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Nuclear condensation

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This work draws an analogy between a heated nucleus breaking up into clusters and a liquid undergoing a phase transition to a gas in which droplets appear. The critical temperature and density in the nucleus are investigated using a Skyrme effective interaction and finite temperature Hartree-Fock theory. The energy and pressure as a function of density are calculated. The effects of compressibility, effective mass, and binding energy per particle on the critical temperature and critical density of nuclear systems is developed. In some cases, analytic expressions for these quantities can be obtained.

[NUCLEAR REACTIONS Phase transitions in hot nuclear matter.]

I. INTRODUCTION

There is much current interest in the behavior of nuclei undergoing violent collisions produced by high energy incident projectiles. Such collisions may produce states of nuclear matter which are far removed from those normally encountered in low energy collisions. For example, at high densities and/or temperatures, nucleons in the nucleus may dissolve into quarks and gluons. At lower temperatures, which can be attained in medium energy heavy-ion reactions, no such exotic states can be produced but there is the possibility of a liquid-gas phase transition leading to the breakup of the heated nucleus into small clusters (or droplets) of nucleons.

The possibility of such a phase transition was previously considered by several authors[1,2] utilizing various approaches. In general, liquid-gas phase transitions occur in systems with short-range repulsive and longer-range attractive forces. The nuclear system satisfies these conditions and is therefore expected to exhibit such a phase transition. Further motivation for this expectation comes from the recent results of a Purdue-Fermilab experiment[3] involving the collision of 80—350 GeV protons with krypton and xenon targets. The isobaric fragment yields of the experiment were found to be given by a power law,

$$Y(A_F) \propto \frac{1}{A_F{}^{\tau}} , \tag{1.1}$$

where A_F is the fragment mass number and $\tau = 2.64$. Such a power law is expected for condensation near the critical temperature as described by the droplet model of Fisher,[4] where the power law describes the size distribution of the resulting droplets and τ is a critical exponent that is predicted to have a value in the interval $2 < \tau < 3$. In particular, $\tau = 2\frac{1}{3}$ for a van der Waals gas. The power law also seems to be valid for heavy fragments in nucleus-nucleus collisions as shown by Gutbrod *et al.*[5]

The present work is concerned with investigating the condensation of a Fermi gas of nucleons interacting through a Skyrme effective interaction. The equation of state of nuclear matter is calculated using finite temperature Hartree-Fock theory. In particular, we will concentrate on the relation of the critical density and critical temperature to properties of the nuclear equation of state. The Skyrme interaction is used because of the simplicity it affords and also because it leads in some cases to analytic results for the critical temperature and density. In particular, it allows us to study their dependence on the nuclear compressibility, effective mass, and binding energy. It should be pointed out that, despite its simplicity, the Skyrme interaction yields an energy density at $T = 0$ that agrees in the region of interest with those derived, for instance, in a renormalized nonlinear relativistic mean-field theory.[6,7]

Previous numerical calculations of a nuclear equation of state with the use of specific versions of the Skyrme interaction have been carried out by Sauer *et al.*[1] and also by Curtin *et al.*[2] In the present work we obtain an analytical expression for the equation of state using a more general version of the Skyrme force. The analytical results provide additional insight in relating the critical temperature to other nuclear properties, and in carrying out various approximations.

II. THE EFFECTIVE INTERACTION AND THE EQUILIBRIUM CONDITION AT $T=0$

The Skyrme interaction has been used extensively in nuclear structure calculations, especially since the work of Vautherin and Brink[8] who showed that, with relatively few parameters, it is possible in nuclear Hartree-Fock calculations to obtain the binding energies, radii, single particle energies, and other gross properties of nuclei to a satisfactory degree of precision and over a wide range of the Periodic Table. The interaction used in the present paper can be expressed as

$$V=-t_0\delta(\vec{r}_{12})+\frac{t_1}{2}[k^2\delta(\vec{r}_{12})+\delta(\vec{r}_{12})k^2]$$
$$+t_2\vec{k}\cdot\delta(\vec{r}_{12})\vec{k}+\frac{t_3}{6}\rho^\sigma\left[\frac{\vec{r}_1+\vec{r}_2}{2}\right]\delta(\vec{r}_{12})\ , \qquad (2.1)$$

with

$$\vec{r}_{12}=\vec{r}_1-\vec{r}_2$$

and

$$\vec{k}=(\vec{\nabla}_1-\vec{\nabla}_2)/2\ .$$

The last term is density dependent, and the usual Skyrme interaction corresponds to $\sigma=1$. By inserting the additional parameter σ a more general density dependence is obtained and this allows us to fit more nuclear properties, such as incompressibility [see below, Eqs. (2.8) and (2.9)].

