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Coupled πNN-NN systems in a Hamiltonian approach and in a relativistic off-mass-shell formalism

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We discuss the scattering theory pertaining to the coupled πNN-NN system and approach the problem in two independent ways. The first one starts from a Hamiltonian formalism and coupled Schrödinger equations, whereas the second one employs an off-mass-shell relativistic theory of classifying perturbation diagrams. Both ways lead to connected equations among transition operators in which πNN vertices, as well as nucleon propagators, are completely dressed and renormalized. Furthermore, the physical amplitudes obey two- and three-body unitarity relations. The resultant equations form a sound theoretical basis for subsequent numerical calculations leading to the evaluation of physical observables in the reactions $\pi+d\rightarrow\pi+d$, $\pi+d\leftrightarrow N+N$, and $N+N\rightarrow N+N$.

NUCLEAR REACTIONS Coupled πNN-NN equations, Hamiltonian approach, off-mass-shell approach, dressed vertex and propagator, last cut lemma.

I. INTRODUCTION

In this paper we intend to present a complete scattering theory of the coupled πNN-NN system. It has been realized that the scattering problem pertaining to these systems plays a crucial role in intermediate energy pion nuclear physics, and therefore, the motivation for our extensive study is very clear.

In a previous work (I) (Ref. 1) we started from a Hamiltonian formalism and applied the Feshbach projection method, as was previously worked out by Mizutani and Koltun (MK).[2] Assuming that the πN interaction in the P_{11} partial wave is due entirely to the nucleon pole term, we then derived integral equations for amplitudes describing the reactions $\pi NN(\pi d)\rightarrow\pi NN(\pi d)$, $\pi NN(\pi d)\leftrightarrow NN$, and $NN\rightarrow NN$ (including inelastic effects due to single pion production). In a second work (II) (Ref. 3) we (i) related I and some previous works, (ii) included the effect of the background contribution in the two body πN P_{11} partial wave (which we termed the nonpole P_{11} contribution or, briefly, NPP_{11}) in such a way that the coupled integral equations retain compact kernels, and (iii) tried to construct a corresponding set of equations with relativistic invariance guaranteed. The interrelation between our results and those based upon the (so called) bound state picture[4] were studied in some detail recently.[5]

The initial motivation for the present work was due to our feeling that the study of the nonpole contribution to the πN P_{11} partial wave (NPP_{11}) needs a further elaboration than the one we gave in II. As is well known, the pole term alone cannot fit the πN P_{11} phase shift δ (P_{11}) [which changes sign at $T_\pi(\text{lab})\approx150$ MeV] so that one basically needs the NPP_{11}. Preliminary numerical results[6] indicate that the inclusion of NPP_{11} substantially affects elastic πd scattering above $T_\pi=200$ MeV. Thus, we were obliged to recast the equations obtained in II [especially Eq. (3.24)] in a form which is easier to handle practically.

Our motivation was recently amplified when we noticed that the Taylor approach to relativistic quantum field theory[7] can be employed to find a set of generalized Bethe-Salpeter equations, giving an off-mass-shell description for the pertinent system. Hence, we found it appropriate to present the two (essentially different) approaches in the present work, since they lead to equations sharing apparently the same basic structure, namely: (i) the equations have a multichannel Lippmann-Schwinger form (nonrelativistic) or Bethe-Salpeter form (reducible to Blankenbecler-Sugar type equations) in the relativistic case; (ii) the $NN\pi$ vertices, as well as the nucleon propagators, are properly dressed; and (iii) all the propagators are renormalized to correctly implement two and three body unitarity to the scattering amplitudes.

In Sec. II we show how to obtain the desired equations starting from the coupled Schrödinger equations of MK. We rely on results obtained in I and II but the presentation is definitely self-contained. The central result of Sec. II is expressed in Eqs. (2.9), which are connected equations for the evaluation of the physical transition operators. In these equations, the πNN vertices and the NN propagator are fully dressed. In fact, these equations, when used in conjunction with the separable approximation (SA), have been *implicitly* obtained in II, namely, Eqs. (3.24). The step which makes the latter more practical is the use of a simple operator identity [see Eq. (2.7)]. It is worth pointing out here that Sec. II contains the only successful attempt so far to obtain the equations starting from a Hamiltonian formulation.

Then, in Sec. III we shall study the relativistic off-mass-shell approach to the problem. As we pointed out in Sec. IV of II, relativistic formulations (and extensions) go much beyond the simple kinematic considerations. Therefore, we present the construction of the relativistic coupled equations based on the Taylor approach.[7] In the context of the coupled πNN-NN systems it was first applied by Mizutani.[8] Here we proceed further along this line and show how the diagrammatic method can be used to derive generalized coupled Bethe-Salpeter equations for the πNN-NN system (with relativistic invariance guaranteed). We note the similarity in the form of these coupled equations [see Eqs. (3.29) and (3.30)] and of those derived in Sec. II (through a Hamiltonian formalism and an off-energy-shell reduction). This similarity is just like the one between the two body Bethe-Salpeter equation and the corresponding nonrelativistic Lippmann-Schwinger equation. The basic input to the relativistic equations includes irreducible vertices and Feynman propagators, while three body forces are retained in the derivation. (In order to have a self-contained presentation we also introduce the basic concept of the Taylor method.)

The equations obtained in this paper are combined with the antisymmetrization procedure (worked out in I), a separable assumption for the two body amplitudes together with the explicit form of the πNN vertices and the two nucleon propagator (their nonrelativistic counterpart is given in the Appendix of II), to form the basis for numerical evaluation of various observables.

There is another approach by Afnan and Blankleider[9] who attacked the problem using a time ordered perturbation theory equipped with the method of classifying diagrams. We shall comment on this method further on, but here we point out that their final equations are *physically equivalent* (in a sense to be clear later on) to Eqs. (3.24) in II and are identical in form with Eqs. (2.9) below.

Before going on, it is useful to point out that although the two approaches presented in Sec. III and IV, respectively, are quite different and have no overlap, we found it appropriate to combine them in order to convey our feeling that the formal theory is now completed. We nevertheless think that, whereas the Hamiltonian approach presented here is a modification of the previous results, the more novel part of the present work is the off-mass-shell approach of Sec. IV.

II. DERIVATION OF THE EQUATIONS FROM THE COUPLED SCHRÖDINGER EQUATIONS OF MK

In this section we shall briefly show how the equations for the coupled πNN-NN system can be derived in the Hamiltonian formalism. It consists of an extension of our previous results (upon which we shall heavily rely), but the ensuing equations are much simpler and physically transparent. We find it encouraging to realize that, despite the complexity of the system under consideration, the nonrelativistic case can be handled on the same footing as ordinary potential scattering problems. The use of more sophisticated tools, such as summing perturbation diagrams, is then reserved solely for the off-mass shell approach which is detailed in Sec. III.

In order to allow for smooth reading, we give here only the main lines of the derivation. Whenever possible we refer the reader to our previous works I and II. We also relegate to the Appendix most of the required definitions, as well as the explicit form of most of the operators.

Our starting point is Eqs. (3.2) and (3.3) of I, which in matrix form read

$$U=B+B\Gamma_0 U \ , \tag{2.1}$$

where $U=\{U_{ab}\}$ is a matrix of transition operators in channel space which includes three kinds of channels: (i) pair-spectator three body channels $\pi(NN)$ and $N(\pi N)$, in which the pair (πN) *does not* interact in the NPP_{11}. These physical channels (which can be reached as asymptotic states) are denoted by $\alpha,\beta,\gamma,\ldots,$ in this sequel; (ii) the two nucleon channel, denoted simply as "N"; (iii) in case the pion and the nucleon can also interact in the NPP_{11} (a situation excluded in I) there are also nonphysical pair-spectator states α_i $(i=1,2)$ corresponding to the clustering $N_j(\pi N_i)$ in which the pion and nucleon N_i *do* interact in the NPP_{11}. The other symbols in Eq. (2.1) are the "potential" matrix B given in Eq. (A1) and the propagator matrix detailed in Eq. (A2).

Equation (2.1) suffers from the following three drawbacks; all of them stem from the occurrence of the nonphysical channels of type (iii). We shall refer to these drawbacks as points (1)−(3) in this sequel. (1) The equation contains disconnected pieces as explained in detail in II. These disconnected terms are shown graphically in Fig. 4(a) in II. The πNN vertices are not dressed with the NPP_{11} interaction. Equation (2.1) contains the undressed vertices $R_i, R_i^\dagger$ defined in (2.3) in I and what we need are the dressed ones y_i, z_i defined in (3.3) in II. (3) The two nucleon propagator is not dressed with the NPP_{11}. Equation (2.1) contains the propagator τ_N defined in (2.15a) in I and what we need is the completely dressed two nucleon propagator Π_{22} which is defined in (3.21d) in II.

We are now going to remedy all the aforementioned flaws in five main steps using the formal theory of scattering from two potentials (which are the disconnected and connected parts of the matrix B, namely B^d and B^c). As it turns out, the removal of the disconnected parts results in both vertex and propagator dressing so that all three points are taken care of simultaneously. Now for the details:

(a) The matrices U and B are decomposed into their respective connected and disconnected parts as

$$U = U^c + U^d \,, \tag{2.2a}$$

$$B = B^c + B^d \,, \tag{2.2b}$$

and it should be kept in mind that only the connected parts have physical meaning. The explicit forms of B^d and B^c are given in Eqs. (A3) and (A4), respectively.

