), i.e., in the expression of (4.7), the symmetrizer $P_{a'}^{(1)}$ appears again, but its operand is not in the form $\Delta\times Y$ but in the form $Y\times Y$. Nevertheless, $Y\times Y$ is again a product of two doubly symmetric matrices, and thus the operation of $P_{a'}^{(1)}$ in (4.7) is understood, as in (4.3). We also remark that there appears in (4.7) a sum over the TD index g. Its origin is Eq. (2.7b). The g's are not necessarily limited to those in T, even if the a's and a''s are.

The superscript (L), attached to $(Y_3)^{(L)}$ in (4.7), stands for *linked-cluster,* and $(Y_3)^{(L)}$ is the linked-cluster part of (Y_3); i.e., the part of (Y_3) which is free from the Δ factor, in contrast to the first term of (4.6), which does contain a Δ factor. A term containing a Δ factor may be called an *unlinked-cluster* term. Thus, what we have achieved in (4.6) was to separate (Y_3) into linked-cluster and unlinked-cluster terms. For larger n, such a separation gets more and more involved, particularly because a variety of Δ factors can be constructed when n is large.

We find, nevertheless, that the result can be presented in a very compact form given as

$$(Y_n)_{aa'}=\sum_{i=0}^{n-2}P_{a'}^{(i)}[\Delta_{12\cdots i;1'2'\cdots i'}(Y_{n-i})_{(i+1)\cdots n;(i+1)'\cdots n'}^{(L)}] , \tag{4.8}$$

combined with a recurrence relation described as

$$(Y_n)_{aa'}^{(L)}=-\frac{2}{n}P_{a'}^{(2)}\left[\sum_g (Y_2)_{1g;1'2'}^{(L)}(Y_{n-1})_{23\cdots n;g3'\cdots n'}^{(L)}+(Y_2)_{12;1'2'}^{(L)}(Y_{n-2})_{3\cdots n;3'\cdots n'}^{(L)}\right] . \tag{4.9}$$

It is understood in (4.8) that $P_{a'}^{(0)}=1$ and $\Delta_{12\cdots 0;1'2'\cdots 0}=1$. We also remark that it is easy to confirm that (4.9) reduces to (4.7) for $n=3$, as it should. Note that, for $n=3$, $P_{a'}^{(2)}=P_{a'}^{(1)}$, and that $(Y_2)=(Y_2)^{(L)}$, i.e., that (Y_2) is purely linked clustered.

In (4.8) (and elsewhere) the symmetrization regarding the $\{a'\}$ indices is explicit, via the presence of $P_{a'}^{(i)}$, but this is not the case for $\{a\}$. The form of the lhs, $(Y_n)_{aa'}$, shows, however, that the rhs of (4.8) is totally symmetric with respect to $\{a\}$. We can thus insert, in the summand in (4.8), a factor $P_a^{(i)}/{}_nC_i$. For simplicity, however, we shall not do this, unless it becomes necessary.

To prove (4.8), we first go back to (4.1), which can be written somewhat more explicitly as

$$(1-Y_n)_{aa'}=(1/n!)\langle 0|B_n\cdots B_2B_1B_{1'}^{\dagger}B_{2'}^{\dagger}\cdots B_{n'}^{\dagger}|0\rangle . \tag{4.10}$$

A possible way to evaluate the fermion matrix element on the rhs of (4.10) is to move the B_i operators to the right, so that they eventually annihilate the vacuum $|0\rangle$. Instead of carrying out this operator for all the B_i's, let us consider moving only the B_1 factor. Each time B_1 is commuted with a $B_j^{\dagger}$, it produces a Kronecker delta, and a term involving the $C^{\dagger}$ operator; cf. (2.7a). We move the $C^{\dagger}$ operator to the right as well. After this operation is done, it is easy to see that we obtain a recurrence relation given as

$$\begin{aligned}(Y_n)_{aa'}=&(1/n)P_{a'}^{(1)}[\Delta_{11'}(Y_{n-1})_{2\cdots n;2'\cdots n'}]+(2/n)P_{a'}^{(2)}[\Delta_{(n-2);(n-2)'}(Y_2)_{n-1,n;(n-1)'n'}^{(L)}]\\&-(2/n)P_{a'}^{(2)}\left[\sum_g (Y_2)_{1g;1'2'}^{(L)}(Y_{n-1})_{23\cdots n;g3'\cdots n'}\right] .\end{aligned} \tag{4.11}$$

The origin of the sum over g that appeared in the last term of (4.11) is the same as that in (4.7).

An inductive proof of (4.8) can be done in the following way. We first rewrite (4.11) by changing n into $(n+1)$. We then insert (4.8) to the newly formed (Y_n) factor (in the first and third terms), and show, by using (4.9), that the rhs of this new (4.11) is reduced to the rhs of (4.8), with n being replaced by $(n+1)$. Since somewhat lengthy algebra is involved in showing this, we shall present details in Appendix A.

Note that the summand of (4.8) is in the form of $\Delta\times Y^{(L)}$, and it will be seen soon that it is very convenient to have $(Y_n)_{aa'}$ in this form. In (4.11), the $(Y_n)_{aa'}$ is not in this convenient form, and this is why we show in Appendix A, that (4.11) is reduced to (4.8).

B. Integer powers of Y_n

In this subsection, we show how the matrix elements of Y_n^k, i.e., of integer powers of Y_n, can be expressed. We claim that we obtain these matrix elements as

$$(Y_n^k)_{aa'}=\sum_{i=0}^{n-2} P_{a'}^{(i)}[\Delta_{(i);(i)'}(Y_{n-i}^k)_{\{n-i\};\{n-i\}'}^{(L)}] , \tag{4.12}$$

combined with the recurrence relation that

$$(Y_m^k)_{(m);(m)'}^{(L)}=\sum_{s,t=0}^{m} {}_{m-t}C_s \sum_{\{m-s-t\}''} P_{(m)'}^{(t)}(Y_{m-s}^{k-1})_{\{m-s\};(t)'\{m-s-t\}''}^{(L)}(Y_{m-t})_{(s)\{m-s-t\}'';\{m-t\}'}^{(L)} . \tag{4.13}$$

As seen clearly in (4.12), we are claiming that the way (Y_n^k) is expressed is very much the same as was (Y_n) given in (4.8). In (4.13), we mean by $\{m-s-t\}''$ a set of indices $1'',2'',\ldots,(m-s-t)''$, and $\sum_{\{m-s-t\}''}$ means a sum over these dummy indices. The meaning of the subscripts including these dummy indices will be clear. For example, in the last factor (Y_{m-t}) of (4.13), the first set of indices $(s)\{m-s-t\}''$ means that there are s unprimed indices $1,2,\ldots s$, together with $m-s-t$ doubly primed indices, so that the total number of the indices $m-t$ matches the rank of the operator Y_{m-t}. Other notation used in (4.12) and (4.13) has appeared previously.

It is evident that (4.12) reduces to (4.8) for $k=1$. We also see that (4.13) becomes an identity for $k=1$. This is because the factor $(Y_{m-s}^{k-1})^{(L)}$ becomes $(1_{m-s})^{(L)}$, and in order to retain the linked cluster nature, we must have $m-s=0$, i.e., $m=s$. The factor $(m-s-t)!$ then requires that $t=0$. With $k=1$, $m=s$, and $t=0$, the rhs of (4.13) reduces to $(Y_m)_{(m);(m)'}^{(L)}$.

Relation (4.12) is proved easily once the following theorem (theorem I) is proved. Suppose there are two doubly-symmetric matrices A_n and B_n such that

$$(A_n)_{aa''}=\sum_{i=0}^{n} P_{a''}^{(i)}[\Delta_{(i);(i)''}(A_{n-i})_{\{n-i\};\{n-i\}''}^{(L)}] ,$$
$$(B_n)_{a''a'}=\sum_{j=0}^{n} P_{a'}^{(j)}[\Delta_{(j)'';(j)'}(B_{n-j})_{\{n-j\}'';\{n-j\}'}^{(L)}] . \tag{4.14}$$

Then we have

$$\sum_{(a'')}(A_n)_{aa''}(B_n)_{a''a'}=\sum_{i=0}^{n} P_{a'}^{(i)}[\Delta_{(i);(i)'}(A\times B)_{\{n-i\};\{n-i\}'}^{(L)}] , \tag{4.15}$$

where

$$(A\times B)_{(m);(m)'}^{(L)}=\sum_{s,t=0}^{n} {}_{m-t}C_s P_{(m)'}^{(t)}[(A_{m-s})_{\{m-s\};(t)'\{m-s-t\}''}^{(L)}(B_{m-t})_{(s)\{m-s-t\}'';\{m-t\}'}^{(L)}] . \tag{4.16}$$

The proof of this theorem is given in Appendix B.

Once theorem I is known, the proof of (4.12) is trivial. We first set $A_n=Y_n^{k-1}$ and $B_n=Y_n$ in theorem I. Then the lhs of (4.15) becomes

$$\sum_{(a'')}(Y_n^{k-1})_{aa''}(Y_n)_{a''a'}=(Y_n^k)_{aa'} ,$$

which is nothing but the lhs of (4.12). We further note that, with $A=Y^{k-1}$ and $B=Y$, the rhs of (4.16) agrees with that of (4.13). Therefore, with the understanding that we simply denote $(Y^{k-1}\times Y)^{(L)}$ as $(Y^k)^{(L)}$, it is seen that Eq. (4.12), together with (4.13), has now been proved.

C. Matrix elements of Z_n and Z_n^{-1}

Since $Z_n=[1-Y_n]^{1/2}$, it can be expanded in a power series of Y_n as

$$Z_n=\sum_{k=0}^{\infty}\beta_k Y_n^k;\quad \beta_k=-[(2k-3)!!]/[2^k k!] , \tag{4.17}$$

with the understanding that $(-3)!!=-1$ and $(-1)!!=1$, which is justified because we can write

$$(2n-3)!!=2^{n-1}\Gamma(n-\tfrac{1}{2})/\sqrt{\pi}\ .$$

We can expand $Z_n{}^{-1}$ similarly as

$$Z_n{}^{-1}=\sum_{k=0}^{\infty}\alpha_k Y_n^k;\quad \alpha_k=[(2k-1)!!]/[2^k k!]\ . \tag{4.18}$$

Note that the expansion coefficients α_k and β_k satisfy the following relations:

$$\sum_{l=0}^{k}\alpha_l\beta_{k-l}=\delta_{k0}\ \text{and}\ \alpha_k-\alpha_{k-1}=\beta_k(k\geq 1)\ . \tag{4.19}$$

From (4.17), we can write the matrix element of Z_n as

$$(Z_n)_{aa'}=\sum_{k=1}^{\infty}\beta_k(Y_n^k)_{aa'}+\Delta_{aa'}\ . \tag{4.20}$$

Note that the $k=0$ term, which is nothing but the second term of (4.20), is treated separately from the $k\geq 1$ terms, because we want to use (4.12), which is valid for $k\geq 1$, in the first term of (4.20). If this is done, we have

$$(Z_n)_{aa'}=\sum_{i=0}^{n-2}P_{a'}^{(i)}\left[\Delta_{(i);(i)'}\sum_{k=0}^{\infty}\beta_k(Y_{n-i}^k)^{(L)}_{\{n-i\};\{n-i\}'}\right]+\Delta_{aa'}\ . \tag{4.21}$$

Note that the lower limit of the summation over k in (4.21) is taken as $k=0$, rather than $k=1$, as it was in (4.20). It is because $(Y^k)^{(L)}=0$, if $k=0$, because of the requirement that it must be a linked cluster term.

We can then use (4.17) and (4.21), bringing $(Z_n)_{aa'}$ into the form that

$$(Z_n)_{aa'}=\sum_{i=0}^{n-2}P_{a'}^{(i)}[\Delta_{(i);(i)'}(Z_{n-i})^{(L)}_{\{n-i\};\{n-i\}'}]+\Delta_{aa'}\ . \tag{4.22}$$

To obtain (4.22) has been the aim of the present section. It is obvious that we obtain $(Z_n{}^{-1})_{aa'}$ by simply replacing Z_{n-i} in (4.22) by $Z_{n-i}{}^{-1}$.