At this point it is convenient to introduce the parameters a_0 and a_3 such that $a_0=\frac{3}{8}t_0$ and $a_3=\frac{3}{48}t_3$. We also introduce $\hat{E}_B$, $\hat{E}_K$, and $\widehat{FR}$ which are, respectively, the binding energy per particle, the kinetic energy per particle, and the finite-range energy per particle. Note that in a Fermi-gas model $\hat{E}_{K_0}=\frac{3}{5}\epsilon_F$, where ϵ_F is the Fermi energy at $T=0$. The $\widehat{FR}$ is the contribution to the energy per particle from the finite range (alternatively velocity-dependent) terms of the Skyrme interaction, i.e., those proportional to t_1 and t_2 in Eq. (2.1). The energy per particle in the nuclear ground state, i.e., at $T=0$, is then given by

$$-\hat{E}_{B_0}=\hat{E}_{K_0}-a_0\rho_0+a_3\rho_0^{1+\sigma}+\widehat{FR}_0\ , \qquad (2.2)$$

where the subscript (0) denotes quantities calculated at zero temperature. The equilibrium condition at $T=0$ yields

$$\rho_0\left[\frac{\partial\hat{E}_B}{\partial\rho}\right]_{\rho_0}=0=\tfrac{2}{3}\hat{E}_{K_0}+\tfrac{5}{3}\widehat{FR}_0$$
$$-a_0\rho_0+a_3(1+\sigma)\rho_0^{1+\sigma}\ . \qquad (2.3)$$

The effective mass is now introduced by the definition

$$\frac{m^*}{m}=\left[1+\frac{\widehat{FR}}{\hat{E}_K}\right]^{-1}\ . \qquad (2.4)$$

Equations (2.2) and (2.3) can then be solved for a_0 and a_3 in terms of $\hat{E}_{B_0}$, $\hat{E}_{K_0}$, and $(m^*/m)_0$:

$$\sigma a_0\rho_0=(1+\sigma)\hat{E}_{B_0}+\hat{E}_{K_0}+(\sigma-\tfrac{2}{3})\left[\frac{m}{m^*}\right]_0\hat{E}_{K_0}\ , \qquad (2.5)$$

$$\sigma a_3\rho_0^{1+\sigma}=\hat{E}_{B_0}+\hat{E}_{K_0}-\tfrac{2}{3}\left[\frac{m}{m^*}\right]_0\hat{E}_{K_0}\ . \qquad (2.6)$$

(Note that if $\sigma=\frac{2}{3}$, a_0 does not depend explicitly on the effective mass.)

A simple expression can be obtained for the nuclear incompressibility

$$K=R^2\left[\frac{\partial^2\hat{E}}{\partial R^2}\right]_{\rho_0}\ ,$$

where $R=\rho^{-1/3}$. One obtains

$$K=9\hat{E}_{B_0}+\left[4\left[\frac{m}{m^*}\right]_0-3\right]\hat{E}_{K_0}$$
$$+\sigma\left\{9\hat{E}_{B_0}-\left[6\left[\frac{m}{m^*}\right]_0-9\right]\hat{E}_{K_0}\right\}\ . \qquad (2.7)$$

It is observed that this expression is linear in σ. Thus σ can be used as a parameter to control the value of K without changing the binding energy or mean square radius of the nucleus. For infinite nuclear matter, using $\hat{E}_{B_0}=16$ MeV and $\hat{E}_{K_0}=24$ MeV, we find that

$$K=168+216\sigma \text{ for } \left[\frac{m^*}{m}\right]_0=1\ ,$$
$$K=216+144\sigma \text{ for } \left[\frac{m^*}{m}\right]_0=\tfrac{2}{3}\ . \qquad (2.8)$$

For finite nuclei, the parameters chosen to fit a wide range of nuclei are $\hat{E}_{B_0}=8$ MeV and $\hat{E}_{K_0}=20$ MeV. These yield

$$K=92+132\sigma \text{ for } (m^*/m)_0=1\ ,$$
$$K=132+72\sigma \text{ for } (m^*/m)_0=\tfrac{2}{3}\ . \qquad (2.9)$$