(b) The disconnected part of U, namely U^d, can be explicitly calculated by solving a matrix Lippmann-Schwinger equation in closed form. The resulting U^d is then employed to construct the Møller operators Ω_d and Ω^d, which are needed in the distortion procedure occurring in the formal theory of scattering from two potentials. The equations determining U^d and the pertinent Møller operators are then [see Eq. (A3′)]

$$U^d = B^d + B^d\Gamma_0 U^d = B^d\Omega^d = \Omega_d B^d \,, \tag{2.3a}$$

$$\Omega^d = 1 + \Gamma_0 U^d, \quad \Omega_d = 1 + U^d\Gamma_0 \,. \tag{2.3b}$$

Since we are going to factorize the Møller operators later on we will be content to give the explicit form of each factor when the factorization is carried out.

(c) The connected part of U, namely U^c (which for physical channels contains the physical transition operators) can now formally be evaluated using the equations for scattering from the sum $B^d + B^c$ of two potentials, namely,

$$U^c = \Omega_d Y^c \Omega^d \,, \tag{2.4a}$$

$$Y^c = B^c + B^c\Gamma Y^c \,, \tag{2.4b}$$

$$\Gamma = \Gamma_0\Omega_d = \Omega^d\Gamma_0 \,. \tag{2.4c}$$

The precise expression for Γ is explained in the Appendix following Eq. (A6). Equations (2.4) describe an algorithm from which it is possible to compute the physical transition operators, since the equations are connected. Thus, point (1) has been remedied but not points (2) and (3). Besides, the new propagator matrix Γ is not diagonal. We therefore need the last two steps which go beyond the results obtained already in II.

(d) It is possible to factorize the Møller operators Ω_d and Ω^d according to

$$\Omega_d = \eta_d\omega_d, \quad \Omega^d = \omega^d\eta^d \tag{2.5}$$

in such a way that the following two conditions are met: (i) As far as *physical* transition operators are concerned, they are given by

$$\bar{U}^c \equiv \omega_d Y^c \omega^d \,, \tag{2.6a}$$

which are related to U^c of Eq. (2.4a) by

$$U^c = \eta_d \bar{U}^c \eta^d \,. \tag{2.6b}$$

In other words, the matrices of operators η_d and η^d, when evaluated *on shell,* deviate from the unit matrix only through elements involving the nonphysical channels α_i. (ii) The nondiagonal propagator matrix Γ appearing in Eqs. (2.4b) and (2.4c) can be written in the form

$$\Gamma = \omega^d\gamma\omega_d \,, \tag{2.7}$$

where γ is *diagonal* and coincides with the original propagator matrix Γ_0 in Eqs. (2.1) and (A2), except for the crucial difference that in γ the two nucleon propagator is the *fully-dressed* one, namely Π_{22}. Thereby, point (3) is corrected. The substantiation of step (d) is straightforward and will not be detailed here. The explicit forms of η_d and ω_d are shown in Eqs. (A5) and (A6), respectively. We point out that the simple procedure detailed in step (d) eluded us in II and when we noticed it, it was a few weeks too late for a "note added in proof." Now for the last step.

(e) If we now sandwich the two sides of Eq. (2.4b) between ω_d on the left and ω^d on the right, we get [employing Eq. (2.7) for Γ] the integral equation for $\bar{U}^c$ directly

$$\bar{U}^c = V^c + V^c \gamma \bar{U}^c , \tag{2.8a}$$

where

$$V^c = \omega_d B^c \omega^d \tag{2.8b}$$

is identical to B^c, except that now all the πNN vertices are *fully dressed* also with the NPP_{11}. Therefore point (2) has been remedied and this completes our quest. The explicit expressions for the "potentials" V^c and the propagators γ are given in Eqs. (A7) and (A8), respectively, so that the input to our final equations is well specified.

We prefer, for clarity, to also write this equation with regard to elements. In doing so, we recall that Eq. (2.8a) is equivalent to several sets of coupled equations; each one couples all the amplitudes with the same initial channel. Since we are interested in the *physical* channels α, β, and N, we will obtain two sets of equations, one for $\bar{U}^c_{a\beta}$ and one for $\bar{U}^c_{aN}$ ($a = \alpha, \beta, \gamma, \ldots, \alpha_1, \alpha_2, N$) in which the elements $\bar{U}^c_{\alpha_i \beta}$ and $\bar{U}^c_{\alpha_i N}$ enter as scaffolding while $\bar{U}^c_{\alpha\beta}$, $\bar{U}^c_{N\beta}$, $\bar{U}^c_{\alpha N}$, and $\bar{U}^c_{NN}$ are the pertinent physical transition operators. Thus we have

$$\begin{aligned}
\bar{U}^c_{\alpha\beta} &= \bar{\delta}_{\alpha\beta} G_0{}^{-1} + \sum_{\gamma \neq \alpha} t_\gamma G_0 \bar{U}^c_{\gamma\beta} + \sum_i t_{\alpha_i} G_0 \bar{U}^c_{\alpha_i \beta} + z_N \Pi_{22} \bar{U}^c_{N\beta} , \\
\bar{U}^c_{N\beta} &= y_N + \sum_\gamma y_N G_0 t_\gamma G_0 \bar{U}^c_{\gamma\beta} + \sum_{i \neq j} y_i G_0 t_{\alpha_j} G_0 \bar{U}^c_{\alpha_j \beta} + w_{NN} \Pi_{22} \bar{U}^c_{N\beta} , \\
\bar{U}^c_{\alpha_i \beta} &= G_0{}^{-1} + \sum_\gamma t_\gamma G_0 \bar{U}^c_{\gamma\beta} + \bar{\delta}_{ij} t_{\alpha_j} G_0 \bar{U}^c_{\alpha_j \beta} + \bar{\delta}_{ij} z_j \Pi_{22} \bar{U}^c_{N\beta} ,
\end{aligned} \tag{2.9a}$$

$$\begin{aligned}
\bar{U}^c_{\alpha N} &= z_N + \sum_{\gamma \neq \alpha} t_\gamma G_0 \bar{U}^c_{\gamma N} + \sum_i t_{\alpha_i} G_0 \bar{U}^c_{\alpha_i N} + z_N \Pi_{22} \bar{U}^c_{NN} , \\
\bar{U}^c_{NN} &= w_{NN} + \sum_\gamma y_N G_0 t_\gamma G_0 \bar{U}^c_{\gamma N} + \sum_{i \neq j} y_i G_0 t_{\alpha_j} G_0 \bar{U}^c_{\alpha_j N} + w_{NN} \Pi_{22} \bar{U}^c_{NN} , \\
\bar{U}^c_{\alpha_i N} &= \bar{\delta}_{ij} z_j + \sum_\gamma t_\gamma G_0 \bar{U}^c_{\gamma N} + \bar{\delta}_{ij} t_{\alpha_j} G_0 \bar{U}^c_{\alpha_j N} + \bar{\delta}_{ij} z_j \Pi_{22} \bar{U}^c_{NN} .
\end{aligned} \tag{2.9b}$$

Equations (2.9) present, in some detail, the central result obtained in this section and employ the nonphysical nature of the operators involving NPP_{11} initial and final states. As already mentioned, no further operation is needed for evaluating the physical amplitudes

$$T_{ab} = \langle \Phi_a \mid \bar{U}^c_{ab} \mid \Phi_b \rangle , \quad a \cup b \neq \alpha_1, \alpha_2 .$$

The relation $U^c = \eta_d \bar{U}^c \eta^d$ is hence redundant and the operators U^c and $\bar{U}^c$ are indeed *physically equivalent*, as we have termed in the Introduction. Since the equations in II were constructed for U^c, we have substantiated our claim that the equations of II and the present equations are physically equivalent.

In order to establish unitarity we need to know the amplitudes involving the channel "0" of the three (πNN) free particles. These are first given in terms of U^c [Eq. (2.4a)], and using straightforward algebra one can relate them to $\bar{U}^c$ of Eqs. (2.9). Thus, for *physical* channels $a \cup b \neq \alpha_1, \alpha_2$ we have

$$\begin{aligned}
U^c_{0b} &= \sum_{c \neq N} t_c G_0 U^c_{cb} + R^\dagger_{N\tau_N} U^c_{Nb} = \bar{U}^c_{0b} \\
&\equiv \sum_{c \neq N} t_c G_0 \bar{U}^c_{cb} + z_N \Pi_{22} \bar{U}^c_{Nb} , \\
&\qquad (a \cup b \neq \alpha_1, \alpha_2) , \\
U^c_{a0} &= \sum_{c \neq N} U^c_{ac} G_0 t_c + U^c_{aN} \tau_N R_N = \bar{U}^c_{a0} \\
&\equiv \sum_{c \neq N} \bar{U}^c_{ac} G_0 t_c + \bar{U}^c_{aN} \Pi_{22} y_N .
\end{aligned} \tag{2.10}$$

Notice that in these expressions c runs on $\alpha, \beta, \gamma, \ldots, \alpha_1, \alpha_2$. With these expressions it is not difficult to show that $\bar{U}^c_{ab}$ ($a \cup b \neq \alpha_1, \alpha_2$) satisfy two ($NN$) and three ($\pi NN$) body unitarity, namely

$$\begin{aligned}
\bar{U}^c_{ab}(E + i\epsilon) - \bar{U}^c_{ab}(E - i\epsilon) = -2\pi i [&\bar{U}^c_{a0}(E + i\epsilon) \delta(E - H_0) \bar{U}^c_{0b}(E - i\epsilon) \\
&+ \bar{U}^c_{aN}(E + i\epsilon) \delta(E - h_0) \bar{U}^c_{Nb}(E - i\epsilon)] .
\end{aligned} \tag{2.11}$$

We have thus completed our quest of deriving the desired equations starting from the Hamiltonian formalism of I and II by making the kernel of the equations connected (compact). Through this (mathematical) procedure, we exposed much physics in the sense that (originally) incompletely dressed nucleon propagators and πNN vertices are now fully dressed.