V. CONCRETE FORMS OF THE $B^\dagger$ AND $C^\dagger$ OPERATORS

A. The $B^\dagger$ operator

As seen in Eq. (3.10), the major part of the work of obtaining the $(B_e^\dagger)_B$ operator explicitly is to obtain an explicit form for the matrix

$$(Z_n/Z_{n-1})_{1\cdots n;e2'\cdots n'}=\sum_{2''\cdots n''}(Z_n)_{1\cdots n;e2''\cdots n''}(Z_{n-1}{}^{-1})_{2''\cdots n'';2'\cdots n'}\ . \tag{5.1}$$

We shall begin this task by first writing the $(Z_n)_{1\cdots n;e2''\cdots n''}$ factor, as a special example of the matrix element given in (4.22), as

$$(Z_n)_{1\cdots n;e2''\cdots n''}\equiv\sum_{i=0}^{n-2}P^{(i)}_{e2''\cdots n''}\Delta_{1\cdots i;e2''\cdots i''}(Z_{n-i})^{(L)}_{(i+1)\cdots n;(i+1)''\cdots n''}+\Delta_{1\cdots n;e2''\cdots n''}\ . \tag{5.2}$$

In the first term of (5.2), we divide the terms that result from the operation of $P^{(i)}_{e2''\cdots n''}$ into two groups, those retaining the index e in the Δ factor, and those having e in the $(Z_{n-i})^{(L)}$ factor. [This is the same technique as used in Appendix A to obtain (A6) from (A5).] It is easy to see that the first group of these terms can be combined with the second term of (5.2), so as to result in a simple term written as $\Delta_{1e}(Z_{n-1})_{2\cdots n;2''\cdots n''}$. After rewriting the second group of the above terms slightly, we thus find that (5.2) is replaced by

$$(Z_n)_{1\cdots n;e2''\cdots n''}=\Delta_{1e}(Z_{n-1})_{2\cdots n;2''\cdots n''}+\sum_{i=0}^{n-2}P^{(i)}_{2''\cdots n''}\Delta_{1\cdots i;2''\cdots(i+1)''}(Z_{n-i})^{(L)}_{(i+1)\cdots n;e(i+2)''\cdots n''}\ . \tag{5.3}$$

The $(Z_{n-1}{}^{-1})$ factor in the summand of (5.1) can be written like the (Z_n) factor in (5.2) as

$$(Z_{n-1}{}^{-1})_{2''\cdots n'';2'\cdots n'}=\sum_{j=0}^{n-3}P^{(j)}_{2'\cdots n'}\Delta_{2''\cdots(j+1)'';2'\cdots(j+1)'}\times(Z_{n-1-j}{}^{-1})^{(L)}_{(j+2)''\cdots n'';(j+2)'\cdots n'}+\Delta_{2''\cdots n'';2'\cdots n'}\ . \tag{5.4}$$

We now insert (5.3) and (5.4) into (5.1). In evaluating the contribution of the first term of (5.3), however, we do not need to use the lengthy form on the rhs of (5.4), because we see easily that

$$\sum_{2''\cdots n''}\Delta_{1e}(Z_{n-1})_{2\cdots n;2''\cdots n''}(Z_{n-1}{}^{-1})_{2''\cdots n'';2'n'}=\Delta_{1\ldots n;e2'\cdots n'}\ . \tag{5.5}$$

We also see that the combination of the second term of (5.3) with that of (5.4) leaves the former unchanged, except that the doubly-primed indices there are replaced by singly-primed indices. Equation (5.1) is thus rewritten as

$$(Z_n/Z_{n-1})_{1\cdots n;e2'\cdots n'}=\Delta_{1\cdots n;e2'\cdots n'}+\sum_{i=0}^{n-2}P^{(i)}_{2'\cdots n'}\Delta_{1\cdots i;2'\cdots(i+1)'}(Z_{n-i})^{(L)}_{(i+1)\cdots n;e(i+2)'\cdots n'}+\sum_{2''\cdots n''}\sum_{i=0}^{n-2}\sum_{j=0}^{n-3}P^{(j)}_{2'\cdots n'}[P^{(i)}_{2''\cdots n''}\Delta_{1\cdots i;2''\cdots(i+1)''}(Z_{n-i})^{(L)}_{(i+1)\cdots n;e(i+2)''\cdots n''}]\times[\Delta_{2''\cdots(j+1)'';2'\cdots(j+1)'}(Z_{n-1-j}{}^{-1})^{(L)}_{(j+2)''\cdots n'';(j+2)'\cdots n'}]\ . \tag{5.6}$$

The third term of (5.6) is somewhat lengthy. However, one easily recognizes that it is essentially in the same form as the rhs of (B1) in Appendix B, which means that we can use theorem I in evaluating this term. In fact, we may set $A_n=Z_n$ and $B_n=Z_{n-1}{}^{-1}$ in theorem I, and obtain the third term of (5.6) in the form (4.15). The third term of (5.6) rewritten this way is found to be nicely combined with the second term of (5.6), to result in a single term. We thus obtain

$$(Z_n/Z_{n-1})_{1\cdots n;e2'\cdots n'}=\Delta_{1\cdots n;e2'\cdots n'}+\sum_{i=0}^{n-2}P^{(i)}_{2'\cdots n'}\Delta_{1\cdots i;2'..(i+1)'}(Z_{n-i}/Z_{n-1-i})^{(L)}_{(i+1)\cdots n;e(i+2)'\cdots n'} \tag{5.7}$$

with

$$(Z_{n-i}/Z_{n-1-i})^{(L)}=(Z_{n-i})^{(L)}_{(i+1)\cdots n;e(i+2)'\cdots n'}\times\sum_{s,t}{}_{n-i-t-1}C_s\sum_{\{n-i-t-1-s\}''}P^{(t)}_{(i+2)'\cdots n'}\times[(Z_{n-i-s})^{(L)}_{(i+s+1)\cdots n;e(i+2)'\cdots(i+t+1)'\{n-i-1-s-t\}''}\times(Z_{n-i-1-t}{}^{-1})^{(L)}_{(i+1)\cdots(i+s)\{n-i-1-s-t\}'';(i+t+2)'\cdots n'}]\ . \tag{5.8}$$

By using the notation of (5.1), the $(B_e^\dagger)_B$ operator of (3.10) is rewritten as

$$(B_e^\dagger)_B=\sum_{n=1}^{\infty}\sum_{a,a'}\frac{1}{(n-1)!}(Z_n/Z_{n-1})_{1\cdots n;e2'\cdots n'}(A_a^\dagger)^n\,|\,0)(0\,|\,(A_{a'})^{n-1}\ . \tag{5.9a}$$

With the notation of Sec. IV, we can rewrite (3.12) for the factor $|\,0)(0\,|$ as

$$|\,0)(0\,|=\sum_{k=0}^{\infty}[(-1)^k/k!]\sum_{(b,b')}\Delta_{b,b'}(A_b^\dagger)^k(A_{b'})^k\ . \tag{5.9b}$$

Insert this into (5.9a), and introduce a new summation variable $l=n+k$. We then get

$$(B_e^\dagger)_B=\sum_{l=1}^{\infty}\sum_{a,a'}\left[\sum_{n=1}^{l}\frac{(-)^{l+n}}{(n-1)!(l-n)!}(Z_n/Z_{n-1})_{1\cdots n;e2'\cdots n'}\Delta_{(n+1)\cdots l;(n+1)'\cdots l'}\right](A_a^\dagger)^l(A_{a'})^{l-1} . \tag{5.10}$$

Note that (5.10) does not contain the symmetrizer that could have appeared to symmetrize the indices in the Δ and the (Z_n/Z_{n-1}) factors. We suppressed it, because the factors $(A_a^\dagger)^l$ and $(A_{a'})^{l-1}$ take care of the symmetrization.

Let us denote by $(B_e^\dagger)_B^{(\mathrm{I})}$ the contribution of the first term of (5.7) to $(B_e^\dagger)_B$. It is given as

$$(B_e^\dagger)_B^{(\mathrm{I})}=\sum_{l=1}^{\infty}\sum_{n=1}^{l}[(-)^{l+n}/\{(n-1)!(l-n)!\}]\Delta_{1\cdots l;e2'\cdots l'}(A_a^\dagger)^l(A_{a'})^{l-1} . \tag{5.11}$$

However,

$$\sum_{n=1}^{l}(-)^{l+n}/\{(n-1)!(l-n)!\}=(-)^{l+1}\sum_{n=0}^{l-1}(-)^n/\{n!(l-1-n)!\}$$
$$=(-)^{l+1}(1+(-1))^{l-1}/(l-1)!=\delta_{l,1} . \tag{5.12}$$

Therefore

$$(B_e^\dagger)_B^{(\mathrm{I})}=\sum_{l=1}^{\infty}\delta_{l,1}\sum_{a,a'}\Delta_{1\cdots l;e2'\cdots l'}(A_a^\dagger)^l(A_{a'})^{l-1}=\sum_{a}\Delta_{1;e}A_1^\dagger=A_e^\dagger . \tag{5.13}$$

The second term of $(B_e^\dagger)_B$, which we write as $(B_e^\dagger)_B^{(\mathrm{II})}$, originates from the second term of (5.7), and is obtained as

$$(B_e^\dagger)_B^{(\mathrm{II})}=\sum_{l=1}^{\infty}\sum_{a,a'}\left[\sum_{n=1}^{l}(-)^{l+n}/\{(n-1)!(l-n)!\}\right.$$
$$\times\sum_{i=0}^{n-2}P_{2'\cdots n'}^{(i)}\Delta_{1\cdots i;2'\cdots(i+1)'}$$
$$\left.\times(Z_{n-i}/Z_{n-i-1})_{(i+1)\cdots n;e(i+2)'\cdots n'}^{(L)}\Delta_{(n+1)\cdots l;(n+1)'\cdots l'}\right]$$
$$\times(A_a^\dagger)^l(A_{a'})^{l-1} . \tag{5.14}$$

In (5.14) the symmetrizer $P_{2'\cdots n'}^{(i)}$ can be replaced by ${}_{n-1}C_i$ because of the presence of the completely symmetric factor $(A_{a'})^{l-1}$. We then introduce a new summation index $k=n-i$, to replace i, and then interchange the order of the summations over n and k. Equation (5.14) is then replaced by

$$(B_e^\dagger)_B^{(\mathrm{II})}=\sum_{l=1}^{\infty}\sum_{a,a'}\sum_{k=2}^{l}\left[\sum_{n=k}^{l}(-)^n/\{(l-n)!(n-k)!\}\right]\{(-)^l/(k-1)!\}$$
$$\times\Delta_{1\cdots(l-k);2'\cdots(l-k+1)'}(Z_k/Z_{k-1})_{(l-k+1)\cdots l;e(l-k+2)\cdots l'}^{(L)}(A_a^\dagger)^l(A_{a'})^{l-1} . \tag{5.15}$$

Very simple algebra, similar to what was given in (5.12), shows that the quantity in the square brackets in (5.15) equals $(-)^l\delta_{k,l}$, i.e., that

$$\sum_{n=k}^{l}(-)^n/\{(l-n)!(n-k)!\}=(-)^l\delta_{k,l} . \tag{5.16}$$

Then (5.15) is replaced by

$$(B_e^\dagger)_B^{(\mathrm{II})}=\sum_{l=2}^{\infty}\sum_{a,a'}[(l-1)!]^{-1}(Z_l/Z_{l-1})_{1\cdots l;e2'\cdots l'}^{(L)}(A_a^\dagger)^l(A_{a'})^{l-1} . \tag{5.17}$$

Combining (5.13) and (5.17), we finally obtain $(B_e^\dagger)_B$ in the following extremely simple form:

$$(B_e^\dagger)_B = A_e^\dagger + \sum_{l=2}^{\infty}\sum_{a,a'} X_{2l-1}(1\cdots l;e2'\cdots l')(A_a^\dagger)^l(A_{a'})^{l-1} , \tag{5.18}$$

with

$$X_{2l-1}(1\cdots l;e2'\cdots l') = [(l-1)!]^{-1}(Z_l/Z_{l-1})^{(L)}_{1\cdots l;e2'\cdots l'};\quad (l\geq 2) . \tag{5.19}$$

It is very gratifying to find that the boson expansion of $(B_e^\dagger)_B$ was obtained in the above very simple form. A more important feature which the $(B_e^\dagger)_B$ has in (5.18), however, is that it is a *linked-cluster* expansion, in that the X_{2l-1} coefficients, as seen in (5.19), are completely free from the Δ factors.