One sees that, in general, K is smaller in finite nuclei than it is in nuclear matter. Using $\sigma=1$, as in the usual Skyrme interaction, tends to give too high an incompressibility; lower values of σ are thus favored. The softest equation of state occurs when $\sigma=0$. In this limit the density-dependent part of the Skyrme interaction has a logarithmic dependence on ρ:

$$c\delta(\vec{r}_{12})\ln\left[\frac{\rho}{\rho_0}\right] . \tag{2.10}$$

III. THE EQUATION OF STATE

Using finite-temperature Hartree-Fock theory, as discussed for example in Fetter and Walecka,[9] it is possible to obtain an equation of state (pressure versus density at finite temperatures) for a Fermi gas of nucleons interacting through the Skyrme force of Eq. (2.1). The details of the derivation are described in the Appendix, but the resulting equation is relatively simple:

$$P=-a_0\rho^2+a_3(1+\sigma)\rho^{2+\sigma} + \left[1-\frac{3}{2}\frac{\rho}{m^*}\frac{dm^*}{d\rho}\right]P_{\mathrm{id}}(m^*) , \tag{3.1}$$

where $P_{\mathrm{id}}(m^*)$ is the "thermal" pressure at temperature T of an ideal Fermi gas with particles having a mass m^*. P_{id} can be calculated from the virial series:

$$P_{\mathrm{id}}=kT\sum_{n=1}^{\infty}B_n\rho_n , \tag{3.2}$$

where the B_n's are given in Eq. (A9) and, apart from a numerical coefficient which decreases rapidly with increasing n, they depend on m^* and T. In particular, we have

$$B_n\sim\left[\frac{2\pi h^2}{m^*kT}\right]^{[3(n-1)]/2}$$

and $B_1=1$. The $n\geq 2$ terms indicate the departure of the quantal Fermi gas from the corresponding classical ideal gas. In addition, the dependence of m^* on ρ must be taken into consideration. For interaction (2.1) the effective mass is given by [see Eq. (A15) in the Appendix]

$$\frac{m^*}{m}=\frac{1}{1+\left[\frac{\rho}{\rho_0}\right]\Delta} , \tag{3.3}$$

where

$$\Delta=\left[\frac{m}{m^*}\right]_0-1 . \tag{3.4}$$

With this form of density dependence the equation of state becomes

$$P=-a_0\rho^2+a_3(1+\sigma)\rho^{2+\sigma} + \frac{\left[1+\frac{5}{2}\frac{\rho}{\rho_0}\Delta\right]}{1+\frac{\rho}{\rho_0}\Delta}P_{\mathrm{id}}(m^*) . \tag{3.5}$$

It is worth noting that the interaction pressure and thermal pressure do not simply add up except for the case $(m^*/m)_0=1$.

The isotherms corresponding to this equation of state resemble those for a Van der Waals gas. This point will be discussed in more detail in Sec. IV A. In particular, at low temperatures each isotherm has two points, a minimum and a maximum, at which $\partial P/\partial\rho$ vanishes. Between the two points the isothermal compressibility has an unphysical negative value, a defect that is remedied by the usual Maxwellian equal-area construction. At the critical temperature these two points merge so that the critical isotherm has a point of inflection where both $\partial P/\partial\rho$ and $\partial^2P/\partial\rho^2$ vanish. This point is the critical point which can be characterized by the values T_c and ρ_c. Moreover, the critical pressure is given by $P(T_c,\rho_c)$. In what follows we attempt to calculate the critical temperature and density (T_c and ρ_c) for the equation of state (3.5) using various approximations.

IV. CRITICAL TEMPERATURE AND CRITICAL DENSITY

A. High temperature, low density approximation

If it is assumed that the critical temperature is high and the critical density is low, then it is possible to keep the $n=1$ term only in the virial series (3.2). In effect, this means that the Fermi gas is completely nondegenerate and thus equivalent to a classical gas. Moreover, the effective mass m^* can be set equal to m for such low densities with very little error. In this approximation the equation of state becomes

$$P=\rho kT-a_0\rho^2+a_3(1+\sigma)\rho^{2+\sigma} , \tag{4.1}$$

and the critical density, temperature, and pressure, given by the conditions

$$\frac{\partial P}{\partial\rho}=0=\frac{\partial^2P}{\partial\rho^2} ,$$

can be easily determined:

$$\rho_c=\left[\frac{a_0}{(1+\sigma)^2\left[1+\frac{\sigma}{2}\right]a_3}\right]^{1/\sigma},$$

$$kT_c=\frac{2\sigma}{1+\sigma}a_0\rho_c\,, \tag{4.2}$$

$$P_c=\frac{\sigma+1}{2(\sigma+2)}\rho_c kT_c\equiv\frac{\sigma+1}{2(\sigma+2)}P_c^{(0)}\,.$$

It is instructive to write ρ_c explicitly in terms of the binding energy per particle and the kinetic energy per particle,

$$\rho_c=\rho_0\left\{\frac{1}{(1+\sigma)^2(1+\sigma/2)}\times\left[1+3\sigma\left[\frac{\hat{E}_{K_0}+\hat{E}_{B_0}}{\hat{E}_{K_0}+3\hat{E}_{B_0}}\right]\right]\right\}^{1/\sigma}. \tag{4.3}$$