In contrast to this result, we shall observe in Sec. III that an approach starting with the dressed vertices and propagators leads directly to the connected set of coupled integral equations without any recourse to the procedure of making the kernel (of the integral equation) compact. To end this section we note that the final set of equations (2.9) is essentially the same as that obtained in Ref. 9 through the classification of perturbation graphs, which is similar in spirit to what we shall study in the next section.

III. OFF MASS-SHELL EQUATIONS

A. Preliminaries

To complete our formal studies in the coupled πNN-NN systems we want to have an off-mass-shell description in terms of generalized coupled Bethe-Salpeter equations. Under certain approximations such equations are then reduced most naturally to the form with Blankenbecler-Sugar propagators and then further to the nonrelativistic set of equations derived previously (both of them are amenable to practical calculations). In accomplishing this task we shall utilize an approach to field theories by Taylor,[7] who established the rules of unambiguously classifying diagrams. These rules then render the analysis of the structure of scattering amplitudes reduced to a combinatorial problem associated with diagrams. The method was applied by one of us[8] to the πNN-NN problem up to the point where the equations obtained are only partially coupled (its nonrelativistic form has been obtained by MK using the projection operator method). In our present study we shall work out this procedure further to find a set of completely coupled equations.

We shall not give here any detailed account for the Taylor method due to lack of space. The interested reader is referred to the original work,[7] whereas here we shall just sketch some necessary background.

(i) We consider an off-mass-shell amplitude with m initial and n final particles obtained from a causal Green's function. To simplify the discussion we consider only scalar fields. Then the Green's function reads

$$G_{nm}=i^{n+m}\langle 0\,|\,T[\Phi_1(x_1)\cdots\Phi_n(x_n)\Phi^\dagger_{n+1}(x_{n+1})\cdots\Phi^\dagger_{n+m}(x_{n+m})\,|\,0\rangle\ .$$

By removing the $n+m$ *dressed* single particle propagators we associate this amplitude with a *diagram* having m initial and n final legs.

(ii) A perturbation graph is defined as being composed of external particle legs, internal lines representing *dressed and renormalized* single particle propagators, and dressed vertices (internal and external).

(iii) A diagram with m initial and n final particle legs is considered the formal summation of all possible (topologically distinct) perturbation graphs with the same number of initial and final external legs.

(iv) A k *cut* is an arc with no multiple points (that is, it does not intersect itself) which intersects k particle lines in a given graph or diagram to separate the initial and final states. It must intersect at least one internal line and should not intersect a particle line more than once (see Fig. 1).

(v) A graph or a diagram is called r *irreducible* in a given channel (that is, initial and final state specified) if it does not admit any k cut with $k\leq r$ (see Fig. 2).

(vi) *The last cut lemma (LCL).* For a given graph or diagram which is $(r-1)$ irreducible there is a unique r cut which is closest either to the initial or the final states *provided the graph (or diagram) is connected.* As usual, a graph is said to be connected if it cannot be split into two (or more) separate pieces unless one breaks a single internal line (see an example in Fig. 3). It is important to keep in mind that the LCL does not hold for disconnected graphs (or diagrams).

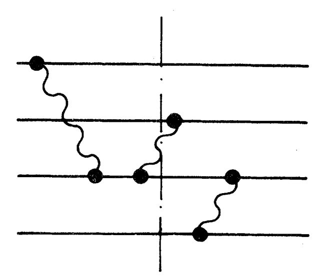

FIG. 1. An example of a K cut for $K=5$.

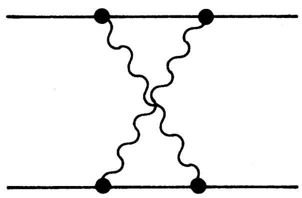

FIG. 2. An example of an r-irreducible graph for $r=3$.

(vii) *Complete unitarity (CU).* Implemented by (ii) and (iii), CU guarantees that diagrams with the same initial and final external leg structure together with the same irreducibility are identical. For example, suppose that by use of the LCL one finds the following decomposition of amplitude A

$$A=B+CGA' ,$$

where both A and A' have the same external leg structure and irreducibility. Then $A=A'$ and the above relation defines an integral equation for A. As may be clear from this example, CU is essentially important to guarantee unitarity of the scattering amplitude and is so termed.[7]

(viii) No specific form of the interaction Lagrangian is needed.

With the LCL and complete unitarity as guiding principles one can expose a definite number of intermediate particle lines in a given diagram step by step, thus expressing a certain amplitude in terms of other amplitudes with higher irreducibility.

Before going on it may be worth mentioning the approach of Ref. 9 which is based on a previous work by Thomas and Rinat.[11] The method used in these works is close in spirit to that of Taylor except that it is based upon a definite time ordered theory without antiparticle degrees of freedom. It should also be mentioned that in order to derive the equations they employ a procedure corresponding to the LCL (without CU), which, however, is also used for amplitudes containing *disconnected* pieces. So far, the work of Taylor is the most fundamental and rigorous algorithm for classifying perturbation diagrams by the LCL. In his papers, he discusses at length how the use of the LCL with disconnected diagrams might lead to erroneous conclusions. The fact that the final equations of Ref. 9 are indeed the correct ones indicates either that there are some fortunate cancellations or that there may be some kind of theorem which covers this special case. It is our judgement that this point needs further substantiation.

B. Study in the πN-N and NN sectors

In order to familiarize the reader with the Taylor method it may be instructive to first apply it to the two body ($\pi N\rightarrow\pi N$) and ($NN\rightarrow NN$) problems as well as the $\pi N\leftrightarrow N$ problems. We write a *connected amplitude* for the process m (initial particles)$\rightarrow n$ (final particles) with irreducibility r as

$$M_{nm}^{(r)}=(\text{connected amplitude having } m \text{ initial and } n \text{ final particle legs and irreducibility } r) , \qquad (3.1)$$

which is diagrammatically depicted in Fig. 4(a). The dressed single particle propagator for particle j ($j=N$ or π in our case) is given by

$$d_j\equiv i\Delta_f'(j) , \qquad (3.2)$$

where $\Delta_F'(j)$ is the usual Feynman propagator for scalar particles (recall that for simplifying the arguments we assume all the particles to be scalar). A dressed single particle Feynman propagator is graphically represented by a straight line and a dot [see Fig. 4(b)].

Now, we consider the following:

(a) *$NN\rightarrow NN$ amplitudes.* Since NN scattering does not go through the single (elementary) particle intermediate state, the amplitude is $M_{22}^{(1)}$ in the

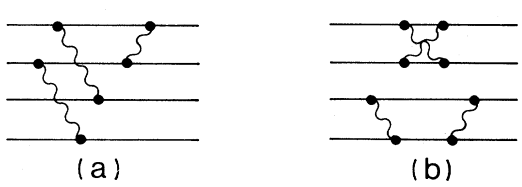

FIG. 3. Example of (a) connected and (b) disconnected graphs.

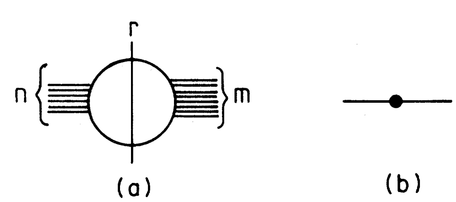

FIG. 4. (a) Symbolic representation of the amplitude (with m initial and n final particles) which is r irreducible [see Eq. (3.1)]. (b) Single particle dressed and renormalized Feynman propagator d_j [see Eq. (3.2)].

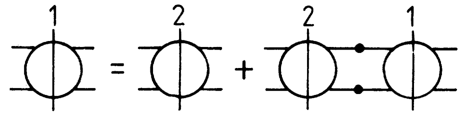

FIG. 5. Graphical description of Eq. (3.3) for the one particle irreducible two nucleon amplitude $M_{22}^{(1)}(NN)$.

sense of Eq. (3.1) and the LCL leads to

$$M_{22}^{(1)}(NN)=M_{22}^{(2)}(NN)[1+d_{N_1}d_{N_2}M_{22}^{(1)}(NN)]$$
$$=[1+M_{22}^{(1)}(NN)d_{N_1}d_{N_2}]M_{22}^{(2)}(NN)\ , \tag{3.3}$$

whose diagrammatic structure is shown in Fig. 5.