Recall that we started from (5.9), in arriving at (5.18), and that the former contained two factors, (Z_n/Z_{n-1}) and $|0)(0|$, which, as seen in (5.7) and (5.9b), were both full of *unlinked-cluster* terms. Their contributions to each X_{2l-1} have, however, been very neatly canceled out, as shown by the two algebraic relations given by (5.12) and (5.16).

We discuss later in more detail the significance of achieving the linked-cluster expansions. In the next subsection, we shall derive $(C_p^\dagger)_B$ also in the form of a linked-cluster expansion.

B. The $C^\dagger$ operator

We first rewrite (3.11) for $(C_p^\dagger)_B$ as follows:

$$(C_p^\dagger)_B = \sum_{n=1}\sum_{(aa')}\left[\sum_{(bb')}(Z_n{}^{-1})_{ab}(Y_{p,n})_{bb'}(Z_n{}^{-1})_{b'a'}\right]|n;a)(n;a'| , \tag{5.20}$$

where

$$(Y_{p,n})_{bb'} = \langle\,|B_n\cdots B_1 C_p^\dagger B_{1'}^\dagger\cdots B_{n'}^\dagger|0\rangle/n! . \tag{5.21}$$

In (5.21), we move the $C_p^\dagger$ operator to the right until it hits the vacuum $|0\rangle$, and each time it is commuted with a $B_{i'}^\dagger$ factor, the commutation relation (2.8b) is used. The result of this operation is written as

$$(Y_{p,n})_{bb'} = n\sum_g P^{(p)}_{b_n',g}(1-Y_n)_{b;g\bar{b}'} = n\sum_g P^{(p)}_{b_n',g}(Z_n{}^2)_{b;g\bar{b}'} . \tag{5.22}$$

Note that the (numerical) coefficient $P^{(p)}_{b_n',g}$ is what was defined in (2.8), and thus should not be confused with the symmetrizer $P_{a'}^{(i)}$ that was introduced in (4.3) and was used repeatedly in the preceding subsection, as well as in Sec. IV. We also note in (5.22), and in a few formulas that appear below, that, e.g., b stands for a set of n indices: $b_1,b_2,\ldots,b_n$, while $\bar{b}$ stands for $(n-1)$ indices: $b_1,\ldots,b_{n-1}$. In (5.22) we could have written the symmetrizer $P_{b'}^{(1)}$ in place of the numerical factor n. We shall restore such a symmetrizer whenever it becomes vital to do so.

Note that the summation index g that appeared in (5.22) runs over the TD elements belonging both to T and $(1-T)$. We find it convenient to divide $(C_p^\dagger)_B$ of (5.20), after (5.22) is inserted into it, into two parts: $(C_p^\dagger)_B^{(\mathrm{I})}$ and $(C_p^\dagger)_B^{(\mathrm{II})}$, the former containing only those g that are in T, and the latter those in $(1-T)$. It is then easy to see that $(C_p^\dagger)_B^{(\mathrm{I})}$ is obtained as

$$(C_p^\dagger)_B^{(\mathrm{I})} = \sum_{n=1} n\sum_{g\in T}\sum_{(aa'bb')} P^{(p)}_{b_n';g}(Z_n{}^{-1})_{ab}(Z_n{}^2)_{b;g\bar{b}'}(Z_n{}^{-1})_{b'a'}|n;a)(n;a'| , \tag{5.23}$$

which can be rewritten further by going through the following algebraic steps:

$$\begin{aligned}(C_p^\dagger)_B^{(\mathrm{I})} &= \sum_{n=1} n\sum_{g\in T}\sum_{(aa'b')} P^{(p)}_{b_n';g}(Z_n)_{a;g\bar{b}'}(Z_n{}^{-1})_{b'a'}|n;a)(n;a'|\\ &= \sum_{n=1} n\sum_{g\in T}\sum_{(aa'b')} P^{(p)}_{g,a_n}(Z_n)_{g\bar{a};b'}(Z_n{}^{-1})_{b'a'}|n;a)(n;a'|\\ &= \sum_{n=1} n\sum_{g\in T}\sum_{(aa')} P^{(p)}_{g,a_n}(\Delta)_{g\bar{a};a'}|n;a)(n;a'| + \delta(C_p^\dagger)_B^{(\mathrm{I})}\\ &= \sum_{n=1}[1/(n-1)!]\sum_{(aa')} P^{(p)}_{1';1}\Delta_{2\cdots n;2'\cdots n'}(A_a^\dagger)^n|0)(0|(A_a')^n + \delta(C^\dagger)_B^{(\mathrm{I})}\\ &= \sum_{a_1a_1'} P^{(p)}_{a_1';a_1}A_{a_1}^\dagger A_{a_1'} + \delta(C_p^\dagger)_B^{(\mathrm{I})} .\end{aligned} \tag{5.24}$$

It will be clear that, in obtaining the first and the third equalities in (5.24), use is made of relations

$$\sum_b (Z_n{}^{-1})_{ab}(Z_n{}^2)_{b;g\bar{b}'} = (Z_n)_{a;g\bar{b}'}$$

and

$$\sum_{b'} (Z_n)_{g\bar{a};b'}(Z_n{}^{-1})_{b'a'} = (\Delta)_{g\bar{a};a'} ,$$

which are valid when $g \in T$. In obtaining the second equality, we made the following replacement:

$$\sum_{g \in T} P^{(p)}_{b_n';g}(Z_n)_{a;g\bar{b}'} \to \sum_{g \in T} P^{(p)}_{g;a_n}(Z_n)_{g\bar{a};b'} , \tag{5.25}$$

which resulted in a correction term $\delta(C_p^\dagger)_B^{(\mathrm{I})}$; this term will be discussed shortly. The fourth equality in (5.24) is trivial. On the other hand, the last equality is very significant. The algebra involved there is the same as what was encountered in going from (5.11) to (5.13). Just as was the case previously, the fifth equality of (5.24) shows that all the unlinked cluster terms in $[(C_p^\dagger)_B^{(\mathrm{I})} - \delta(C_p^\dagger)_B^{(\mathrm{I})}]$ are neatly canceled out, retaining only one surviving linked cluster term, the first term in the last expression of (5.24).

We shall now explain $\delta(C_p^\dagger)_B^{(\mathrm{I})}$. Obviously, it can be written as

$$\delta(C_p^\dagger)_B^{(\mathrm{I})} = \sum_{n=1} n \sum_{g \in T} \sum_{(aa'b')} [P^{(p)}_{b_n',g}(Z_n)_{a;g\bar{b}'} - P^{(n)}_{g,a_n}(Z_n)_{g\bar{a};b'}](Z_n{}^{-1})_{b'a'} \, |n;a)(n;a'| \ . \tag{5.26}$$

For our present purpose, it is not necessary to attempt to rewrite this expression further. We just want to make a few remarks. The first is that, although we wrote the lower limit of the summation over n as $n=1$, the summation actually begins with $n=2$, because the [ ] factor in (5.26) vanishes for $n=1$; note that $Z_1 = 1$. The second is that, although (5.26) is written as a sum over g in T, the rhs of (5.26) can be rewritten so that it contains a (very complicated) sum over ξ, where we denote by ξ the TD components that are outside T: $\xi \in (1-T)$. From here on, we shall express this fact by saying that, e.g., the rhs of (5.26) consists only of the ξ sum(s). Suppose we choose T very large, still permitting the presence of $(Z_n{}^{-1})$. Then the $(1-T)$ space will be all but empty, making the ξ sum very small. Thus a ξ sum term may in general be regarded as a small term, although this may not always be the case if T is taken very small.

The proof of what we stated as the second remark above is somewhat lengthy. We shall thus explain only the outline of how it can be done. In doing this, we first note that we can prove an equality given as

$$\sum_g P^{(p)}_{b_n',g}(Y_n)_{b;g\bar{b}'} = \sum_g P^{(p)}_{g,b_n}(Y_n)_{g\bar{b};b'} \ . \tag{5.27}$$

Previously, we obtained $(Y_{p,n})_{bb'}$ in the form of (5.22), by moving $C_p^\dagger$ in (5.21) to the right. We can also obtain another expression of $(Y_{p,n})_{bb'}$, however, this time moving $C_p^\dagger$ to the left. By equating the previously obtained expression to that in (5.22), one sees that (5.27) holds. What we need to do next is expand in powers of Y_n the (Z_n) factors in the [ ] expression of (5.26). By using (5.27) repeatedly, one finds that only the ξ sum remains for each power of Y_n. Thus our statement is proved.

The $(C_p^\dagger)_B^{(\mathrm{II})}$ is given by (5.23), if we replace the $g \in T$ sum there by a ξ sum. We also note that, once this replacement is made, the $n=1$ term vanishes again, because the factor

$$(Z_1{}^2)_{b_1,\xi} = (\Delta)_{b_1;\xi} = 0 ,$$

since $b_1 \in T$ and $\xi \notin T$. Therefore, the terms $\delta(C_p^\dagger)_B^{(\mathrm{I})}$ and $(C_p^\dagger)_B^{(\mathrm{II})}$ behave very similarly. We may thus combine them into a single term $\delta(C_p^\dagger)_B$, obtaining $(C_p^\dagger)_B$ as

$$(C_p^\dagger)_B = \sum_{a_1 a_1'} P_{a_1';a_1} A^\dagger_{a_1} A_{a_1'} + \delta(C_p^\dagger)_B \ . \tag{5.28}$$

Just as we expressed $(B_e^\dagger)_B$ in (5.18), we may write $(C_p^\dagger)_B$ as

$$(C_p^\dagger)_B = \sum_{l=1} \sum_{(aa')} X^{(p)}_{2l}(1 \cdots l; 1' \cdots l') \times (A_a^\dagger)^l (A_{a'})^l \ . \tag{5.29}$$

Since the $\delta(C_p^\dagger)_B$ term in (5.29) contributes only the $l \geq 2$ terms in (5.30), one sees that

$$X_2^{(p)}(1;1') = P^{(p)}_{1'1} \ . \tag{5.30a}$$

This agrees with what was obtained in KT-1. If we also decide to suppress all the ξ-sum terms, we may further write

$$X^{(p)}_{2l} = 0; \quad (l \geq 0) , \tag{5.30b}$$

again in agreement with what we obtained in KT-1, although they were obtained there in quite a different context. (In order to avoid any misunderstanding of what we have just said above, the reader is strongly urged to read Ref. 16.)

As will be discussed in some detail soon, there is no *a priori* reason to justify suppressing the ξ sums always. It then makes it desirable to give explicit forms to the X_{2l} terms with $l \geq 2$, in particular in the form of linked cluster expansion. Instead of doing this, we shall be content to give only the leading term, the X_4 term. It reads

$$X_4^{(p)}(12;1'2') = \sum_{1''2''} \left[\sum_{g \in T} \{ P^{(p)}_{2'',g} (Z_2)_{12;g1''} - P^{(p)}_{g,2} (Z_2)_{g1;1''2''} \} - \sum_{\xi} \sum_{\bar{1},\bar{2}} P^{(p)}_{2'',\xi} (Z_2{}^{-1})_{12;\bar{1}\bar{2}} (Y_2)_{\bar{1}\bar{2},\xi 1''} \right] (Z_2{}^{-1})_{1''2'';1'2'} \,. \tag{5.31}$$

Combining the presentations of the present and the preceding subsections, we see that the bosonization of the fermion pair operators, and thus of any fermion operator, has now been completed. As we stressed in Sec. I, they are all in linked-cluster forms, and are permitted to be used directly in the space of ideal bosons. They were derived through rigorous mathematical procedures, and yet, as they stand, are in a form that can be used for practical calculations.

The reader may wonder how the ξ sums find their way into our formalism, and why we are so conscious of them, in spite of the fact that we have decided to truncate the (TD) space in which to work. The origin of the ξ sums is of course the commutation relation (2.7b), or more simply the presence of the scattering operator $C_p^\dagger$ in the original fermion theory. Even if we choose the T space to start with, the $C_p^\dagger$ scatters the pairs in T into the $(1-T)$ space, and there is no way to prevent this from taking place.