The expression is especially simple in the limit $\sigma\rightarrow 0$:

$$\rho_c(\sigma\rightarrow 0)=\rho_0\exp\left[-\frac{5}{2}+3\frac{\hat{E}_{K_0}+\hat{E}_{B_0}}{\hat{E}_{K_0}+3\hat{E}_{B_0}}\right], \tag{4.4}$$

while for $\sigma=1$ one obtains

$$\rho_c(\sigma=1)=\frac{\rho_0}{6}\left[1+3\frac{\hat{E}_{K_0}+\hat{E}_{B_0}}{\hat{E}_{K_0}+3\hat{E}_{B_0}}\right]. \tag{4.5}$$

Using nuclear matter values $\hat{E}_{B_0}=16$ MeV and $\hat{E}_{K_0}=24$ MeV one obtains

$$\begin{aligned}\sigma=0:\ & \rho_c=\rho_0 e^{-5/6}=0.435\rho_0\,,\\ & kT_c=20.9\text{ MeV}\,,\\ \sigma=1:\ & \rho_c=\tfrac{4}{9}\rho_0=0.444\rho_0\,,\\ & kT_c=28.4\text{ MeV}\,.\end{aligned} \tag{4.6}$$

The critical density varies only slightly in going from $\sigma=0$ to 1 while the σ dependence of the critical temperature is more dramatic. Using finite nucleus values $\hat{E}_{B_0}=8$ MeV and $\hat{E}_{K_0}=20$ MeV one obtains

$$\begin{aligned}\sigma=0:\ & \rho_c=0.554\rho_0\quad kT_c=16.1\text{ MeV}\,,\\ \sigma=1:\ & \rho_c=0.485\rho_0\quad kT_c=20.5\text{ MeV}\,.\end{aligned} \tag{4.7}$$

The rationale behind using finite nucleus values for $\hat{E}_{B_0}$ and $\hat{E}_{K_0}$ is that it may account, in a crude way, for the surface and Coulomb energies that are not taken into consideration in the equation of state. More values for T_c and ρ_c calculated in the same approximation are listed in the first column of Table I. It is noted that in all cases the critical density is not low enough to justify the approximation made in the calculation. This suggests that further terms in the virial expansion (3.2) are important. The effective mass must also be treated properly. These corrections will be discussed in the following sections.

Despite the approximations involved in arriving

TABLE I. kT_c (in MeV) and ρ_c/ρ_0 are listed as ordered pairs for various parametrizations of the Skyrme interaction and for various approximations used in the calculations.

		Nondegenerate Gas approximation	First order degeneracy correction with $m_0^*=m$	Degeneracy treated exactly with $m_0^*=m$	Degeneracy treated exactly with $m_0^*=\frac{2}{3}m$
Infinite	$\sigma=1$	28.4, 0.444	22.9, 0.399	22.9, 0.40	19.2, 0.40
nuclear	$\sigma=\frac{2}{3}$	26.2, 0.430	20.5, 0.372	20.5, 0.375	18.0, 0.39
matter	$\sigma=0.1$	21.7, 0.426	16.0, 0.323	16.0, 0.325	15.5, 0.36
	$(a_3=0,\ m_0^*=\frac{2}{5}m)$				19.8, 0.49
Finite	$\sigma=1$	20.5, 0.485	13.1, 0.386	13.4, 0.39	10.0, 0.35
nuclei	$\sigma=\frac{2}{3}$	19.2, 0.481	11.5, 0.353	11.9, 0.36	9.5, 0.33
	$\sigma=0.1$	16.5, 0.524	8.1, 0.272	8.5, 0.29	8.1, 0.30
	$(a_3=0,\ m_0^*=\frac{10}{21}m)$				8.3, 0.30

at the equation of state (4.1), its simple analytic form makes it interesting to compare with the equation of state for a van der Waals gas. To make this comparison, we use the corresponding states form which involves a change of variables in the equation of state to the dimensionless quantities

$$t=\frac{T}{T_c}, \quad v=\frac{V}{V_c}=\frac{\rho_c}{\rho}, \quad p=\frac{P}{P_c} . \tag{4.8}$$

If all gases obeyed the van der Waals equation of state their corresponding state representations would be identical. For a van der Waals gas the equation of state becomes

$$p=\frac{8t}{3v-1}-\frac{3}{v^2} . \tag{4.9}$$

For the equation of state (4.1), one obtains

$$p=2\frac{\sigma+2}{\sigma+1}\frac{t}{v}-\frac{\sigma+2}{\sigma}\frac{1}{v^2}+\frac{2}{\sigma(\sigma+1)}\frac{1}{v^{\sigma+2}} . \tag{4.10}$$

Figure 1 is a plot of p vs v for the Skyrme and van der Waals systems with $\sigma=1$. The two theories agree at the critical point $p=1$, $t=1$, and $v=1$ by construction. From the figure we see that the two descriptions agree qualitatively. However, for $t<1$ the van der Waals system has much deeper minima. The dots represent points where the pressure is a maximum or minimum. Note that the Skyrme equation of state has no excluded volume effect incorporated into it.