(b) *$\pi N \rightarrow \pi N$ amplitudes.* There is an s channel nucleon pole contribution, hence the amplitude should be one particle reducible (that is, it has an intermediate state with a single particle line). The LCL actually gives

$$M_{22}(\pi N)=M_{22}^{(1)}(\pi N)+M_{21}^{(1)}(\pi N)d_N M_{12}^{(1)}(\pi N)\ , \tag{3.4}$$

as seen in Fig. 6. Obviously, the separation of the dressed (direct) pole term and the nonpole term is clearly demonstrated here. For the nonpole part $M_{22}^{(1)}(\pi N)$ we find an equation similar to the one satisfied by $M_{22}^{(1)}(NN)$ [see Eq. (3.3)]. Note that these equations for $M_{22}^{(1)}$ are just of the form of the Bethe-Salpeter equation, where the two particle irreducible amplitudes $M_{22}^{(2)}$ serve as two body potentials.

(c) *$\pi N \leftrightarrow N$ amplitudes (pion nucleon vertex).* Since the *dressed propagators* are always taken off the Green's function, this vertex is at least one-particle irreducible. One then finds

$$M_{21}^{(1)}(\pi N)= M_{21}^{(2)}(\pi N)+M_{22}^{(1)}(\pi N)d_N d_\pi M_{21}^{(2)}(\pi N)$$
$$= M_{21}^{(2)}(\pi N)+M_{22}^{(2)}(\pi N)d_N d_\pi M_{21}^{(1)}(\pi N) \tag{3.5}$$

with the corresponding diagrams drawn in Fig. 7.

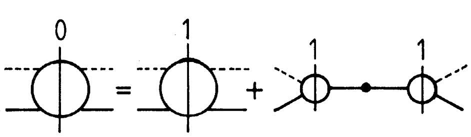

FIG. 6. Graphical form of Eq. (3.4), giving the decomposition of the $\pi N \rightarrow \pi N$ amplitude $M_{22}(\pi N)$.

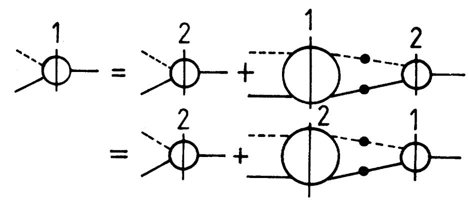

FIG. 7. Graphical form of the integral Eq. (3.5) for the $\pi N \leftrightarrow N$ vertices $M_{21}^{(1)}(\pi N)$.

Similar decomposition holds for $M_{12}^{(1)}(\pi N)$. Clearly, $M_{12}^{(1)}(\pi N)$ and $M_{21}^{(1)}(\pi N)$ correspond to the dressed vertices y_i and z_i, respectively [see Eq. (2.16)]. Also, they should be properly renormalized to give the πNN strong coupling constant when all the particles are put on the mass shell. We note here that this result has been derived in the Green's function approach by Nutt and Shakin[12] in the *PS-PS* theory of πN scattering, where $M_{21}^{(2)}(\pi N)$ is set equal to the strong interaction coupling constant $g_{\pi NN}$.

(d) The nucleon propagator admits the following decomposition

$$d_N=d_N^{(0)}+d_N^{(1)}\ ,$$
$$d_N^{(1)}=d_N^{(2)}+M_{12}^{(2)}(\pi N)d_\pi d_N M_{21}^{(1)}(\pi N)\ , \tag{3.2'}$$

in which $d_N^{(0)}$ is the bare propagator.

In order to study the structure of the NN "potential" $M_{22}^{(2)}(NN)$ we should expose its three particle intermediate states. This will be done in the next subsection.

C. Study in the coupled πNN-NN sector

Since our interest is in the pion interaction with the two nucleon system, any three body intermediate states to be exposed by the cutting procedure will be restricted to πNN states. In other words, heavy mesons are regarded as multipion states. Alternatively, it is always possible to also consider "elementary" heavy mesons; in this case, one should expose various three particle states so that amplitudes like $\pi NN \leftrightarrow \rho NN$, $NN\rho \rightarrow NN\rho$, etc., as well as two body inputs $\pi N \leftrightarrow \rho N$, should also be considered.

For convenience we introduce the following notations:

(i) The two nucleons are labeled "1" and "2," while the pion is labeled "3."

(ii) In the presence of the spectator particle i the amplitudes for $j+k \rightarrow j+k$ ($j \neq i \neq k \neq j$) and $j+k \leftrightarrow j$ ($j \neq 3$ in the $2 \leftrightarrow 1$ case) introduced in the previous subsection are redefined as

$$\begin{aligned} &m^{(r)}(i)\equiv M_{22}^{(r)}(jk)d_i^{-1} ,\\ &\gamma_+^{(r)}(i)\equiv M_{21}^{(r)}(jk)d_i^{-1} ,\\ &\gamma_-^{(r)}(i)\equiv M_{12}^{(r)}(jk)d_i^{-1} ,\\ &(j\neq i\neq k\neq j) . \end{aligned} \tag{3.6}$$

This notation is in line with the "odd-man-out" prescription extensively used in many body problems. Note that $\gamma_\pm^{(r)}(i)$ is defined only for $i=1,2$. We extend it simply by also admitting $i=3$, with the understanding that $\gamma_\pm^{(r)}(3)\equiv 0$.

(iii) Finally (for the notation) we define free propagators

$$G_2\equiv d_1d_2 ,$$

$$G_3\equiv d_1d_2d_3 .$$

With the above preparation we go on to the analysis of three particle amplitudes.

(a) $\pi NN\to\pi NN$. First we notice that it is one-particle irreducible. The LCL leads to

$$\begin{aligned} M_{33}^{(1)}&=M_{33}^{(2)}+M_{32}^{(1)}G_2M_{23}^{(2)}+\sum_i\gamma_+^{(1)}(i)G_2M_{23}^{(2)}+\sum_i M_{32}^{(1)}G_2\gamma_-^{(1)}(i)\\ &=M_{33}^{(2)}+M_{32}^{(2)}G_2M_{23}^{(1)}+\sum_i\gamma_+^{(1)}(i)G_2M_{23}^{(1)}+\sum_i M_{32}^{(2)}G_2\gamma_-^{(1)}(i) , \end{aligned} \tag{3.7}$$

as we see in Fig. 8.

(b) $\pi NN\leftrightarrow NN$. We find

$$\begin{aligned} M_{23}^{(1)}&=M_{23}^{(2)}+M_{22}^{(1)}(NN)G_2M_{23}^{(2)}+\sum_i M_{22}^{(1)}(NN)G_2\gamma_-^{(1)}(i)\\ &=M_{23}^{(2)}+M_{22}^{(2)}(NN)G_2M_{23}^{(1)}+\sum_i M_{22}^{(2)}(NN)G_2\gamma_-^{(1)}(i) . \end{aligned} \tag{3.8}$$

An analogous result holds for $M_{32}^{(1)}$.

(c) *Two particle irreducible amplitudes for $\pi NN\to\pi NN$.* With the LCL one easily finds

$$\begin{aligned} M_{33}^{(2)}&=M_{33}^{(3)}+M_{33}^{(2)}G_3M_{33}^{(3)}+\sum_i M_{33}^{(2)}G_3m^{(2)}(i)+\sum_i m^{(1)}(i)G_3M_{33}^{(3)}+\sum_{i\neq j}m^{(1)}(i)G_3m^{(2)}(j)\\ &=M_{33}^{(3)}+M_{33}^{(3)}G_3M_{33}^{(2)}+\sum_i M_{33}^{(3)}G_3m^{(1)}(i)+\sum_i m^{(2)}(i)G_3M_{33}^{(2)}+\sum_{i\neq j}m^{(2)}(i)G_3m^{(1)}(j) . \end{aligned} \tag{3.9}$$

In analogy with $M_{22}^{(2)}$ (appearing in $2\to 2$ amplitudes) one may identify $M_{33}^{(3)}$ as the three body force.

(d) *Two particle irreducible amplitudes for $\pi NN\leftrightarrow NN$ ($M_{23}^{(2)},M_{32}^{(2)}$).* Here one easily obtains

$$\begin{aligned} M_{23}^{(2)}&=M_{23}^{(3)}+M_{23}^{(2)}G_3M_{33}^{(3)}+\sum_i\gamma_-^{(1)}(i)G_3M_{33}^{(3)}+\sum_i M_{23}^{(2)}g_3m^{(2)}(i)+\sum_{i\neq j}\gamma_-^{(1)}(i)G_3m^{(2)}(j)\\ &=M_{23}^{(3)}+M_{23}^{(3)}G_3M_{33}^{(2)}+\sum_i\gamma_-^{(2)}(i)G_3M_{33}^{(2)}+\sum_i M_{23}^{(3)}G_3m^{(1)}(i)+\sum_{i\neq j}\gamma_-^{(2)}(i)G_3m^{(1)}(j) . \end{aligned} \tag{3.10}$$

(e) *Two particle irreducible amplitudes for $NN\to NN$ [$M_{22}^{(2)}(NN)$].* The exposure of three particle states leads to

$$M_{22}^{(2)}(NN)=M_{22}^{(3)}(N)+M_{23}^{(2)}G_3M_{32}^{(3)}+\sum_i[\gamma_-^{(1)}(i)G_3M_{32}^{(3)}+M_{23}^{(2)}G_3\gamma_+^{(2)}(i)]+\sum_{i\neq j}\gamma_-^{(1)}(i)G_3\gamma_+^{(2)}(j) . \tag{3.11}$$

Here $M_{22}^{(3)}(NN)$ may be identified as multipion exchange or single heavy meson exchange NN interactions, while the last term is obviously responsible for one-pion exchange (OPE). The 2nd, 3rd, and 4th terms include some complicated interactions which so far have not been considered in previous works (I, II, AB, and Ref. 11), but do contribute to three body (πNN) unitarity. The amplitude $M_{22}^{(2)}(NN)$ is schematically expressed in Fig. 9. For our present purpose we need not expose intermediate states with more than three particles, although this process could certainly go further.