We may thus say that whether to retain the ξ sums or not is a matter of choice. As we stressed in TWP,[12] once one decides to use BET, it means that one has decided to introduce an approximation, one way or another. In this regard we note here that the problem of the ξ sum became apparent only when we started to discuss the bosonization of the $C_p^\dagger$ operator in the present subsection; it was not encountered in the preceding subsection in which we discussed the bosonization of the $B_e^\dagger$ operator. Actually, the problem was simply hidden there. See, e.g., Eq. (4.7), in which we first encountered the g sum, which includes both the $g \in T$ and the ξ sums. To retain the latter may improve the accuracy of evaluating the matrix elements that are eventually used to construct the $(B_e^\dagger)_B$ operators. In this regard, we also remark here that the same g sum already appeared in (Sec. 5 of) KT-1. We have retained the ξ sums which appeared there in our later calculations.[3–7] In these calculations, however, we set $X_{2n}^{(p)} = 0$ for $n \geq 2$, which means that we had suppressed all the ξ-sum terms when they appeared in these coefficients.

We want to stress here that it should not be misunderstood that we are encountering the ξ-sum term problem because we have bosonized the calculation. It is a problem that is encountered in a purely fermion calculation. Note, first of all, that the need for constructing $(Z_n{}^{-1})$, and thus the need for truncation, is encountered in the fermion system. One may also consider evaluating a fermion matrix element $\langle n;a \mid C_p^\dagger \mid n;a' \rangle$, as an example. By using the definition (3.3) for $\mid n;a \rangle$, one immediately sees that the above matrix element is nothing but what was given in the square brackets in (5.20). It is then clear that the same ξ-term problem appears in the fermion calculations. When one does a fermion calculation, one would set up a strategy on how to treat ξ-sum terms. One may then know exactly what to do when one switches to the boson description.

VI. PROPERTIES OF THE $B^\dagger$ AND $C^\dagger$ OPERATORS

In Sec. V, we obtained the $(B^\dagger)_B$ and $(C^\dagger)_B$ operators in rather compact forms. [The $(B)_B$ operator is obtained simply as the Hermitian conjugate of $(B^\dagger)_B$.] In the present section, we discuss a few interesting properties of these operators. In subsection A, we prove that they satisfy the bosonized versions of the commutation relations in (2.7) within the ξ sums. Had we not introduced a truncation, it would have been unnecessary to go through such a proof; satisfaction of the commutation relations is guaranteed by the construction of the bosonized operators. It is thus interesting to see to what extent the commutation relations are violated because of the truncation. In subsection B, we shall see how the step-by-step operation is done by using the $(B^\dagger)_B$'s obtained above.

A. Commutation relations

We may make Eqs. (5.18) and (5.29) more explicit to obtain

$$(B_a)_B = A_a + \sum_{122'} X_3(12;a2')A_{2'}^\dagger A_1 A_2 + \sum_{1232'3'} X_5(123;a2'3')A_{2'}^\dagger A_{3'}^\dagger A_1 A_2 A_3 + \cdots \ ,$$

$$(B_b^\dagger)_B = A_b^\dagger + \sum_{122'} X_3(12;b2')A_1^\dagger A_2^\dagger A_{2'} + \sum_{1232'3'} X_5(123;b2'3')A_1^\dagger A_2^\dagger A_3^\dagger A_{2'} A_{3'} + \cdots \ , \tag{6.1}$$

$$(C_p^\dagger)_B = \sum_{22'} X_2^{(p)}(2,2')A_2^\dagger A_{2'} + \sum_{232'3'} X_4^{(p)}(23;2'3')A_2^\dagger A_3^\dagger A_{2'} A_{3'} + \cdots \ ,$$

and insert them into

$$[(B_a)_B,(B_b^\dagger)_B] = \delta_{ab} - \sum_p P_{a,b}^{(p)}(C_p^\dagger)_B \ , \tag{6.2a}$$

$$[(C_p^\dagger)_B,(B_a^\dagger)_B] = \sum_g P_{a,g}^{(p)}(B_g^\dagger)_B \ . \tag{6.2b}$$

Both sides of (6.2) are expressed as sums of normal products of the $A^\dagger$ and A operators, and we equate the coefficients of the same products on both sides. This results in a set of equations that are to be satisfied by the $X_2^{(p)}, X_3, \ldots,$ coefficients in (6.1). These equations are exactly the same (save for a slight change of notation, as is easily figured out) as are those obtained in KT-1. In KT-1, however, these coefficients were totally unknown, and were obtained by solving these equations. (This point will be discussed further in Sec. VII.) Here we show that these coefficients obtained in Sec. V do satisfy these equations.

It is trivial to show that the X_2 equation, i.e., the equation to be satisfied by $X_2^{(p)}$, makes $X_2^{(p)}(1,1') = P_{1';1}^{(p)}$, in agreement with (5.30a). We then obtain the X_3 equation as

$$2X_3(a2;b2') + \sum_{1''2''} X_3(a2;1''2'')X_3(1''2'';b2') + (Y_2)_{a2;b2'} = 0 \ . \tag{6.3}$$

Note that we used (2.9) and (4.5) to obtain the last term on the lhs of (6.3). It is easy to see, by using (5.19) and then (4.1), that the second term of (6.3) is rewritten as

$$\begin{aligned}\sum_{1''2''} (Z_2)_{a2;1''2''}^{(L)}(Z_2)_{1''2'';b2'}^{(L)} &= \sum_{1''2''} [(Z_2)_{a2;1''2''} - \Delta_{a2;1''2''}]\cdot[(Z_2)_{1''2'';b2'} - \Delta_{1''2'';b2'}] \\ &= (Z_2{}^2)_{a2;b2'} - 2(Z_2)_{a2;b2'} + \Delta_{a2;b2'} \\ &= -(Y_2)_{a2;b2'} - 2X_3(a2;b2') \ .\end{aligned} \tag{6.4}$$

With (6.4), it is easy to see that the lhs of (6.3) vanishes. Thus the consistency of (5.19) with (6.3) has been proven.

The X_4 equation is given as

$$\begin{aligned}&2\Big[X_4^{(p)}(12;a2') + \sum_{1''2''} X_4^{(p)}(12;1''2'')X_3(1''2'';a2')\Big] \\ &\quad + 2\sum_{1''} P_{1'',1}^{(p)}X_3(1''2;a2') - \sum_{1''} P_{2;1''}^{(p)}X_3(12;a1'') - \sum_g P_{a,g}^{(p)}X_3(12;g2') = 0 \ .\end{aligned} \tag{6.5}$$

Note that the summation indices $1''$ and $2''$ are restricted to lie within the truncated space $[T]$, while g is not in general; see the discussion of Sec. V B.

We first note that the two terms in the square brackets in (6.5) are combined into a single term, causing the first term of (6.5) to be replaced by

$$2\sum_{1''2''} X_4^{(p)}(12;1''2'')(Z_2)_{1''2'';a2'} \ . \tag{6.6}$$

For the purpose we have here, we find it convenient to express the $X_4^{(p)}$ factor in (6.6), not as in (5.31), but as

$$X_4^{(p)}(12;1''2'') = \sum_{3''4''5''6''g} (Z_2{}^{-1})_{12;3''4''}P_{6'';g}^{(p)}(1-Y_2)_{3''4'';g5''}(Z_2{}^{-1})_{5''6'';1''2''} - \sum_{22''} P_{1'';1}^{(p)} \ . \tag{6.7}$$

Equation (6.7) is obtained by noting that $X_4^{(p)}$ is nothing but the linked-cluster part of the coefficient of the $n=2$ term of (5.20), and that the factor $(Y_{p,n})_{bb'}$ there can be expressed as in (5.22).

Insert (6.7) into (6.6), and combine the factors $(Z_2{}^{-1})_{5''6'';1''2''}$ and $(Z_2)_{1''2'';a2'}$ to obtain

$$\Delta_{5''6'';a2'} = (\delta_{5''a}\delta_{6''2'} + \delta_{5''2'}\delta_{6''a})/2 \ .$$

One then sees rather easily that (6.6), i.e., the first term of (6.5), is rewritten as

$$\sum_{g\in T} P^{(p)}_{2';g}(Z_2)_{12;ag}+\sum_{g\in T} P^{(p)}_{a;g}(Z_2)_{12;g2'}-\sum_{1''} P^{(p)}_{1'',1}(Z_2)_{1''2;a2'}$$
$$-\sum_{\xi}\sum_{1''2''}(Z_2{}^{-1})_{12;1''2''}\{P^{(p)}_{2',\xi}(Y_2)_{1''2'',\xi a}+P^{(p)}_{a,\xi}(Y_2)_{1''2'',\xi 2'}\}\ . \tag{6.8}$$

The other three terms in (6.5) are easy to manipulate, and one finds that in combination they cancel the first three terms of (6.8), and add a ξ-sum term. It is thus seen that we eventually obtain the lhs of Eq. (6.5) to be equal to

$$-\sum_{\xi}\left[\sum_{1''2''}(Z_2{}^{-1})_{12;1''2''}\{P^{(p)}_{2',\xi}(Y_2)_{1''2'',\xi a}+P^{(p)}_{a,\xi}(Y_2)_{1''2'';\xi 2'}\}+P^{(p)}_{a,\xi}X_3(12;\xi 2')\right]\ . \tag{6.9}$$

This shows that the coefficient equation (6.5) is satisfied to within the ξ sums.

The equation to be satisfied by X_5 is somewhat lengthy, and we find it convenient to write it as

$$A_5+A_4+A_3=0\ . \tag{6.10}$$

Here A_5 stands for a sum of all the terms that contain the X_5 coefficient, while A_4 is linear in $X_4^{(p)}$ and A_3 is quadratic in X_3. We first write A_5 in full:

$$A_5=6\sum_{1''2''3''}X_5(1''2''3'';a23)X_5(1''2''3'';b2'3')+6X_5(a23;b2'3')$$
$$+6\sum_{1''2''}[X_3(1''2'';a2)X_5(1''2''3;b2'3')+X_5(1''2''3';a23)X_3(1''2'';b2')]\ . \tag{6.11}$$

The A_4 and A_3 terms are much shorter, and are given as

$$A_4=\sum_p P^{(p)}_{a;b}X_4^{(p)}(23;2'3')\ , \tag{6.12a}$$

$$A_3=4\sum_{1''}X_3(1''3',a2)X_3(1''3;b2')-\sum_{1''}X_3(2'3',a\,1'')X_3(23;b\,1'')\ . \tag{6.12b}$$

In (6.11), we note that X_5 can be written as

$$X_5(1''2''3'',a23)=\tfrac{1}{2}(Z_3/Z_2)^{(L)}_{1''2''3'';a23}$$
$$=\tfrac{1}{2}[(Z_3/Z_2)_{1''2''3'',a23}-2\delta_{3'3''}(Z_2)^{(L)}_{1''2'';a2}-\Delta_{1''2''3'';a23}]\ . \tag{6.13}$$

See Eq. (5.19) and then Eq. (5.7). If (6.13) is inserted, it is seen, after short algebra, that A_5 is rewritten in the following form:

$$A_5=(\tfrac{3}{2})\sum_{1''2''3''}(Z_3/Z_2)_{1''2''3'';a23}(Z_3/Z_2)_{1''2''3'';b2'3'}$$
$$-(\tfrac{3}{2})\sum_{1''2''3''}[2\delta_{33''}(Z_2)^{(L)}_{1''2'';a2}+\Delta_{1''2''3'';a23}]$$
$$\times[2\delta_{3'3''}(Z_2)^{(L)}_{1''2'';b2'}+\Delta_{1''2''3'';b2'3'}]\ . \tag{6.14}$$

It is significant that the (Z_3/Z_2) factors are not in the linked-cluster forms, because we can write, e.g.,

$$(Z_3/Z_2)_{1''2''3'';a23}=\sum_{4''5''}(Z_3)_{1''2''3'';a4''5''}(Z_2{}^{-1})_{4''5'',23}\ . \tag{6.15}$$

It is easy to show than that the first term of (6.14) is rewritten as

$$(\tfrac{3}{2})\sum_{1''2''3''4''}(Z_2{}^{-1})_{23;1''2''}(Z_3{}^2)_{a\,1''2'';b\,3''4''}(Z_2{}^{-1})_{3''4'';2'3'} \tag{6.16}$$

Since $Z_3{}^2=1-Y_3$, as shown in (4.1), and since Y_3 is expressed in terms of Y_2's, as seen from (4.6) and (4.7), we see that the first term of (6.14) is expressed in terms only of Y_2's and Z_2's. The same is also true for the second term of (6.14). The A_3 term is, of course, expressed in terms of Z_2 only. If we use (6.7), the A_4 term is also seen to be expressed in terms of Y_2's and Z_2's. This means that the algebra involved in the evaluation of $A_5+A_4+A_3$ is reduced to that of adding and subtracting terms that are expressed entirely in terms of Y_2's and Z_2's (including $Z_2{}^{-1}$'s). Thus the rest of the algebra is rather straightforward, though lengthy. After this lengthy algebra one finds that

$$A_5+A_4+A_3=\sum_{\xi}\left[\sum_{efe'f'}(Z_2{}^{-1})_{23;ef}(Y_2)_{ef;\xi b}(Y_2)_{a\xi;e'f'}(Z_2{}^{-1})_{e'f';2'3'}\right], \tag{6.17}$$

i.e., that $A_5+A_4+A_3=0$, to within a ξ sum.