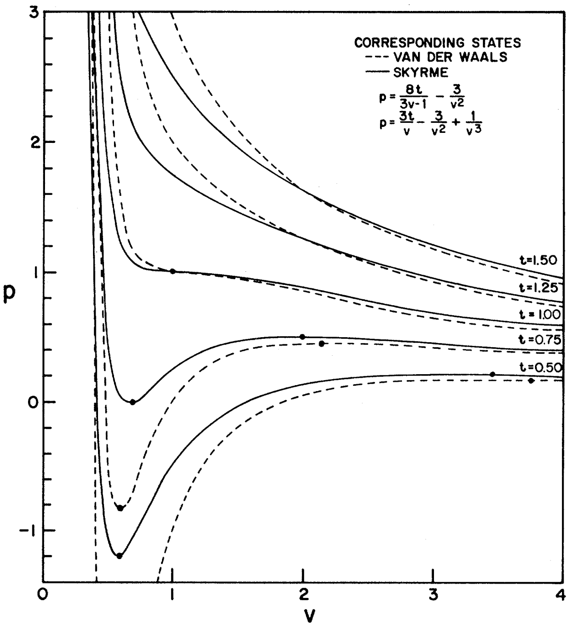

FIG. 1. Law of corresponding states. Comparison of the equation of state for a van der Waals gas and for a system interacting through a Skyrme force (with $\sigma=1$) when written in the corresponding state form.

B. Lowest-order degeneracy correction

When the B_2 term in the virial series (3.2) is also included the resulting equation of state becomes [for the case $(m^*/m)_0=1$, i.e., $t_1=t_2=0$ in (2.1)]

$$P=-a_0\rho^2+a_3(1+\sigma)\rho^{2+\sigma}+kT\rho + \left[\frac{2\pi\hbar^2}{m}\right]^{3/2}\frac{1}{(kT)^{1/2}}\frac{\rho^2}{2^{5/2}g_{S,I}} , \tag{4.11}$$

where $g_{S,I}$ is the spin-isospin degeneracy factor

$$(2S+1)(2I+1)=4 .$$

This equation has the same form as (4.1) with a_0 replaced by

$$a_0'=a_0-\left[\frac{2\pi\hbar^2}{m}\right]^{3/2}\frac{(kT)^{-1/2}}{2^{5/2}g_{S,I}} . \tag{4.12}$$

Note, however, that a_0' is T dependent. The critical density and temperature can be obtained by iteration using Eqs. (4.2) and starting with the values of kT_c obtained in Sec. IV A to get a first estimate for a_0'. This procedure converges rapidly and only 3–4 iterations are required. The critical densities and temperatures calculated with this equation of state are shown in the second column of Table I. It is obvious from these results that the B_2 term is very important since it causes a sizable reduction in both the critical temperature and critical density. The most drastic change occurs for $\sigma=0.1$ where, for finite nuclei, kT_c drops from 16.6 to 8.1 MeV and ρ_c is reduced from $0.524\rho_0$ to $0.272\rho_0$.

C. Exact treatment of degeneracy for the case $(m^*/m)_0=1$

For the case $(m^*/m)_0=1$, the equation of state (3.1) becomes

$$P=-a_0\rho^2+a_3(1+\sigma)\rho^{2+\sigma}+kT\sum_{n=1}^{\infty}B_n\rho^n , \tag{4.13}$$

where we have made use of the virial series (3.2). This series converges very rapidly because of the rapid decrease of the B_n's as n increases [see Eq. (A9) in the Appendix]. In most cases it is practically sufficient to sum up to $n=3$ for $\rho\le\rho_0/2$ and $kT\ge4$

MeV, or for $\rho \leq \rho_0$ and $kT \geq 8$ MeV. However, calculations in this section have been carried out by summing up to $n=6$. The accuracy attained in this approximation can be checked by comparing with the exact values of the Fermi-Dirac integrals. For $\rho \leq \rho_0$ and $kT \geq 8$ MeV the errors involved are $<0.17\%$, while for $\rho \leq \rho_0/2$ and $kT \geq 4$ MeV the errors are $<0.24\%$. Even at $\rho=\rho_0$ and $kT=4$ MeV the error is less than 3%.