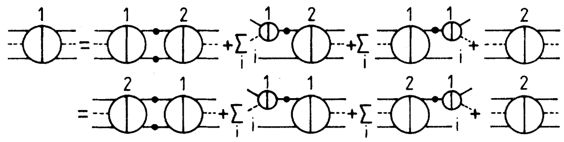

FIG. 8. Diagrammatic form of Eq. (3.7) for the one particle irreducible $\pi NN \to \pi NN$ amplitude $M_{33}^{(1)}$.

The next step is to rewrite $M_{33}^{(2)}$, etc., in a more transparent form. This is carried out by first adding disconnected contributions (m, $\gamma_\pm$, etc.) to connected amplitudes M's.

$$F^{(2)} \equiv M_{33}^{(2)} + \sum_i m^{(1)}(i) , \tag{3.12a}$$

$$\Gamma_+^{(2)} \equiv M_{32}^{(2)} + \sum_i \gamma_+^{(1)}(i) , \tag{3.12b}$$

$$\Gamma_-^{(2)} \equiv M_{23}^{(2)} + \sum_i \gamma_-^{(1)}(i) , \tag{3.12c}$$

and

$$F^{(1)} \equiv M_{33}^{(1)} + \sum_i m^{(1)}(i) + \sum_{i,j} \gamma_+^{(1)}(i) G_2 \gamma_-^{(1)}(j) , \tag{3.13a}$$

$$\Gamma_+^{(1)} \equiv M_{32}^{(1)} + \sum_i \gamma_+^{(1)}(i) , \tag{3.13b}$$

$$\Gamma_-^{(1)} \equiv M_{23}^{(1)} + \sum_i \gamma_-^{(1)}(i) . \tag{3.13c}$$

Furthermore, we shall introduce the following quantities:

$$v_0 \equiv M_{33}^{(3)} \tag{3.14a}$$

(πNN three body potential),

$$v_i \equiv m^{(2)}(i), \quad i = 1,2,3 \tag{3.14b}$$

(two body potentials for πN and NN pairs),

$$v \equiv \sum_{\mu=0}^{3} v_\mu , \tag{3.14c}$$

$$m^{(1)}(0) \equiv v_0 + v_0 G_3 m^{(1)}(0) , \tag{3.14d}$$

$$\Gamma_{\pm i}^{(3)} \equiv \gamma_\pm^{(2)}(i) , \tag{3.14e}$$

$$\Gamma_{\pm 0}^{(3)} \equiv \begin{cases} M_{32}^{(3)} \\ M_{23}^{(3)} \end{cases} , \tag{3.14f}$$

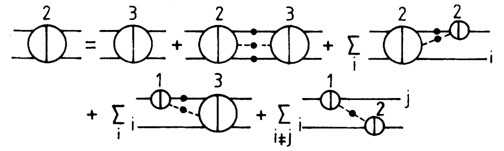

FIG. 9. The two particle irreducible contribution to the two nucleon amplitude, denoted as $M_{22}^{(2)}(NN)$ in Eq. (3.11).

and

$$\Gamma_\pm^{(3)} \equiv \sum_{\mu=0}^{3} \Gamma_{\pm\mu} . \tag{3.14g}$$

In terms of $\Gamma_{\pm 0}^{(3)}$ we can now introduce the dressed irreducible $\pi NN \leftrightarrow NN$ vertices, which are the (three body connected) counterpart of $\gamma_\pm^{(1)}(i)$ [$i=1,2,3$, but $\gamma_\pm^{(1)}(3) \equiv 0$] recalling the statement following Eq. (6). Thus, we have

$$\gamma_\pm^{(1)}(0) \equiv \begin{cases} [1 + m^{(1)}(0) G_3] \Gamma_{+0}^{(3)} \\ \Gamma_{-0}^{(3)} [1 + G_3 m^{(1)}(0)] \end{cases} , \tag{3.14$'$}$$

and from Eqs. (5) and (6) we find

$$\gamma_\pm^{(1)}(\mu) = \begin{cases} [1 + m^{(1)}(\mu) G_3] \Gamma_{+\mu}^{(3)} \\ \Gamma_{-\mu}^{(3)} [1 + G_3 m^{(1)}(\mu)] \end{cases} . \tag{3.14$''$}$$

Then we can easily obtain from Eqs. (3.4), (3.5), and (3.7)–(3.11) the following equalities:

$$\begin{aligned} F^{(2)} &= v + v G_3 F^{(2)} \\ &= v + F^{(2)} G_3 v , \end{aligned} \tag{3.15a}$$

$$\Gamma_+^{(2)} = (1 + F^{(2)} G_3) \Gamma_+^{(3)} , \tag{3.15b}$$

$$\Gamma_-^{(2)} = \Gamma_-^{(3)} (1 + G_3 F^{(2)}) , \tag{3.15c}$$

$$\Gamma_+^{(1)} = \Gamma_+^{(2)} [1 + G_2 M_{22}^{(1)}(NN)] , \tag{3.15d}$$

$$\Gamma_-^{(1)} = [1 + M_{22}^{(1)}(NN) G_2] \Gamma_-^{(2)} , \tag{3.15e}$$

$$\begin{aligned} F^{(1)} &= F^{(2)} + \Gamma_+^{(1)} G_2 \Gamma_-^{(2)} \\ &= F^{(2)} + \Gamma_+^{(2)} [G_2 + G_2 M_{22}^{(1)}(NN) G_2] \Gamma_-^{(2)} . \end{aligned} \tag{3.15f}$$

Equations (3.15) are just the relativistic off mass shell analog of those obtained by MK. The physical amplitude for $\pi NN \leftrightarrow NN$ and $\pi NN \to \pi NN$ are identified as $\Gamma_\pm^{(1)}$ and $F^{(1)}$, respectively. (Note that these amplitudes do contain disconnected parts, in the same way as $3 \to 3$ amplitudes in potential scattering.) Equation (3.15f) decomposes $F^{(1)}$ (for $\pi NN \to \pi NN$) into $F^{(2)}$ (which contains no intermediate pion absorption channel) and an "absorption correction" term $\Gamma_+^{(1)} G_2 \Gamma_-^{(2)}$. Still we want to obtain coupled equations which are more easily handled than Eq. (3.15).

D. Derivation of the coupled equations for the πNN-NN system

Based upon the preceding results in Secs. III B and C we are now in the position to find coupled sets of equations among the amplitudes pertaining to the πNN-NN systems.

(a) *Coupled equations for* $NN\to NN$ *and* $NN\to\pi NN$. First we abbreviate the NN "potential" and amplitude as

$$v_{NN}\equiv M_{22}^{(2)}(NN)\ , \qquad (3.16a)$$

$$U_{NN}\equiv M_{22}^{(1)}(NN)\ . \qquad (3.16b)$$

Then, from Eqs. (3.3), (3.11), (3.12), and (3.14) we find

$$U_{NN}=v_{NN}+v_{NN}G_2U_{NN} =v_{NN}+U_{NN}G_2v_{NN}\ , \qquad (3.3')$$

$$v_{NN}=M_{22}^{(3)}(NN)+\Gamma_-^{(2)}G_3\Gamma_+^{(3)} - \sum_{i=1}^{3}\gamma_-^{(1)}(i)G_3\gamma_+^{(2)}(i)\ , \qquad (3.11')$$

where the subtraction of the last term in (3.11′) ensures that v_{NN} is connected. This last term is nothing but the nucleon self-energy term due to virtual pion production. We shall regard the first term $M_{22}^{(3)}(NN)$ as the heavy meson (ρ, ω, etc.) exchange NN interaction. Thus we define

$$v_{\rm HM}\equiv M_{22}^{(3)}(NN)\ , \qquad (3.11''a)$$

$$\mathrm{SE}(op)\equiv \sum_{i=1}^{3}\gamma_-^{(1)}(i)G_3\gamma_+^{(2)}(i)\ . \qquad (3.11''b)$$

Now, we shall rewrite the expressions for U_{NN} [Eq. (3.3′)] and $\Gamma_+^{(1)}$ [Eq. (3.15)] using Eqs. (3.11′), (3.11″), and (3.15):

$$\Gamma_+^{(1)}=[1+F^{(2)}G_3]\Gamma_+^{(3)}(1+G_2U_{NN})\ , \qquad (3.17a)$$

$$U_{NN}=[v_{\rm HM}-\mathrm{SE}(op)](1+G_2U_{NN}) +\Gamma_-^{(3)}G_3\Gamma_+^{(1)}\ . \qquad (3.17b)$$

The above set of equations couples the amplitudes U_{NN} and $\Gamma_+^{(1)}$, but is still far from being practical as it contains explicitly the nucleon self-energy terms SE(op), $F^{(2)}$ (the absorption free $\pi NN\to\pi NN$ amplitude), etc. Thus, a further modification is necessary, which we achieve by decomposing $F^{(2)}$, $\Gamma_\pm^{(1)}$, etc., into "channels." In accordance with the "odd-man-out" notation (introduced earlier) and the definitions of v_0 and $\Gamma_{\pm 0}^{(3)}$ [Eqs. (3.14a) and (3.14f)], we define channel 0 and other channels as follows:

Channel 0 is the πN_1N_2 interacting through three particle irreducible interactions;

Channel i, $i=1,2$ is the interacting πN_j with spectator N_j, $j\neq 1$;

Channel 3 is the interacting N_1N_2 with spectator π;

Channel N is the interacting N_1N_2 with no spectator π.