It is rather difficult to continue similar proof indefinitely for still higher order coefficients $X_6, X_7, \ldots$, but it will be a reasonable guess that the equations for these coefficients are also satisfied to within ξ-sum terms. In this regard, it will be worthwhile to emphasize that the problem of the ξ-sum terms is never encountered in the X_1, X_2, and X_3 coefficients. In the terminology of KT-1, a BET which terminates the boson expansion at X_3 is called a fourth order theory, Therefore, what is meant by the remark we just made is that the fourth order BET can be carried out exactly, within a fixed truncation. Note that we have experienced in the past that the fourth order theory was already a rather good approximation.[3-7]

B. Step-by-step operation

We have discussed in TWP,[12] and in its recapitulation in Sec. II C, the significance of the step-by-step operation (SSO). We shall discuss it here again, this time by using the explicit form of the $(B_e^\dagger)_B$ operator obtained in Sec. V B. By using the $(B_e^\dagger)_B$'s, as given by (5.18), it will be easy to see that the results of the first and the second operations take the following forms:

$$(B_e^\dagger)_B\,|\,0)=A_e^\dagger\,|\,0)=(Z_1)_{ee}A_e^\dagger\,|\,0)\;, \tag{6.18a}$$

$$\begin{aligned}(B_f^\dagger)_B(B_e^\dagger)_B\,|\,0)&=\left[A_f^\dagger+\sum_{122'}X_3(12;f2')A_1^\dagger A_2^\dagger A_{2'}\right]\times A_e^\dagger\,|\,0)\\&=\sum_{e'f'}[\Delta_{ef;e'f'}+X_3(ef;e'f')]\times A_{e'}^\dagger A_{f'}^\dagger\,|\,0)\;,\\&=\sum_{e'f'}(Z_2)_{ef,e',f'}A_{e'}^\dagger A_{f'}^\dagger\,|\,0)\;.\end{aligned} \tag{6.18b}$$

It may be expected that this pattern is repeated with any further step of operation, and it is rather easy to prove by induction that we obtain the following general relation:

$$(B_a^\dagger)_B^n\,|\,0)=\sum_{(b)}(Z_n)_{a;b}(A_b^\dagger)^n\,|\,0)\;, \tag{6.19}$$

where we again used the notation

$$a=\{a_1,a_2,\ldots,a_n\}\;.$$

Relation (6.19) can be inverted to give

$$\sum_{(b)}(Z_n{}^{-1})_{a;b}[1/\sqrt{n!}](B_b^\dagger)_B^n\,|\,0)=[1/\sqrt{n!}](A_a^\dagger)^n\,|\,0)\;. \tag{6.20}$$

In the terminology of Sec. II, the rhs of (6.20) is the normalized ideal-boson state $|\,n;\bar{a})$, while the lhs of (6.20) is the orthonormal physical boson state, which may be denoted as $|\,n;a)$ in analogy to the orthonormal fermion state $|\,n;a\rangle$ introduced in (3.3). Equation (6.20) thus means that we have established an equality

$$|\,n;a)=|\,n;\bar{a})\;. \tag{6.21}$$

Equation (6.21) is a generalization of Eq. (14) of TWP, which was proved only for the case in which we had one kind of monopole bosons.

To have (6.21) is very significant, because we can now combine Eqs. (2.14) and (2.16) to give

$$\langle m;b\,|\,O_F\,|\,n;a\rangle=(m;b\,|\,O_B\,|\,n,a)=(m;\bar{b}\,|\,O_B\,|\,n;\bar{a})\;. \tag{6.22}$$

As was stressed above, we use the third version of (6.22), i.e., we work in the ideal-boson space, in actual calculations. However, the second equality of (6.22) shows that in effect we work in the physical boson space. Since the meaning of the first equality of (6.22) is well understood [see text following Eq. (2.14)], to have the second equality of (6.22) is an ample justification of the use of the ideal-boson space in the calculations.

It is interesting to see how the matter changes if we use PPR instead of TDR. For simplicity, take $n=2$, and write the equivalent of (6.19) appropriate for PPR. As is well known,[9] it is given as

$$(B_{12}^\dagger)_B(B_{34}^\dagger)_B\,|\,0)=(1/\sqrt{3})(A_{12}^\dagger A_{34}^\dagger-A_{13}^\dagger A_{24}^\dagger+A_{14}^\dagger A_{23}^\dagger)\,|\,0)\;. \tag{6.23}$$

To invert (6.23) in the same way as we did (6.19) into (6.20) is not possible, however. Note that the indices 1 through 4 in (6.23) denote fermion orbits, and thus, when we choose for the lhs the states such as $B_{12}^{\dagger}B_{24}^{\dagger}|0)$ and $B_{14}^{\dagger}B_{23}^{\dagger}|0)$, we simply see on the rhs that the expression as seen in (6.23) appears repeatedly. Therefore, the corresponding Z_2^{-1} matrix is singular, which prevents one from writing an equation that corresponds to (6.20). This then means that we cannot write the second equality of (6.22) either, and are thus forced to work always in the physical boson space. This is another way to look at the difficulty one encounters when one utilizes PPR.

VII. FAST CONVERGENCE OF THE LINKED CLUSTER EXPANSIONS

In Sec. V, we showed very explicitly that the basic pair of operators, i.e., $(B_e^{\dagger})_B$ and $(C_p^{\dagger})_B$, were obtained as linked cluster expansions, and thus in forms that were extremely compact. In Sec. I, on the other hand, we stressed that the linked cluster expansions would converge very fast. With the form given in Sec. V for the above operators, however, the reader may not yet see very clearly in what way the given expansions embody this fast convergence. In this section, we shall thus rewrite the results of Sec. V still further, so that the fast convergence becomes evident. In the course of this, it will also be shown that the forms we obtain in this way for the expansion coefficients agree with those we had obtained earlier, and use in many calculations.[3–7]

As for the $(C_p^{\dagger})_B$ operator, we may choose to ignore the ξ sums, thus obtaining $X_{2n}=0$ for $n\geq 2$. Then $(C_p^{\dagger})_B$ consists of a single term, and thus no problem of convergence is encountered. We shall therefore concentrate on the $(B_e^{\dagger})_B$ operator from now on, and show that $X_{2n-1}=0(Y_2^{n-1})$, where Y_2 is a quantity defined, e.g., by (4.5). In other words, we intend to show that the expansion of $(B_e^{\dagger})_B$ is in the form of a Taylor series, taking $|Y_2|$ as a smallness parameter. We shall then show that $|Y_2|$ is indeed very small, if we truncate the system to the collective component.

We find it most convenient to first recapitulate our previous formalism; and for this purpose we can use the coefficient equations given explicitly in Sec. VI. The X_3 equation is given by (6.3), and one easily recognizes that this equation is the same as the equation given as (3.5c) in KT-1 (apart from a slight change of notation). In KT-1, this equation was solved exactly, retaining all the TD components. [See Eq. (3.13) of KT-1, and notice that it in fact contains an unlinked-cluster term.]

As emphasized in Sec. I, however, the solution of (3.13) in KT-1 is not what we used for our calculations. As was explained in KT-2, we first truncated the TD system to the collective component (of quadrupole nature), and then solved (6.3). To redo it here explicitly, let us first rewrite (Y_2) and X_3 as

$$(Y_2)_{a2;b2'}=\sum_{im}(y_i/2)(a2\,|\,m)(b2'\,|\,m)\;, \qquad (7.1a)$$

$$X_3(a2;b2')=\sum_{im}(x_3)_i(a2\,|\,m)(b2'\,|\,m)\;. \qquad (7.1b)$$

Here, e.g., $(a2\,|\,m)$ is an abbreviation of the Clebsch-Gordan coefficient which reads $(2\mu_a 2\mu_2\,|\,im)$ in full. It is obvious that $i=0$, 2 , and 4, while m is the projection of i.

If (7.1) is inserted into (6.3), the latter becomes a quadratic equation for the unknown $(x_3)_i$:

$$2(x_3)_i+(x_3)_i^{\,2}+(y_i/2)=0\;, \qquad (7.2)$$

with a solution that

$$(x_3)_i=[1-\tfrac{1}{2}y_i]^{1/2}-1=-\tfrac{1}{4}y_i-\tfrac{1}{32}y_i^{\,2}-\cdots\;. \qquad (7.3)$$

This shows that $(x_3)_i=O(y_i)$, i.e., that $X_3=O(Y_2)$, as we stated above.

We remark that we chose the plus sign in front of the square root in (7.3). The reason for this choice is that the choice of the minus sign makes $X_3=O(1)$. We also remark that the lowest order solution of (7.3), i.e., $(x_3)_i=-(\frac{1}{4})y_i$, which results in

$$X_3(a2;b2')=\sum_{im}(-1/4)y_i(a2\,|\,m)(b2'\,|\,m)\;, \qquad (7.4)$$

is obtained from (7.2), if we first suppress the $(x_3)_i^{\,2}$ term, which is $O(y_i^{\,2})$, compared with other two terms which are both $O(y_i)$. If we are to be satisfied with (7.4), the procedure explained above, with the use of (7.2), was necessary. We can just suppress the second term in (6.3). Then (6.3) is solved trivially and gives (7.4) again. We shall use this simplifying techinque in solving the X_5 equation.

The x_5 equation is given by Eq. (6.10), together with (6.11) and (6.12). Anticipating that $X_5=O(y_i^{\,2})$, we retain only terms in (6.10) that are $O(y_i^{\,2})$. [We also set $X_4=0$ in (6.12a).] We then immediately see that the X_5 equation is solved as

$$X_5(a23;b2'3')=-A_3/6\;, \qquad (7.5)$$

where A_3 was defined in (6.12b). We shall use

$$X_3(a2;b2')=-\tfrac{1}{2}(Y_2)_{a2;b2'}\;,$$

which is nothing but (7.4), in (6.12b), to express A_3 in terms of (Y_2). We then find rather easily that

$$X_5(a\,23;b\,2'3') = -(\tfrac{1}{8})\sum_{3''}(Y_2)_{23;b3''}(Y_2)_{a3'';2'3'} \,. \tag{7.6}$$

The X_5 of (7.6) is obviously linked clustered, and is $O(y_i^2)$.

Combining (5.18), (5.19), (7.4), and (7.6) together, we can write down $(B_e^\dagger)_B$ very explicitly as

$$(B_e^\dagger)_B = A_a^\dagger + \sum_{122'}\sum_{im}[-\tfrac{1}{4}y_i](12\,|\,m)(e2'\,|\,m)A_1^\dagger A_2^\dagger A_{2'}$$
$$+\sum_{1232'3'}\sum_{ii'mm'}[-\tfrac{1}{32}y_iy_{i'}]\sum_{3''}(23\,|\,m)(e3''\,|\,m)(13''\,|\,m')(2'3'\,|\,m')A_1^\dagger A_2^\dagger A_3^\dagger A_{2'}A_{3'} \,. \tag{7.7}$$

The expansion given in (7.7) is exactly what was used in our calculations. Note that (7.7) is also a reproduction of the expansion given as Eq. (14) in Ref. 17.