With the thermal pressure calculated numerically the critical temperature and density are found by plotting the successive isotherms and finding the point of inflection. The results are tabulated in the third column of Table I and are found to be almost identical with those of the second column even for the case where the predicted critical temperature is close to 8 MeV. This indicates that, for the case $(m^*/m)_0=1$, the lowest order degeneracy correction is sufficient.

D. Exact treatment of degeneracy for the case $(m^*/m)_0<1$

The virial series is again summed up to $n=6$ to give the equation of state

$$P=-a_0\rho^2+a_3(1+\sigma)\rho^{2+\sigma}+\frac{\left[1+\frac{5}{2}\frac{\rho}{\rho_0}\Delta\right]}{\left[1+\frac{\rho}{\rho_0}\Delta\right]}kT\sum_{n=1}^{6}\tilde{B}_n\rho^n\,, \qquad (4.14)$$

where the notation $\tilde{B}_n$ is used to indicate the dependence on m^*,

$$\tilde{B}_n \sim (m^*)^{-[3(n-1)]/2}$$

or, using (3.3),

$$\tilde{B}_n=\left[1+\frac{\rho}{\rho_0}\Delta\right]^{[3(n-1)]/2}B_n\,,$$

the B_n's being those used for the case $(m^*/m)_0=1$. Because of the effective mass factor the virial series does not now converge as fast as before, but the convergence is still fast enough for the summation up to $n=6$ to be quite adequate. In particular, it is not sufficient to sum to $n=2$ or 3 only, except for the case $(m^*/m)\approx 1$ or $\Delta \ll 1$. For the case $(m^*/m)_0=\frac{2}{3}$, corresponding to $\Delta=0.5$, the calculated T_c and ρ_c are displayed in the fourth column of Table I. The effect of changing the effective mass from 1 to $\frac{2}{3}$ is seen to be a lowering of the critical temperature, with the change being more pronounced for higher values of σ. The effect on the critical densities is mixed but it appears to be in the direction that makes the σ dependence of ρ_c rather weak.

Note that for an effective mass of $\frac{2}{3}$ all the critical densities fall in the range $0.30<\rho_c/\rho_0<0.40$. This is in contrast with the higher densities $0.43 \lesssim \rho_c/\rho_0 \lesssim 0.52$ predicted for a nondegenerate gas.

It must be pointed out, however, that the effective mass cannot be made arbitrarily small because then the density-dependent t_3 term of the Skyrme interaction (2.1) becomes negative. The lowest value for the effective mass is thus obtained by setting $t_3=0$ in which case the repulsion necessary for saturation is provided solely by the velocity-dependent terms involving t_1 and t_2. The condition t_3 (or a_3)$=0$ leads, from Eq. (2.6), to the following value for the effective mass:

$$\frac{m^*}{m}=\frac{2}{3\left[1+\hat{E}_{B_0}/\hat{E}_{K_0}\right]}\,. \qquad (4.15)$$

One also has

$$a_0\rho_0=\tfrac{5}{2}\hat{E}_{B_0}+\tfrac{3}{2}\hat{E}_{K_0}\,. \qquad (4.16)$$

For infinite matter ($\hat{E}_{B_0}=16$ MeV and $\hat{E}_{K_0}=24$ MeV),

$$\left[\frac{m^*}{m}\right]_0=\tfrac{2}{5},\quad \Delta=1.5,\quad a_0\rho_0=76\text{ MeV}\,. \qquad (4.17)$$

For finite nuclei ($\hat{E}_{B_0}=8$ MeV and $\hat{E}_{K_0}=20$ MeV),

$$\left[\frac{m^*}{m}\right]_0=\tfrac{10}{21},\quad \Delta=1.1,\quad a_0\rho_0=50\text{ MeV}\,. \qquad (4.18)$$

The critical temperatures and densities obtained with purely velocity-dependent repulsive forces are shown in the fourth and eighth rows of Table I.

Finally, we point out that the critical temperatures calculated here for infinite nuclear matter are in good agreement with those calculated earlier in Refs. 1 and 2. We also note that, in general, the results agree with the rough argument that $kT_c \sim$ binding energy per particle.

V. CRITICAL EXPONENTS FOR THE SKYRME INTERACTION

Critical exponents for the equation of state (4.1) can be easily derived. These critical exponents provide information about the thermodynamic properties of the nuclear system near the critical point. In the following, the critical exponents are defined and then their values for the Skyrme equation of state are presented. In a liquid-gas phase transition the order parameter is $\rho_L-\rho_G$, the liquid-gas density difference at the end points of the Maxwell construction line. In the theory of critical phenomena the variation of the order parameter with temperature near T_c is written as

$$\rho_L - \rho_G \sim (-\epsilon)^{\beta} , \tag{5.1}$$

where

$$\epsilon = \frac{T - T_c}{T_c} < 0$$

and β is a critical exponent.