Now we may decompose

$$F^{(2)}=\sum_{\mu\nu}F_{\mu\nu}^{(2)}\quad(\mu=0,1,2,3)\ , \qquad (3.18)$$

where $F_{\mu\nu}^{(2)}$ satisfy the Faddeev equations

$$F_{\mu\nu}^{(2)}=m^{(1)}(\mu)\delta_{\mu\nu}+\sum_{\eta}m^{(1)}(\mu)\bar{\delta}_{\mu\eta}F_{\eta\nu}^{(2)}\ . \qquad (3.18')$$

Likewise we decompose $\Gamma_+^{(1)}$, noting Eqs. (3.14f), (3.17a), and (3.18),

$$\Gamma_+^{(1)}\equiv\sum_{\mu}\Gamma_{+\mu}^{(1)}\ , \qquad (3.19a)$$

$$\Gamma_{+\mu}^{(1)}=\sum_{\nu}\left[\delta_{\mu\nu}+\sum_{\eta}F_{\mu\eta}^{(2)}G_3\right]\times\Gamma_{+\nu}^{(3)}(1+G_2U_{NN}) =\sum_{\nu}\left[\delta_{\mu\nu}+\sum_{\eta}F_{\mu\eta}^{(2)}G_3\bar{\delta}_{\mu\nu}\right]\gamma_+^{(1)}(\nu)\times(1+G_2U_{NN})\ , \qquad (3.19b)$$

where the last equality employs Eqs. (3.18′) and (3.14″). Applying the above result to $\Gamma_-^{(3)}G_3\Gamma_+^{(1)}$ in Eq. (3.17b) we find after some algebra

$$\Gamma_-^{(3)}G_3\Gamma_+^{(1)}=\sum_{\mu}\Gamma_{-\mu}^{(3)}G_3\gamma_+^{(1)}(\mu)(1+G_2U_{NN}) +\sum_{\mu\eta}\gamma_-^{(1)}(\mu)G_3\bar{\delta}_{\mu\eta}\Gamma_{+\eta}^{(1)}\ . \qquad (3.20)$$

Since we have [recall Eqs. (3.14e), (3.14′), and (3.14″)]

$$\sum_{\mu}\Gamma_{-\mu}^{(3)}G_3\gamma_+^{(1)}(\mu)=\Gamma_{-0}^{(3)}G_3\gamma_+^{(1)}(0) +\sum_{i=1}^{3}\gamma_-^{(2)}(i)G_3\gamma_+^{(1)}(i)\ , \qquad (3.21)$$

we get

$$U_{NN}=[v_{\rm HM}+\Gamma_{-0}^{(3)}G_3\gamma_+^{(1)}(0)](1+G_2U_{NN}) +\sum_{\mu\eta}\gamma_-^{(1)}(\mu)G_3\bar{\delta}_{\mu\eta}\Gamma_{+\eta}^{(1)}\ . \qquad (3.17b')$$

We remark that the contribution $\Gamma_{-0}^{(3)}G_3\gamma_+^{(1)}(0)$ to the NN potential (composed of three particle irreducible $\pi NN\leftrightarrow NN$ vertices and the πNN three body potential) has pure three body (πNN) intermediate states and hence does contribute to three body unitarity (see Fig. 10). It has been overlooked in all previous works.

Two more steps are to be taken before arriving at our coupled equations. First, for algebraic simplicity we introduce vector and matrix notations.

$$\underline{F}^{(2)}\equiv\{F^{(2)}_{\mu\nu}\},\ \ \underline{m}^{(1)}\equiv\{m^{(1)}(\mu)\delta_{\mu\nu}\},\ \ \underline{G}_3\equiv\{G_3\delta_{\mu\nu}\},\ \ \underline{K}\equiv\{\bar{\delta}_{\mu\nu}\}\ ,$$
$$\underline{\gamma}^{(1)}_{-}\equiv[\gamma^{(1)}_{-}(0),\gamma^{(1)}_{-}(1),\gamma^{(1)}_{-}(2),\gamma^{(1)}_{-}(3)],\ \ \underline{\gamma}^{(1)}_{+}=[\underline{\gamma}^{(1)}_{-}]^{T^+},\ \text{etc.} \tag{3.22}$$

(where T^+ means transpose and replace $-$ by $+$). Thus, Eqs. (3.17b′), (3.18′), and (3.19) can be written as

$$U_{NN}=[v_{\text{HM}}+\Gamma^{(3)}_{-0}G_3\gamma^{(1)}_{+}(0)](1+G_2U_{NN})+\underline{\gamma}^{(1)}_{-}\underline{G}_3\underline{K}\underline{\Gamma}^{(1)}_{+}\ , \tag{3.17b″}$$

$$\underline{\Gamma}^{(1)}_{+}=(\underline{1}+\underline{F}^{(2)}\underline{K}\,\underline{G}_3)\underline{\gamma}^{(1)}_{+}(1+G_2U_{NN})\ , \tag{3.19′}$$

$$\underline{F}^{(2)}=\underline{m}^{(1)}+\underline{m}^{(1)}\underline{K}\,\underline{G}_3\underline{F}^{(2)}\ . \tag{3.18″}$$

By eliminating $\underline{F}^{(2)}$ from Eq. (3.19′) one finds

$$\underline{\Gamma}^{(1)}_{+}=\gamma^{(1)}_{+}(1+G_2U_{NN})+\underline{m}^{(1)}\underline{K}\,\underline{G}_3\underline{\Gamma}^{(1)}_{+}\ . \tag{3.19″}$$

Second, we substitute Eq. (3.19″) into Eq. (3.17b″) to explicitly show the OPE NN interaction. This leads to

$$U_{NN}=(v_{\text{HM}}+\underline{\Gamma}^{(3)}_{0}G_3\gamma^{(1)}_{+}(0)+\underline{\gamma}^{(1)}_{-}\underline{G}_3\underline{K}\underline{\gamma}^{(1)}_{+})\times(1+G_2U_{NN})+\underline{\gamma}^{(1)}_{-}\underline{G}_3\underline{m}^{(1)}\underline{G}_3\underline{\Gamma}^{(1)}_{+}\ . \tag{3.17b‴}$$

It is instructive, before going on, to observe the structure of $\underline{\gamma}^{(1)}_{-}\underline{G}_3\underline{K}\underline{\gamma}^{(1)}_{+}$ (see Fig. 11). Beside the usual one-pion exchange potential (OPEP), there are other contributions which do affect πNN three body unitarity. Like $\Gamma^{(3)}_{-0}G_3\gamma^{(1)}_{+}(0)$ (discussed previously, see Fig. 10), these terms have also been overlooked in previous works.

Equations (3.19″) and (3.17b‴) are the coupled set of equations describing the processes $NN\to NN$, $NN\to\pi NN$ which we have been seeking.

(b) *Coupled equations for $\pi NN\to NN$ and $\pi NN\to\pi NN$ amplitudes.* Utilizing again the matrix notation we find, from Eqs. (3.15e) and (3.15f)

$$\underline{F}^{(1)}=\underline{F}^{(2)}+(\underline{1}+\underline{F}^{(2)}\underline{K}\,\underline{G}_3)\underline{\gamma}^{(1)}_{+}G_2\underline{\Gamma}^{(1)}_{-}$$
$$=\underline{F}^{(2)}+\Gamma^{(1)}_{+}G_2\underline{\gamma}^{(1)}_{-}(\underline{1}+\underline{G}_3\underline{K}\,\underline{F}^{(2)})\ , \tag{3.23a}$$

$$\underline{\Gamma}^{(1)}_{-}=(1+U_{NN}G_2)\gamma^{(1)}_{-}(\underline{1}+\underline{G}_3\underline{K}\,\underline{F}^{(2)})\ . \tag{3.23b}$$

Eliminating $\underline{F}^{(2)}$ from Eq. (3.23) and using Eq. (3.18″) one finds

$$\underline{F}^{(1)}=\underline{m}^{(1)}+\underline{m}^{(1)}\underline{K}\,\underline{G}_3\underline{F}^{(1)}+\underline{\gamma}^{(1)}_{+}G_2\underline{\Gamma}^{(1)}_{-}\ . \tag{3.23a′}$$