We shall now show that the expansion of (7.7) is easily reproduced by using the results we obtained in Sec. V. First of all, Eq. (5.19) shows (noting that $Z_1 = 1$) that

$$\begin{aligned} X_3(12;e2') &= (Z_2)^{(L)}_{12;e2'} \\ &= (Z_2)_{12;e2'} - \Delta_{12;e2'} \\ &= \sum_{im}\{[1-\tfrac{1}{2}y_i]^{1/2}-1\} \\ &\qquad \times(12\,|\,m)(e2'\,|\,m) \,. \end{aligned} \tag{7.8}$$

It is obvious that (7.8) is an exact reproduction of (7.1b), together with (7.3).

In evaluating X_5, we find it convenient to use the expression given by the second equality in (6.13), rather than by the first equality. In any case, the algebra to show that X_5 of (6.13) reduces (to the lowest power of y_i) to (7.6) is somewhat lengthy, but straightforward. Therefore, we shall not give every detail of it. It will be sufficient here to give only a few basic ideas on how to carry it out.

The most complicated term in (6.13) is $(Z_3/Z_2)_{123;e2'3'}$, which is of course rewritten as

$$\sum_{2''3''}(Z_3)_{123;e2''3''}(Z_2^{-1})_{2''3'';2'3'} \,.$$

We first rewrite the Z_3 factor here as

$$Z_3 = 1-(Y_3)/2-(Y_3^2)/8-\cdots \,,$$

and then note in (4.6) that (Y_3) consists of two terms: One is $O(y_i)$ and the other $O(y_i^2)$. By using (4.6), we express the above Z_3 up to terms of $O(y_i^2)$. We next expand Z_2^{-1}, also to $O(y_i^2)$, and combine it with the expansion of Z_3. In the resultant $(Z_3/Z_2)_{123;e2'3'}$, the term of $O(1)$ is $\Delta_{123;e2'3'}$. The term of $O(y^i)$ is of the form

$$P^{(1)}_{e2'3'}[\Delta_{1;e}(Y_2)_{23;2'3'}] \,.$$

There are two terms that are $O(y_i^2)$. The one is of the form

$$P^{(1)}_{e2'3'}[\Delta_{1;e}(Y_2^2)_{23;2'3'}] \,,$$

while the other is of the form of (7.6). Thus the (Z_3/Z_2) term produces one linked and three unlinked cluster terms. These unlinked cluster terms are, however, very neatly canceled out, as they should be, by the two remaining terms in (6.13). We thus see that X_5 indeed reduces to a linked-cluster term, and further, that the thus reduced X_5 agrees exactly with what was given in (7.6).

This completes the proof that the new method of the present paper results in the bosonized fermion pair operators which agree exactly with those obtained earlier[2,17] and were used in calculations.[3-7] As seen from the explanations given above, however, our earlier work did not contain a proof that the thus obtained operators can be used in the ideal-boson space. This missing proof has now been presented by showing that the results with the new and old methods are the same. One thus sees that our earlier calculations were in fact performed based on formulas that can be well justified.

In the above proof of the equivalence, we expanded X_{2n-1} in powers of Y_2, and retained only the leading terms. For the calculations we have done so far, Y_2 was indeed small, and thus the above comparison is justified. With the forms given by the present method, however, an X_{2n-1} can be evaluated without expanding the square roots in powers of Y_2. In other words, the use of the new formulas allows us to evaluate the X_{2n-1}'s with an increased accuracy, when such becomes necessary.

It is easy to see why Y_2 is so small if we restrict ourselves, as we did in our earlier calculations, to the collective TD component. Let us denote by $(B_C^\dagger)\,|\,0\rangle$ the collective TD state, and assume that there are involved N_p particle pairs (or particle-hole pairs) to define $B_c^\dagger$, so that the summation in (2.2) contains N_p terms. For the collective state, the expansion coefficients $\Psi^{(C)}_{j_1j_2}$ may be very weakly dependent on $\{j_1j_2\}$, and roughly equal to $1/(N_p)^{1/2}$. By keeping these facts in mind, and then constructing

$$\langle 0 | (B_C)^2 (B_C^\dagger) | 0 \rangle = (2!)(1-Y_2) ;$$

cf. (4.1), it is easy to see that $Y_2 = O(N_p^{-1})$, which may be of the order of 0.1 or less. (The smallness parameter that appeared in our actual calculations was still smaller than this, because we used RPA-type, rather than TD-type bosons, as we stressed in Sec. I, as well as in Refs. 2 and 7.)

For the noncollective states, the corresponding Y_2 is very close to unity. This can be seen easily from the fact that, for a PPR operator $B_{12}^\dagger$, we have

$$(B_{12}^\dagger)^2 | 0 \rangle = 0 ,$$

making $Y_2 = 1$. However, one should not consider that the boson expansion theory breaks down because of this. It should be kept in mind that, in practice, we will seldom consider basis states in which a particular noncollective component appears more than once (although the collective component may appear multiply). Then what happens is the situation we encountered in Eq. (6.18a). As seen, we can let the theory remain exact, even if we retain only the first term of the expansion. Thus the slow convergence of the $B^\dagger$ operators corresponding to noncollective components should not be bothersome.

We finally explain why the presence of unlinked-cluster terms slows down the convergence. We showed above that, with

$$X_{2n-1} = (Z_n / Z_{n-1})^{(L)} ,$$

we had

$$X_{2n-1} = O(Y_2^{n-1})$$

and thus a fast convergence. Suppose we had

$$X_{2n-1} = (Z_n / Z_{n-1})$$

instead. Then we see, from Eq. (5.7), that we have

$$X_{2n-1} = O(1) + \sum_{i=0}^{n-2} O(Y_n^{n-i-1}) = O(1) .$$

Obviously we cannot have a fast convergence, even if Y_2 is very small.

VIII. FORMULATION FOR ODD-MASS NUCLEI

All the formalism presented so far in the preceding sections was for even-even nuclei. Such a restriction was also the case with MYT, LH, our earlier papers, and many other related papers.[22]

Several authors have discussed BET, intending to use it for odd-A nuclei.[23] However, the presentations made in these papers remained rather formal, largely because the development was made in PPR. As we emphasized above, a BET in PPR is all but useless. We shall show in this section that it is possible to extend the TDR formalism of the preceding sections, which were developed for even-even nuclei, to the use for odd-A nuclei. It will be seen that this results in formulas that can be used for very practical calculations.

We begin again by introducing an operator U_{odd}, defined as

$$U_{\text{odd}} = \sum_n \sum_{(a,j)} | n;aj)(n;aj | , \tag{8.1}$$

as a straightforward extension of what was given in (3.1). Here,

$$| n;aj \rangle = \sum_{(a'j')} (\mathring{Z}_n^{-1})_{aj;a'j'} | n;a'j' \rangle\rangle , \tag{8.2}$$

with

$$| n;aj \rangle\rangle = [1/\sqrt{n!}] d_j^\dagger (B_a^\dagger)^n | 0 \rangle , \tag{8.3}$$

which is a nonorthonormal state with a fermion in an orbit j, together with n TDR-type fermion pairs, specified by

$$a = \{ a_1, \ldots, a_n \} .$$

The $\mathring{Z}_n$ matrix, and the related $\mathring{Y}_n$ matrix, are defined through a relation that

$$(\mathring{Z}_n{}^2)_{aj;a'j'} = (1 - \mathring{Y}_n)_{aj;a'j'} = \langle 0 | (B_a)^n d_j d_{j'}^\dagger (B_{a'}^\dagger)^n | 0 \rangle / n! . \tag{8.4}$$

Further in (8.1), $| n;aj)$ is a normalized, ideal fermion-boson state given as

$$| n;aj) = [1/\sqrt{n!}] c_j^\dagger (A_a^\dagger)^n | 0) , \tag{8.5}$$

the ideal fermion operator[23] $c_j^\dagger$ commuting with A_a's and $A_a^\dagger$'s, and satisfying a commutation relation

$$\{ c_j, c_{j'}^\dagger \} = \delta_{jj'} .$$

Once the U_{odd} operator is defined, the procedure to be taken for bosonizing the fermion operators and the states is essentially the same as it was for even-even nuclei. We shall thus give immediately the boson images of the $B_e^\dagger$ and the $C_p^\dagger$ operators:

$$(B_e^\dagger)_B = \sum_{n=1} \sqrt{n} \sum_{(aa'a'')} \sum_{jj'j''} [(\mathring{Z}_n)_{aj;ea''j''} (\mathring{Z}_{n-1}^{-1})_{a''j'',a'j'}] | n;aj)(n-1;a'j' | \tag{8.6}$$

and

$$(C_p^\dagger)_B = \sum_{n=0} [1/n!] \sum_{(aa'bb')} \sum_{jj'ii'} (\mathring{Z}_n^{-1})_{aj;bi} \langle 0 | d_i (B_b)^n C_p^\dagger (B_{b'}^\dagger)^n d_{i'}^\dagger | 0 \rangle (\mathring{Z}_n^{-1})_{b'i';a'j'} | n;aj)(n;a'j' | . \tag{8.7}$$

The reader will find no difficulty in understanding these two expressions.

Contrary to what we have done above for even-even nuclei, we shall here be content with obtaining explicitly only the first two terms of the series in the above expressions, i.e., the $n=1$ and 2 terms in (8.6) and the $n=0$ and 1 terms in (8.7).

In doing this, we shall first go back to Eq. (8.4), and note that the factor $d_j d_{j'}^\dagger$ that appeared there can be rewritten as

$$d_j d_{j'}^\dagger = \delta_{jj'} - \sum_{kq} (m'\tilde{m} \mid q) C_{p'}^\dagger \ , \tag{8.8}$$

where

$$(m'\tilde{m} \mid q) = (j'm'\tilde{j}m \mid kq) \ ,$$

while $p' = \{j'jkq\}$. [See the text following Eq. (2.7).] By inserting (8.8) into (8.4), and then using (5.22), we see that

$$(1-\mathring{Y}_n)_{aj;a'j'} = \delta_{jj'}(1-Y_n)_{aa'} - P_{a'}^{(1)} \sum_{kq} (m'\tilde{m} \mid q) \sum_{g} (1-Y_n)_{a;g\bar{a}'} P_{1';g}^{(p')} \ , \tag{8.9}$$

from which we obtain

$$(\mathring{Y}_0)_{jj'} = 0 \ , \tag{8.10a}$$

$$(\mathring{Y}_1)_{1j;1'j'} = \sum_{kq} (m'\tilde{m} \mid q) P_{1';1}^{(p')} \ , \tag{8.10b}$$

$$(\mathring{Y}_2)_{12j;1'2'j'} = \delta_{jj'}(Y_2)_{12;1'2'} + P_{1'2'}^{(1)} \Delta_{22'} (\mathring{Y}_1)_{1j;1'j'} - P_{1'2'}^{(1)} \sum_{kq} \sum_{g} (m'\tilde{m} \mid q)(Y_2)_{12;g2'} P_{1';g}^{(p')} \ . \tag{8.10c}$$

Note that $(\mathring{Y}_1) \neq 0$, while $(Y_1) = 0$.