The critical isotherm near the critical point is described by the equation

$$P - P_c \sim |\rho - \rho_c|^{\delta} \mathrm{sgn}(\rho - \rho_c) , \tag{5.2}$$

where a second critical exponent δ is thus introduced. Two more exponents α and γ are defined by the variation of the specific heat at constant volume and the isothermal compressibility

$$K_T{}^{-1} = -V \left[\frac{\partial P}{\partial V} \right]_T$$

above and near the critical temperature ($\epsilon \gtrsim 0$):

$$C_V \sim (-\epsilon)^{\alpha}, \;\; K_T \sim (\epsilon)^{-\gamma} . \tag{5.3}$$

The values obtained for these exponents with the use of Eq. (4.1) are

$$\alpha = 0, \;\; \beta = \tfrac{1}{2}, \;\; \gamma = 1, \;\; \delta = 3 , \tag{5.4}$$

as expected for a mean-field theory.

VI. COMPOSITE PARTICLE EMISSION AND THE CRITICAL POINT

Fisher, in his droplet model,[4] calculates the probability of forming a cluster of A particles. Assuming the binding energy of such a cluster to have a bulk and a surface term

$$E_b(A) = a_V A - ws$$

[where s is a measure of the surface area ($s = a_0 A^{\sigma}$)], and with a similar expression for the entropy

$$S(A) = S_0 A + \omega s ,$$

Fisher finds that the probability of forming a cluster of A particles is given by

$$P(A) \propto \frac{x^{A^{\sigma}} y^{A}}{A^{\tau}} , \tag{6.1}$$

where

$$y = e^{\beta[\mu + a_V + TS_0]} , \quad x = e^{[-a_0 \beta(w - \omega T)]}$$

and μ is the chemical potential. At the critical temperature $x = 1$, $y = 1$, and

$$P(A) \propto A^{-\tau} . \tag{6.2}$$

Moreover, τ is related to the critical exponent δ, introduced in Sec. V, by the equation

$$\tau = 2 + \frac{1}{\delta} . \tag{6.3}$$

For a mean-field theory, $\delta = 3$, and therefore the exponent τ has the value $2\frac{1}{3}$.

Equation (6.2) was used by Minich *et al.*[3,10] to analyze the isobaric fragment yields of ultrarelativistic proton-nucleus collisions [see the discussion preceding Eq. (1.1)]. They fit their data using an exponent $\tau = 2.64$ (similar values are also found for τ by Gutbrod *et al.*[5]). Moreover, they obtain values for the critical density and temperature $\rho_c \sim \rho_0/3$ and $kT_c \sim 3$–4 MeV. For comparison, our calculated values for the critical density are generally in the range $0.3\rho_0 < \rho_c < 0.4\rho_0$. The experimentally determined critical temperature is a factor of 2–3 below our calculated results for finite nuclei. However, fluctuations are expected to reduce the calculated T_c. This point is currently being investigated.

VII. CLOSING REMARKS

The problem of critical behavior of nuclei at the liquid-gas point is drawing considerable attention both experimentally and theoretically. A good overview of the subject was provided by Bertsch.[11] It is not clear whether phase equilibrium can be established in a heavy ion collision.[11] Even if it cannot be established it is still of value to have the equilibrium quantities P_c, T_c, and ρ_c as guideposts.

The work presented here may be regarded as a preliminary effort to obtain the dependence of the critical temperature, density, and pressure in terms of gross properties of nuclei: effective mass, compressibility, binding energy, Fermi energy, etc. In the future, the calculations could be improved by dealing with the nuclear surface in a less phenomenological way, by considering the effects of nuclear correlations more explicitly, and by considering a wider class of equations of state.

Perhaps the main point of the present work is to show that there is a big difference in the results when finite nucleus parameters are used instead of those of infinite nuclear matter (in contrast to ordinary liquids where the surface effects do not alter the critical temperature). It is surprising that this has not been pointed out before. Note that in infinite nuclear matter the binding energy per particle is 16 MeV, but in a finite nucleus it is about 8 MeV. Clearly the much smaller binding energy per particle in a finite nucleus should make it much easier to cause the nucleus to boil off the nucleons.