Then, we eliminate U_{NN} from Eqs. (3.23b) employing Eq. (3.17b″). The result is

$$\underline{\Gamma}^{(1)}_{-}=\{1+[1-(v_{\text{HM}}+\Gamma^{(3)}_{-0}G_3\gamma^{(1)}_{+}(0))G_2]^{-1}\times[(v_{\text{HM}}+\Gamma^{(3)}_{-0}G_3\gamma^{(1)}_{+}(0))+\underline{\gamma}^{(1)}_{-}\underline{G}_3\underline{K}\underline{\Gamma}^{(1)}_{+}]G_2\}\underline{\gamma}^{(1)}_{-}(\underline{1}+\underline{G}_3\underline{K}\,\underline{F}^{(2)})\ , \tag{3.24}$$

which, after some algebra [noting Eq. (3.23b)], reduces as

$$\underline{\Gamma}^{(1)}_{-}=(v_{\text{HM}}+\underline{\Gamma}^{(3)}_{0}G_3\gamma^{(1)}_{+}(0))G_2\underline{\Gamma}^{(1)}_{-}+\underline{\gamma}^{(1)}_{-}(\underline{1}+\underline{G}_3\underline{K}\,\underline{F}^{(1)})\ . \tag{3.24′}$$

Furthermore, substitution of Eq. (3.23a′) into Eq. (3.24′) now yields explicit OPE structure

$$\underline{\Gamma}^{(1)}_{-}=\underline{\gamma}^{(1)}_{-}+\underline{\gamma}^{(1)}_{-}\underline{G}_3\underline{K}\,\underline{m}^{(1)}(\underline{1}+\underline{G}_3\underline{K}\,\underline{F}^{(1)})+[v_{\text{HM}}+\Gamma^{(3)}_{-0}G_3\gamma^{(1)}_{+}(0)+\underline{\gamma}^{(1)}_{-}\underline{G}_3\underline{K}\underline{\gamma}^{(1)}_{+}]G_2\Gamma^{(1)}_{-}\ . \tag{3.25}$$

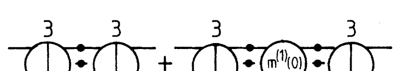

FIG. 10. The contribution $\Gamma^{(3)}_{-0}G_3\gamma^{(1)}_{+}(0)$ [appearing in Eq. (3.17b′)] to the NN potential. It contributes to three body unitarity but has been disregarded in previous works.

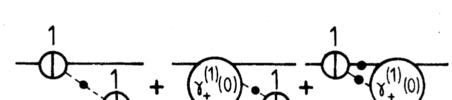

FIG. 11. The contribution $\underline{\gamma}^{(1)}_{-}\underline{G}_3\underline{K}\underline{\gamma}^{(1)}_{+}$ [appearing in Eq. (3.21)] to the two nucleon interaction. Beside the usual OPEP term (a) there are other terms (b) affecting three body unitarity which have not been considered in previous works.

Equations (3.23a′) and (3.25) form the second set of coupled integral equations which we have been seeking.

In order to make closer contact with what we have obtained in the first half of the current work we shall modify Eqs. (3.17b‴), (3.19″), (3.23a′), and (3.25) into the Alt-Grassberger-Sandhas AGS (Ref. 10) form by recalling the amplitudes from the representation of transition operators in terms of channel wave functions. Specifically, we write

$$\Gamma^{(1)}_{-\mu} \equiv \langle \chi_{NN} \mid U_{N\mu} \mid \psi_\mu \rangle \ , \tag{3.26a}$$

$$F^{(1)}_{\mu\nu} \equiv \langle \psi_\mu \mid U_{\mu\nu} \mid \psi_\nu \rangle \ , \tag{3.26b}$$

where ψ_μ is the wave function for channel μ and χ_{NN} is the two nucleon plane wave. It is then straightforward to show that (regarding $\Gamma^{(1)}_{\pm}$ and $F^{(1)}$ as operators)

$$U_{\mu\nu} = G_3{}^{-1}\bar{\delta}_{\mu\nu} + \sum_{\alpha\beta} \bar{\delta}_{\mu\alpha} F^{(1)}_{\alpha\beta} \delta_{\beta\nu} \ , \tag{3.27a}$$

$$U_{N\mu} = \sum_{\alpha} \Gamma^{(1)}_{-\alpha} \bar{\delta}_{\alpha\mu} \ , \tag{3.27b}$$

$$U_{\mu N} = \sum_{\alpha} \bar{\delta}_{\mu\alpha} \Gamma^{(1)}_{+\alpha} \ . \tag{3.27c}$$

Defining the matrices

$$\underline{U} \equiv \{ U_{\mu\nu} \}, \quad \underline{U}^{(+)} \equiv \{ \underline{U}_{\mu N} \}, \quad \underline{U}^{(-)} \equiv \{ U_{N\mu} \} \ ,$$

and combining OPE plus $v_{\rm HM}$ with the additional forces (discussed previously) to form

$$\tilde{v}_{\rm HM} \equiv v_{\rm HM} + \Gamma^{(3)}_{-0} G_3 \gamma^{(1)}_{+}(0) + \underline{\gamma}^{(1)}_{-} \underline{G}_3 \underline{K} \underline{\gamma}^{(1)}_{+} \ , \tag{3.28}$$

we can easily obtain the following equations:

$$\left.\begin{aligned} U_{NN} &= \tilde{v}_{\rm HM}(1 + G_2 U_{NN}) + \underline{\gamma}^{(1)} \underline{G}_3 \underline{K}\, \underline{m}^{(1)} \underline{G}_3 \underline{U}^{(+)} \\ \underline{U}^{(+)} &= \underline{\gamma}^{(1)}_{+}(1 + G_2 U_{NN}) + \underline{m}^{(1)} \underline{G}_3 \underline{U}^{(+)} \end{aligned}\right\} \quad \text{for } NN \rightarrow \begin{cases} NN \\ \pi NN \end{cases} , \tag{3.29}$$

and

$$\left.\begin{aligned} \underline{U} &= \underline{G}_3{}^{-1} \underline{K} + \underline{K}\, \underline{m}^{(1)} \underline{G}_3 \underline{U} + \underline{K} \underline{\gamma}^{(1)}_{+} G_2 \underline{U}^{(-)} \\ \underline{U}^{(-)} &= \underline{\gamma}^{(1)}_{-}(\underline{1} + \underline{G}_3 \underline{K}\, \underline{m}^{(1)} \underline{G}_3 \underline{U}) + \tilde{v}_{\rm HM} G_2 \underline{U}^{(-)} \end{aligned}\right\} \quad \text{for } \pi NN \rightarrow \begin{cases} \pi NN \\ NN \end{cases} . \tag{3.30}$$

Clearly, one can observe the formal similarity between the above equations and those we obtained from the nonrelativistic coupled channel approach and also those obtained by AB. However, the equations obtained currently are for the relativistic off-mass-shell amplitudes. Our coupled Bethe-Salpeter type equations for the πNN-NN systems preserve the antiparticle degrees of freedom.

As for the unitarity structure, it has been shown explicitly[8] that $F^{(1)}$, $\Gamma^{(1)}_{\pm}$, and U_{NN} do satisfy off-mass-shell two (NN) and three (πNN) body unitarity (as well as off-mass-shell subenergy unitarity) *provided* that dressed single particle propagators are properly renormalized. Since Eqs. (3.29) and (3.30) have resulted from formally rearranging the expressions for $F^{(1)}$, $\Gamma^{(1)}_{\pm}$, and U_{NN}, the unitarity proof proceeds along the same line, which is not difficult although rather lengthy. As remarked before, the second and the non-OPE parts of the third contribution to $\tilde{v}_{\rm HM}$ [see Eq. (3.28)] do contribute to three body (πNN) unitarity. It can be shown that neglecting these parts does preserve the unitarity structure of the equations provided that one consistently drops the three body πNN potential v^0 and three particle irreducible $\pi NN \leftrightarrow NN$ vertices $\Gamma^{(3)}_{\pm}$. At this stage the formal correspondence to our previous derivation (through the Hamiltonian plus projection operator approach) becomes more transparent; one could then establish the following correspondence:

$$U\text{'s} \leftrightarrow U\text{'s} \ , \tag{3.31a}$$

$$t_i \leftrightarrow m^{(1)}(i) \ , \tag{3.31b}$$

$$R_\alpha, R^\dagger_\alpha \leftrightarrow \gamma^{(2)}_{-}(\alpha), \gamma^{(2)}_{+}(\alpha) \ , \tag{3.31c}$$

$$y_\alpha, z_\alpha \leftrightarrow \gamma^{(1)}_{-}(\alpha), \gamma^{(1)}_{+}(\alpha) \ , \tag{3.31d}$$

$$G_0 \leftrightarrow G_3 \ , \tag{3.31e}$$

$$\Pi_{22} \leftrightarrow G_2 \ . \tag{3.31f}$$

For a practical use of Eqs. (3.29) and (3.30) one may adopt the procedure by Freedman *et al.*[13] to introduce isobar approximation to the two body amplitudes and also to eliminate the relative energy component (see also the work of Aaron *et al.*[14]). The elimination of relative energy in problems involving true pion absorption have already been worked out in II. This corresponds to introducing the Blankenbecler-Sugar propagators. Some care must be taken in deriving the two-nucleon propagator from G_2 [the product of two (dressed) single nucleon propagators]. The simplest way is to evaluate the discontinuity of G_2 in s (the total NN c.m. energy squared) to keep up to the value corresponding to the πNN elastic scattering and then disperse it with respect to s. We believe that this is the most natural and transparent way to arrive at practical formulations implementing relativity in the manner of Blankenbecler-Sugar.