Since $(\mathring{Y}_0) = 0$, we have $(\mathring{Z}_0) = 1$. Therefore, for the $n=1$ term in (8.6), we simply have

$$(\mathring{Z}_1/\mathring{Z}_0)_{1j;ej'} = \delta_{jj'} \Delta_{1e} + (\mathring{Z}_1)_{1j;ej'}^{(L)} \ . \tag{8.11}$$

For the $n=2$ term of (8.6), we obtain, after a little algebra, the result that

$$(\mathring{Z}_2/\mathring{Z}_1)_{12j;e2'j'} = \delta_{jj'} (Z_2)_{12;e2'}^{(L)} + \Delta_{12'} (\mathring{Z}_1)_{2j;ej'} + (\mathring{Z}_2/\mathring{Z}_1)_{12j;e2'j'}^{(L)} \ . \tag{8.12}$$

In the discussion which follows, it is not necessary to have an explicit form of the last term in (8.12). We shall nevertheless give it here, in order to give an idea of how the linked-cluster terms look for odd-A nuclei:

$$(\mathring{Z}_2/\mathring{Z}_1)_{12j;e2'j'}^{(L)} = (\mathring{Z}_2)_{12j;e2'j'}^{(L)} + \sum_{2''} (Z_2)_{12;e2''}^{(L)} (\mathring{Z}_1)_{2''j;2'j'}^{(L)} + \sum_{j''} (\mathring{Z}_1)_{2j;ej''}^{(L)} (\mathring{Z}_1{}^{-1})_{1j'',2'j'}^{(L)} + \sum_{2''j''} (\mathring{Z}_2)_{12j;e2''j''}^{(L)} (\mathring{Z}_1{}^{-1})_{2''j'';2'j'}^{(L)} \ . \tag{8.13}$$

We can now insert (8.11) and (8.12) into (8.6). Since we consider only two terms in (8.6), we may set

$$\mid 0)(0 \mid = 1 - \sum_{1} A_1^\dagger A_1$$

in the $n=0$ term, and $\mid 0)(0 \mid = 1$ in the $n=2$ term. We then see that the unlinked-cluster terms neatly cancel out, and that we obtain the following linked cluster expansion of the $(B_e^\dagger)_B$ operator as

$$(B_e^\dagger)_B = \{A_e^\dagger + \sum_{122'} X_3(12;e2') A_1^\dagger A_2^\dagger A_{2'} + \cdots \} \sum_{j} (C_j^\dagger c_j) + \sum_{jj'1} (\mathring{Z}_1)_{1j;ej'}^{(L)} (A_1^\dagger c_j^\dagger c_{j'}) + \sum_{jj'122'} (\mathring{Z}_2/\mathring{Z}_1)_{12j;e2'j'}^{(L)} A_1^\dagger A_2^\dagger c_j^\dagger A_{2'} c_{j'} + \cdots \ . \tag{8.14}$$

The expression in the curly brackets in the first term of (8.14) is nothing but what we have obtained for $(B_e^\dagger)_B$, when we considered an even-even nucleus. Therefore, the first term of (8.14) is interpreted as the term that describes the motion of the core part of an odd-A nucleus, and the last (unpaired) nucleon is just playing the role of a spectator.

The other terms in (8.14) evidently describe the coupling of the last nucleon and the core. The significance of obtaining the expansion of (8.14) is that the two kinds of terms, the core terms and the coupling terms, were obtained from a single starting point, and thus in a completely consistent manner. It may also be evident from the derivation explained above, that all the nucleons, whether paired or unpaired, were treated on a completely equal footing.

The situation is very much the same regarding the operator $(C_p^\dagger)_B$. We shall give the first two terms of its boson expanded form, without going through the derivation. (Note that $p=\{jj'kq\}$.)

$$(C_p^\dagger)_B=\sum_{mm'}(m'\widetilde{m}\mid q)c_j^\dagger c_{j'}+\sum_{11'j_1j_1'}\left[\sum_{mm'}(m'\widetilde{m}\mid q)\left\{\delta_{jj_1}(\mathring{Z}_1{}^{-1})^{(L)}_{1j';1'j_1'}+\delta_{j'j_1'}(\mathring{Z}_1)^{(L)}_{1j_1;1'j}+\sum_{1''}(\mathring{Z}_1)^{(L)}_{1j_1'1''j}(\mathring{Z}_1{}^{-1})^{(L)}_{1''j';1'j_1'}\right\}+\sum_{g1''j''}P^{(p)}_{1'',g}(\mathring{Z}_1)^{(L)}_{1j_1;gj''}(\mathring{Z}_1{}^{-1})^{(L)}_{1''j'';1'j_1'}\right]A_1^\dagger c_{j_1}^\dagger c_{j_1'}A_{1'}\ . \tag{8.15}$$

It will be evident that a relation very similar to what was given in (6.22) will hold again here. Since, as shown above, the bosonized operators have been obtained in a linked-cluster form, and since they can be used in the ideal fermion-boson space, they satisfy all the requirements for them to be amenable to practical numerical calculations. We may thus conclude that we now have obtained a very practical BET that can be used for odd-A nuclei. It does not seem that such a BET was obtained in the past, at least not in a form as general and as flexible as is our formalism.

During the past decade or so, Paar and his coworkers performed rather extensive analyses of collective odd-A nuclei; see Ref. 24 and earlier papers cited therein. In their calculations, however, the even-even core part was treated in a phenomenological way. Extensive analyses of odd-A nuclei were also done by Kuriyama *et al.*,[25] with a particular emphasis of the "dressing" of the odd nucleons of the collective field produced by the even-even core. The formulas given above may be considered to offer a possibility either to improve or extend these earlier calculations, although some additional algebra would have to be worked out before such calculations could be performed.

IX. CONCLUDING REMARKS

We shall first summarize what we have done in the present paper.

(i). We showed that to construct a BET based on PPR is very unlikely to lead us to any formulas for practical purposes. To have such a BET means that we can at best repeat, normally in a much more difficult way, the calculations which can be done in the original fermion form.

(ii). Once TDR, or its equivalent, is introduced, the above problem of PPR is largely removed. However, the theory can become useful, if and only if, the formulation is done in such a way that the bosonized operators, thus derived, are in the forms of the linked-cluster expansions, and are allowed to be used in the ideal boson space. The major part of the present paper was devoted to show how such a practically useful theory can be constructed.

(iii). The guide we took in constructing such a theory can be expressed in terms of an extremely compact equation; Eq. (6.22). Actually, the necessity of taking the first equality in (6.22) was pointed out by several authors[22] in the past, but in most of these cases the significance of retaining the second equality as well seems to have been overlooked to a large extent. One reason for this, we believe, was the use of PPR in such discussions. We showed that, with PPR, there is no chance to retain the second equality of (6.22).

(iv). To use criteria other than that of (6.22) could be gravely misleading. For example, one often evaluated various BET's by asking whether a bosonized operator annihilates unphysical boson states, or equivalently, whether it produces a physical boson state when it is operated on an ideal boson state. As is well known, the MYT operators meet this criterion, while those of KT-1 and KT-2, for example, do not. However, once we take as the new criterion the satisfaction of the equalities in (6.22), which we believe is much more meaningful than is the above criterion, we see that there is no chance that we ever encounter an unphysical state. In other words, there remains no room to apply his criterion. We have shown that the BET of KT-2, which was used for calculations,[3–7] (approximately) satisfies (6.22). Thus, Marshalek's doubt[8] about our calculations is removed. We have also shown that the original MYT failed to satisfy the second equality of (6.22).

(v). Above we made a few negative remarks about MYT. However, MYT has in it an outstanding feature in that it begins with introducing the modified Usui operator U. If the original form of U used in MYT is retained, the above problems remain. However, if a revised U is used, and the line set forth by MYT is followed, a very useful BET theory emerges, and to demonstrate this is what has been done in the present paper. As a by-product of constructing a rather flexible new BET, we found that the results of KT-2 and of LH are obtained as limiting cases of the present results. We take this as convincing evidence to indicate that our previous calculations were done correctly.

(vi). We showed that the technique developed in the present paper can be used to derive formulas that can be used to describe not only collective even-even nuclei, but also collective odd-A nuclei.

After summarizing our achievement in the present paper this way, we shall now turn to a discussion of what should be or could be done in the future. Since we have found that the present results, if truncated to a single quadrupole component, agree with those of KT-2, it is unnecessary to redo the earlier calculations,[3–7] so long as we decide to stay with the above truncation. However, in spite of the very good general agreement with experiment we have achieved, there remained certain features which called for improvements. These features almost always indicated that it was necessary to remove the severe truncation to a single collective component.

We have been rather hesitant in the past, however, in moving in this new direction, because we were not certain of how to properly describe the coupling between different modes, particularly when the new modes that are to be integrated into the calculations are of a noncollective nature. With the new formalism of the present paper, however, this missing information has now been given. We may thus attempt to start such extended calculations. The situation is very much the same as regards odd-A nuclei. Previously we felt that we did not have well formulated expressions to work with, but we now feel we do have them.

In all the calculations we have reported,[3–7] we have treated the pairing field by solving the BCS (Bardeen-Cooper-Schriefer) equation in a spherical basis, and kept it unchanged, even when the resultant system turned out to be well deformed. This we believed to be another feature, for which an improved treatment was necessary, but we were unable to find a proper method for doing this. In this sense, it is very pleasing to find that a recipe proposed recently by Suzuki *et al.*[26] appears to be precisely what we have been looking for. Note that, among the calculations we have performed, we were most successful regarding the Ru-Pd (Ref. 4) and Os-Pt (Ref. 5) isotopes, all of these isotopes being relatively close to the spherical limit, but for the Sm isotopes,[3] where we encountered a rapid transition from spherical to deformed elements, the fit to data we achieved was much less satisfactory. It will thus be very interesting to see whether the problems in Sm isotopes are solved by integrating the method of Suzuki *et al.* into the framework developed above. Such an approach may also help us in removing a few problems we have encountered in our previous analysis of the Ge isotopes.[6]

We have compared above, at various stages, the present formalism with those of MYT, KT-1, and KT-2. Since there have appeared a number of formal papers on BET in the past, as we mentioned above, it would be desirable to extend our comparison with these papers as well. This will be done in a separate paper.[19]

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APPENDIX A: INDUCTIVE PROOF OF EQUATION (4.8)

We first replace n by $n+1$ in (4.11), and denote the three terms in the thus obtained (Y_{n+1}) simply by [I], [II], and [III].

We then see, first of all, that [with $a'=\{1',\ldots,(n+1)'\}$]

$$[\mathrm{I}]=(n+1)^{-1}P_{a'}^{(1)}[\Delta_{1,1'}(Y_n)_{2\cdots(n+1);2'\cdots(n+1)'}]\ . \tag{A1}$$

We then rewrite the (Y_n) matrix on the rhs of (A1) by using (4.8), obtaining

$$[\mathrm{I}]=(n+1)^{-1}\sum_{i=0}^{n-2}[P_{a'}^{(1)}\Delta_{1;1'}P_{2'\cdots(n+1)'}^{(i)}\Delta_{2\cdots(i+1);2'\cdots(i+1)'}(Y_{n-i})_{(i+2)\cdots(n+1);(i+2)'\cdots(n+1)'}^{(L)}]\ . \tag{A2}$$

It is not difficult to see in (A2) that the product of two symmetrizers $P^{(1)}P^{(i)}$ can be replaced by $(i+1)P^{(i+1)}_{1'\cdots(n+1)'}$. After making this replacement we change the summation index i into $i-1$. We then obtain [I] as

$$[\mathrm{I}]=\sum_{i=1}^{n-1}[i/(n+1)]P_{a'}^{(i)}[\Delta_{(i);(i)'}(Y_{n+1-i})^{(L)}_{\{n+1-i\};\{n+1-i\}'}] . \tag{A3}$$

This is the expression we need to have for [I]. The corresponding expression for [II] can be obtained, without performing any algebra, as

$$[\mathrm{II}]=[2/(n+1)]P_{a'}^{(n-1)}[\Delta_{(n-1);(n-1)'}(Y_2)^{(L)}_{n,n+1;n'(n+1)'}] . \tag{A4}$$

In obtaining [III] in a similar form, we start with the form that

$$[\mathrm{III}]=-[2/(n+1)]P_{a'}^{(2)}\sum_g (Y_2)_{1g;1'2'}\sum_{i=0}^{n-2}P^{(i)}_{g3'\cdots(n+1)'}$$
$$\times[\Delta_{2\cdots(i+1);g3'\cdots(i+1)'}(Y_{n-i})^{(L)}_{(i+2)\cdots(n+1);(i+2)',,,(n+1)'}] . \tag{A5}$$

Note that (4.8) has been used to obtain (A5). The symmetrizer $P^{(i)}_{g3'\cdots(n+1)'}$ of course produces ${}_nC_i$ terms, and it is very important to separate these terms into two groups, those retaining the index g in the Δ factor, and those in which g has been shifted into the (Y_{n-1}) factor. Having this fact in mind, one sees that [III] is rewritten as

$$[\mathrm{III}]=-[2/(n+1)]\sum_{i=0}^{n-2}P_{a'}^{(2)}(Y_2)^{(L)}_{1g;1'2'}$$
$$\times\{\Delta_{2g}P^{(i-1)}_{3'\cdots(n+1)'}\Delta_{3\cdots(i+1);3'\cdots(i+1)'}(Y_{n-i})^{(L)}_{(i+2)\cdots(n+1);(i+2)'\cdots(n+1)'}$$
$$+P^{(i)}_{3'\cdots(n+1)'}\Delta_{2\cdots(i+1);3'\cdots(i+2)'}(Y_{n-i})^{(L)}_{(i+2)\cdots(n+1);g(i+3)'\cdots(n+1)'}\} . \tag{A6}$$

For the first term in the curly brackets in (A6), we first note that the $i=0$ term vanishes because of the presence of the symmetrizer $P^{(i-1)}$. Therefore, if we change the summation index i into $i+1$, the range of summation for the new index i becomes from 0 to $n-3$, rather than from 0 to $n-2$, as it was in (A6). We can nevertheless set the upper limit of i as $n-2$, because this term contains a factor $(Y_1)=0$. We can thus retain the same range of summation for both the first and second terms on (A6).