To end this section, one remark seems to be necessary. It has been pointed out by Kowalski *et al.*[15] that in order to completely eliminate the overcounting of the π exchange effect, the πN two body amplitudes $m^{(1)}(i)$, $i=1,2$ (appearing in the Faddeev amplitudes $F^{(2)}_{\mu\nu}$), must be void of the crossed nucleon pole term. The reason why our derivation has not apparently met this requirement is that we have applied the LCL only in the s channel for the coupled πNN-NN systems, but not in u and t channels. One could, in principle, carry out the cutting procedure in all channels at the same time but the resulting equations would (if obtained in closed form) necessarily become highly nonlinear and basically impractical. Fortunately, compared with the important role played by the direct nucleon pole term in nuclear π absorption, the possible overcounting of the crossed pole term should hardly affect the essential physics. We note that in time-ordered approaches like those using the Blankenbecler-Sugar reduction this overcounting problem is absent (see Kowalski *et al.*[15]).

IV. CONCLUSIONS

We have ended our quest for presenting a realistic theory for the coupled πNN-NN systems. Combined with our previous works, it completes our formal development, and we hope that most of the theoretical questions have been answered. [Assuming that unitarity is obtained as pointed out after Eqs. (3.30), this will be proved in a future communication.]

In the nonrelativistic case, Eqs. (2.9) form the complete solution to the scattering problem initiated by the set of coupled Schrödinger equations (3.2) and (3.3) in I. In achieving this goal, we had to overcome numerous subtle problems resulting from few body dynamics, absorption phenomena, and the peculiar nature of the two body πN P_{11} channel. Nevertheless, the equations obtained here possess various attractive properties such as (i) containing only dressed and renormalized πNN vertices and nucleon propagators, (ii) satisfying two and three body unitarity, and (iii) being practical and amenable for numerical solution, with no substantial effort required beyond the solution of ordinary three body problems. (The antisymmetrization procedure as well as the explicit form of the vertices and propagators have already been worked out in I and II.) We note in passing that Eqs. (2.9) have also been suggested in AB, but to our knowledge, the present work is the first one to solve the scattering problem in the Hamiltonian formalism.

In the relativistic approach we have utilized the powerful Taylor method and obtained the coupled Bethe-Salpeter type equations (3.29) and (3.30). Thus we have presented an off-mass-shell theory for which the basic input includes Feynman (completely dressed and renormalized) propagators and vertices as well as two and three body forces. The basic tool in the derivation procedure was an algorithm for classifying perturbation diagrams based on the last cut lemma and complete unitarity. We point out, however, that much care is needed in carrying this procedure out since the last cut lemma can be used only for connected diagrams with fully dressed (and renormalized) vertices and propagators.

Unlike the nonrelativistic case, one more step is required here before arriving at the practical phase, namely, the elimination of relative energies. As we have already pointed out, one might, for this purpose, use the techniques developed by Freedman *et al.*[13] and Aaron *et al.*[14] which have been slightly modified in II for problems involving absorption. As a result one arrives (after employing the separable approximation) at three dimensional integral equations with Blankenbecler-Sugar type propagators. These equations form the basis for numerical evaluation of physical observables associated with the πNN-NN systems. Much care is required in deriving the explicit form of all the input quantities (especially the dressed πNN vertices, and the dressed and renormalized two nucleon propagator). This task has already been accomplished together with numerical results for the elastic $\pi d \rightarrow \pi d$ scattering.[6]

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APPENDIX

This appendix is composed mainly of definitions and the detailed forms of operators appearing in Sec. II. Its sole purpose is to allow for a smooth reading of Sec. II without being bothered by lengthy expressions. First we introduce (in order of their appearance) some operators, most of which are defined in I and II (in which case the reader is referred to the appropriate equation).

G_0 is the πNN free propagator [(2.11b) in I].

R_N, R_i, $R_N^\dagger$, and $R_i^\dagger$ are the pion emission and absorption operators [(23) in I].

$v_{NN}=v_0+Z_{NN}$ is the two nucleon interaction. See, e.g., (2.14) in I without d_N therein.

t_a is the two body amplitude in three body space [(2.11f) in I].

τ_N is the two nucleon propagator without the NPP_{11} dressing [(2.15a) in I].

Π_{22} is the completely dressed two nucleon propagator [(3.21d) in II].

$\omega_{22}=1+T_{22}\tau_N, T_{22}=q_N+q_N\Pi_{22}q_N$,

and

$$q_N=\sum_{i=1}^{2} R_i G_0 t_{\alpha_i} G_0 R_i^\dagger .$$

y_i,z_i are the fully dressed pion absorption and emission operators for nucleon N_i, $i=1,2$.

$$y_N=y_1+y_2,\ z_N=z_1+z_2 .$$

$w_{NN}=v_0+y_1G_0z_2+y_2G_0z_1$ is the NN interaction with fully dressed πNN vertices.

We are now in a position to give the explicit expressions of operators introduced in Sec. II. The matrices B and Γ_0 in Eq. (2.1) are given by

$$B_{ab}=\begin{cases} G_0{}^{-1}\bar{\delta}_{ab} & a\cap b\neq N \\ R_N & a=N\ \ b\neq N \\ R_N^\dagger & a\neq N\ \ b=N \\ v_{NN} & a=b=N \end{cases}, \tag{A1}$$

$$(\Gamma_0)_{ab}=\begin{cases} G_0 t_a G_0 \delta_{ab} & a\cup b\neq N \\ \tau_N & a=b=N \end{cases}. \tag{A2}$$

The disconnected and connected parts of B (in $\alpha_1\alpha_2N$ channel subspace) read

$$B_{ab}^d=\begin{bmatrix} 0 & 0 & R_1^\dagger \\ 0 & 0 & R_2^\dagger \\ R_1 & R_2 & 0 \end{bmatrix}, \tag{A3}$$

$$(a,b=\alpha_1,\alpha_2,N)\ ,$$

$$B_{ab}^c=\begin{bmatrix} 0 & G_0{}^{-1} & R_2^\dagger \\ G_0{}^{-1} & 0 & R_1^\dagger \\ R_2 & R_1 & v_{NN} \end{bmatrix}. \tag{A4}$$

For the sake of completeness we give also the disconnected part of U, namely U^d, which is introduced in Eq. (2.3a)

$$U_{ab}^d=\begin{cases} 0 & a\cup b\neq \alpha_1,\alpha_2,N\ , \\ \begin{bmatrix} R_1^\dagger\Pi_{22}R_1 & R_1^\dagger\Pi_{22}R_2 & R_1^\dagger\omega^{22} \\ R_2^\dagger\Pi_{22}R_1 & R_2^\dagger\Pi_{22}R_2 & R_2^\dagger\omega^{22} \\ \omega_{22}R_1 & \omega_{22}R_2 & T_{22} \end{bmatrix} & a\cap b=\alpha_1,\alpha_2,N\ . \end{cases} \tag{A3$'$}$$

Next we need to know the factors η_d and ω_d whose product leads to the Møller operator Ω_d appearing in Eq. (2.5). It is found that

$$\eta_d\equiv\begin{cases} \delta_{ab} & a\cup b\neq\alpha_1,\alpha_2,N\ , \\ \begin{bmatrix} 1 & 0 & R_1^\dagger\Pi_{22} \\ 0 & 1 & R_2^\dagger\Pi_{22} \\ 0 & 0 & \omega_{22} \end{bmatrix} & a\cap b=\alpha_1,\alpha_2,N\ , \end{cases} \tag{A5}$$

$$\omega_d=\begin{cases} \delta_{ab} & a\cup b\neq\alpha_1,\alpha_2,N\ , \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ R_1G_0t_{\alpha_1}G_0 & R_2G_0t_{\alpha_2}G_0 & 1 \end{bmatrix} & a\cap b=\alpha_1,\alpha_2,N\ . \end{cases} \tag{A6}$$

Although not needed directly we give below the nondiagonal propagator matrix Γ appearing in Eq. (2.4c)

$$\Gamma_{ab}=\delta_{ab}G_0t_aG_0 \quad (a\cup b\neq\alpha_1,\alpha_2,N)\ ,$$

$$\Gamma_{\alpha_i b}=G_0t_{\alpha_1}G_0(\Omega_d)_{\alpha_i b} \quad (b=\alpha_1,\alpha_2,N)\ ,$$

$$\Gamma_{a\alpha_i}=(\Omega^d)_{a\alpha_i}G_0t_{\alpha_i}G_0,\quad \Gamma_{N\alpha_i}=\Pi_{22}R_iG_0t_{\alpha_i}G_0,\quad \Gamma_{NN}=\Pi_{22}\ .$$

Finally, the matrix of dressed "potential" V^c and the diagonal matrix of dressed propagators γ appearing in our final equations (2.8a) are detailed below:

$$V^c_{ab}=(\omega_d B^c\omega^d)_{ab}=\begin{cases}\bar{\delta}_{ab}G_0{}^{-1} & a\cap b\neq N\ ,\\ z_N & b=N \quad a\neq\alpha_1,\alpha_2,N\ ,\\ y_N & a=N \quad b\neq\alpha_1,\alpha_2,N\ ,\\ \begin{bmatrix}0 & G_0{}^{-1} & z_2\\ G_0{}^{-1} & 0 & z_1\\ y_z & y_1 & w_{NN}\end{bmatrix} & \end{cases} \tag{A7}$$

$$\gamma_{ab}=\begin{cases}\delta_{ab}G_0t_aG_0 & a\cup b\neq N\\ \Pi_{22} & a=b=N\end{cases}\ . \tag{A8}$$

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