In both terms, we modify the roles of the products of the two symmetrizers slightly, and find that [III] is rewritten as

$$[\mathrm{III}]=-[2/(n+1)]\sum_{i=0}^{n-2}P_{a'}^{(i)}\Delta_{(i);(i')}[P^{(2)}_{(i+1)'\cdots(n+1)'}$$
$$\times\{(Y_2)^{(L)}_{(i+1)(i+2);(i+1)'(i+2)'}(Y_{n-1-i})^{(L)}_{(i+3)\cdots(n+1);(i+3)'..(n+1)'}$$
$$\times\sum_g (Y_2)^{(L)}_{(i+1)g;(i+1)'(i+2)'}(Y_{n-i})^{(l)}_{(i+2)\cdots(n+1);g(i+3)'\cdots(n+1)'}\}] . \tag{A7}$$

By looking at Eq. (4.9), we see that the expression in the curly brackets in (A7) is nothing but

$$-[(n+1-i)/2](Y_{n+1-i})^{(L)}_{(i+1)\cdots(n+1);(i+1)'\cdots(n+1)'} .$$

In other words, we found that [III] can now be written in the following very compact form:

$$[\mathrm{III}]=[(n+1-i)/(n+1)]P_{a'}^{(i)}[\Delta_{(i);(i)'}(Y_{n+1-i})^{(L)}_{\{n+1-i\};\{n+1-i\}'}] . \tag{A8}$$

As seen in (A3), (A4), and (A8), all three terms [I], [II], and [III] are brought into very similar forms, and they can be easily summed together, thus giving

$$(Y_{n+1})_{aa'}=\sum_{i=0}^{n-1}P_{a'}^{(i)}[\Delta_{(i);(i)'}(Y_{n+1-i})^{(L)}_{\{n+1-i\};\{n+1-i\}'}] , \tag{A9}$$

which agrees with (4.8), if n in the latter is replaced by $n+1$. This completes the inductive proof of Eq. (4.8).

APPENDIX B: PROOF OF THEOREM I

From (4.14), we first obtain

$$\sum_{(a'')} (A_n)_{aa''}(B_n)_{a''a'} \equiv (A\cdot B)_{aa'}$$

$$= \sum_{i,j} P_{a'}^{(j)} [P_{a''}^{(i)} \Delta_{(i);(i)''}(A_{n-i})^{(L)}_{\{n-i\};\{n-i\}''}][\Delta_{(j)'';(j)'}(B_{n-j})^{(L)}_{\{n-j\}'';\{n-j\}'}] \; . \quad \text{(B1)}$$

In (B1), we first emphasize that the symmetrizer $P_{a''}^{(i)}$ operates only upon the doubly-primed indices attached to quantities that appear inside the first square brackets, and not those appearing in the second set of square brackets.

We shall now claim that the result of operating $P_{a''}^{(i)}$ reduces (B1) into

$$(A\cdot B)_{aa'} = \sum_{i,j} P_{a'}^{(j)} \sum_{m} ({}_jC_m \cdot {}_{n-j}C_{i-m})$$

$$\times \sum_{\{n-m\}''} \left[\sum_{(m)''} \Delta_{(i);m''\{i-m\}''} \Delta_{(m)''\{j-m\}'';(j)'} \right]$$

$$\times (A_{n-i})^{(L)}_{\{n-i\};\{j-m\}''\{m+n-i-j\}''}(B_{n-j})^{(L)}_{\{i-m\}''\{m+n-i-j\}'';\{n-j\}'} \; . \quad \text{(B2)}$$

Equation (B2) was obtained in the following way. We pick up terms in which m doubly primed indices, denoted by $(m)''$, are common in the two Δ factors, the rest of the doubly primed indices in these factors, denoted respectively, by $\{i-m\}''$ and $\{j-m\}''$, having no matching elements in common. It is easy to find that the number of such terms equals ${}_jC_m \cdot {}_{n-j}C_{i-m}$, which is the weight factor appearing just after the summation symbol $\sum_m$. The doubly primed indices in the (A_{n-i}) factor are divided into two groups; the first $\{j-m\}''$ being the same as the $\{j-m\}''$ that appeared in the second Δ factor and the rest whose number equals $(n+m-i-j)$ are simply denoted by $\{m+n-i-j\}''$. A similar division was made in the (B_{n-j}) factor also.

We now rewrite the two Δ factors in (B2) somewhat more explicitly as

$$\Delta_{(i);(m)''\{i-m\}''} = [{}_iC_m]^{-1} P^{(m)}_{1\cdots i} [\Delta_{(m);(m)''} \Delta_{\{i-m\};\{i-m\}''}] \; ,$$

$$\Delta_{(m)''\{j-m\}'';(j)'} = [{}_jC_m]^{-1} P^{(m)}_{1'\cdots j'} [\Delta_{(m)'';(m)'} \Delta_{\{j-m\}'';\{j-m\}'}] \; . \quad \text{(B3)}$$

If we insert (B3) into (B2), most of the summations over the doubly primed indices can be carried out, obtaining

$$(A\cdot B)_{aa'} = \sum_{i,j} \sum_{m} [{}_{n-j}C_{i-m}] P_{a'}^{(j)} P^{(m)}_{1'\cdots j'} \Delta_{(m);(m)'}$$

$$\times \sum_{\{n+m-i-j\}''} (A_{n-i})^{(L)}_{\{n-i\};\{j-m\}'\{n+m-i-j\}''}(B_{n-j})^{(L)}_{\{i-m\}\{n+m-i-j\}'';\{n-j\}'} \; . \quad \text{(B4)}$$

A factor $[1/{}_iC_m]P^{(m)}_{1\cdots i}$ that could have appeared in (B4) was replaced by unity, following our general practice of not writing the symmetrization over the nonprimed indices explicitly.

The two symmetrizers $P_{a'}^{(j)}$ and $P^{(m)}_{1'\cdots j'}$ symmetrize first the primed indices in the Δ and the (A_{n-i}) factors, and then symmetrize them with those in the (B_{n-j}) factor. We shall now modify this symmetrization into a form so that the indices in the (A_{n-i}) and (B_{n-j}) factors are symmetrized first, and then with those in the Δ factor. The result is written as

$$(A\cdot B)_{aa'} = \sum_{i,j} \sum_{m} [{}_{n-j}C_{i-m}] P_{a''}^{(m)} \Delta_{(m);(m)'}$$

$$\times \sum_{\{n+m-i-j\}''} P^{(j-m)}_{(m+1)'\cdots n'} (A_{n-i})^{(L)}_{\{n-i\};\{j-m\}'\{n+m-i-j\}''}(B_{n-j})^{(L)}_{\{i-m\}\{n+m-i-j\}'';\{n-j\}'} \; . \quad \text{(B5)}$$

In (B5), let us change the summation indices m, i, and j, into i, l, and m, respectively, and then set $l=i+s$ and $m=i+t$. We then find that (B5) is nothing but (4.15), together with (4.16). Theorem I has thus been proved.

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[1]T. Kishimoto, and T. Tamura, Nucl. Phys. A192, 246 (1972).

[2]T. Kishimoto and T. Tamura, Nucl. Phys. A270, 317 (1976).

[3]R. Tamura, K. J. Weeks, and T. Kishimoto, Phys. Rev. C 20, 307 (1979).

[4]K. J. Weeks and T. Tamura, Phys. Rev. C 22, 888 (1980).

[5]K. J. Weeks and T. Tamura, Phys. Rev. Lett. 44, 533 (1980); Phys. Rev. C 22, 1323 (1980).

[6]K. J. Weeks, T. Tamura, T. Udagawa, and F. J. W. Hahne, Phys. Rev. C 24, 703 (1981).

[7]T. Tamura, K. J. Weeks, and T. Kishimoto, Nucl. Phys. A347, 359 (1980).

[8]E. R. Marshalek, Nucl. Phys. A347, 253 (1980); Phys. Lett. 95B, 337 (1980).

[9]T. Marumori, M. Yamamura, and A. Tokunaga, Prog. Theor. Phys. 31, 1009 (1964).

[10]K. A. Brueckner, Phys. Rev. 97, 1353 (1955); 100, 36 (1955).

[11]J. Goldstone, Proc. R. Soc. London 239, 267 (1957).

[12]T. Tamura, K. J. Weeks, and V. G. Pedrocchi, Phys. Rev. C 23, 1297 (1981).

[13]S. G. Lie and G. Holzwarth, Phys. Rev. C 12, 1035 (1975): See also T. Marumori, K. Takada, and F. Sakata, Prog. Theor. Phys. Suppl. 71, 1 (1982).

[14]G. Holzwarth, D. Jansen, and R. V. Jolos, Nucl. Phys. A261, 1 (1976).

[15]B. Sorensen, Nucl. Phys. A97, 1 (1967); A119, 65 (1968); A142, 392, 411 (1970).

[16]V. G. Pedrocchi and T. Tamura, Prog. Theor. Phys. (in press).

[17]K. J. Weeks and T. Tamura, Phys. Rev. C 21, 1160 (1980).

[18]B. Silvestre-Brac and R. Piepenbring, Nucl. Phys. A288, 408 (1977); Phys. Rev. C 20, 1161 (1979).

[19](a) V. G. Pedrocchi and T. Tamura (unpublished). (b) Since we consider the nuclear collective motions as due to the creation of the particle-hole pairs, these fermions $d^\dagger$ and d are to be intepreted to mean the quasiparticles in the sense of the BCS theory.

[20]N. Fukuda, K. Sawada, and M. Taketani, Prog. Theor. Phys. (Kyoto) 12, 156 (1954): S. Okubo, *ibid.* 12, 102 (1954); 12, 603 (1954).

[21]B. H. Brandow, *Lecture Notes in Physics* (Springer, Berlin, 1975), Vol. 40; H. Hyuga and H. Otsubo, Nucl. Phys. A294, 348 (1978).

[22]E. R. Marshalek, Nucl. Phys. A161, 401 (1971); D. Janssen, F. Donau, S. Frauendorf, and R. V. Jolos, *ibid.* A172, 145 (1971); E. R. Marshalek and G. Kolzwarth, *ibid.* A191, 438 (1972); F. Donau and D. Janssen, *ibid.* A209, 109 (1973); D. Janssen, R. V. Jolos, and F. Donau, *ibid.* A224, 93 (1974); P. Ring and O. Schuck, Phys. Rev. C 16, 801 (1977); J. P. Blaozot and E. R. Marshalek, Nucl. Phys. A309, 422 (1978); A309, 453 (1978); C. T. Lie and A. Klein, Phys. Rev. C 19, 2023 (1979); A. Klein, Comm. Nucl. Part. Phys. 10, 9 (1981).

[23]F. Dönau and D. Janssen, Nucl. Phys. A209, 109 (1973); E. R. Marshalek, *ibid.* A224, 221 (1974); A224, 245 (1974); S. Okubo, Phys. Rev. C 10, 2048 (1974); H. B. Geyer and F. J. W. Hahne, Nucl. Phys. A363, 45 (1981).

[24]K. Allaart, P. Hofstra, and U. Paar, Nucl. Phys. A366, 384 (1981).

[25]A. Kriyama, T. Marumori, F. Sakata, and T. Suzuki, Prog. Theor. Phys. Suppl. 58, 9 (1975).

[26]T. Suzuki, M. Fuyuki, and K. Matsuyanagi, Prog. Theor. Phys. (Kyoto) 61, 1682 (1979); 62, 690 (1979); 65, 1667 (1981).