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APPENDIX

For an ideal Fermi gas at temperature T, chemical potential μ, and volume V the density is given by

$$\rho=\frac{N}{V}=\frac{g_{S,I}}{(2\pi)^3}\int d^3q\, n_q\ , \tag{A1}$$

where $g_{S,I}$ is the spin-isospin degeneracy factor

$$n_q=\{\exp[(\epsilon_q^0-\mu)/kT]+1\}^{-1}\ , \tag{A2}$$

$$\epsilon_q^0=\frac{\hbar^2q^2}{2m}\ . \tag{A3}$$

If the temperature is high enough, one can expand about the nondegenerate solution to get

$$\rho=\frac{g_{S,I}}{\lambda_T{}^3}f_{3/2}(z)\ ,\qquad \lambda_T{}^3=\left[\frac{2\pi\hbar^2}{mkT}\right]^{3/2}\ , \tag{A4}$$

$$f_{3/2}(z)=\sum_n(-1)^{n+1}z^n/n^{3/2}\ ,\qquad z=e^{\mu/kT}\ . \tag{A5}$$

The pressure can be determined from the relationship

$$P=\frac{2}{3}\frac{E}{V} \tag{A6}$$

valid for any nonrelativistic ideal gas, which gives

$$\frac{P}{kT}=\frac{2}{3}\frac{E}{VkT}=\frac{2}{3}\frac{1}{VkT}\frac{g_{S,I}V}{(2\pi)^3}\int d^3q\,\epsilon_q^0n_q =\frac{g_{S,I}}{\lambda_T{}^3}f_{5/2}(z)\ , \tag{A7}$$

where

$$f_{5/2}(z)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n^{5/2}}\ . \tag{A8}$$

By combining (A4) and (A7), one can eliminate Z to get the virial series

$$\frac{g_{S,I}}{\lambda_T{}^3}f_{5/2}(z)=\frac{P}{kT}=\sum_{n=1}^{\infty}B_n\rho^n\ , \tag{A9}$$

where

$$B_1=1,\ \ B_2=\frac{1}{2^{5/2}}\frac{\lambda_T{}^3}{g_{S,I}}\ ,$$

$$B_3=\left[\frac{1}{8}-\frac{2}{9\sqrt{3}}\right]\frac{\lambda_T{}^6}{g_{S,I}{}^2}$$

$$B_4=\left[\frac{3\sqrt{6}+5\sqrt{3}-16}{32\sqrt{6}}\right]\frac{\lambda_T{}^9}{g_{S,I}{}^3},\ \text{etc.}\ldots.$$

For an interacting Fermi gas in the Hartree-Fock approximation, Eqs. (A1) and (A2) still hold, but the single particle energies are given by

$$\epsilon_q=\frac{\hbar^2q^2}{2m}+\hbar\Sigma(\vec{q})\ , \tag{A10}$$

where

$$\hbar\Sigma(\vec{q})=\frac{g_{S,I}v(0)}{(2\pi)^3}\int d^3q'n_{q'}-\frac{1}{(2\pi)^3}\int d^3q'n_{q'}v(\vec{q}-\vec{q}\,') \tag{A11}$$

and $v(\vec{q})$ is the Fourier transform of the two-body interparticle potential $v(\vec{r})$,

$$v(\vec{q})=\int d^3\vec{r}\,v(\vec{r})e^{-i\vec{q}\cdot\vec{r}}=\int d^3\vec{r}\,v(\vec{r})[1-i\vec{q}\cdot\vec{r}-\tfrac{1}{2}(\vec{q}\cdot\vec{r})^2+\cdots]\ ,$$

$$v(\vec{q})=v(0)[1-\tfrac{1}{6}(qa)^2+\cdots]\ , \tag{A12}$$

where

$$a^2=\int d^3\vec{r}\,v(\vec{r})r^2\Big/\int d^3\vec{r}\,v(\vec{r})\ ,$$

and the expansion is possible if $v(\vec{r})$ has a finite range. It is useful to note the similarity between (A12) and the Skyrme interaction which is used in the main part of this paper. Using (A12) the energy spectrum can also be expanded in powers of q^2,

$$\epsilon_q=\epsilon_0+\frac{\hbar^2q^2}{2m^*}\ , \tag{A13}$$

where

$$\epsilon_0=\frac{g_{S,I}-1}{(2\pi)^3}v(0)\int d^3q\,n_q+\frac{1}{(2\pi)^3}\frac{v(0)a^2}{6}\int d^3q\,q^2n_q\ , \tag{A14}$$

$$m^*=m\left[1+\frac{v(0)a^2m}{3\hbar^2 g_{S,I}}\rho\right]^{-1}\ . \tag{A15}$$

Using (A1), (A7), and (A15), one can rewrite (A14) as

$$\epsilon_0=\frac{g_{S,I}-1}{g_{S,I}}v(0)\rho-\frac{3}{2}\frac{kT}{m^*}\frac{dm^*}{d\rho}\frac{g_{S,I}}{\lambda_T{}^3(m^*)}f_{5/2}(\tilde{z})\ , \tag{A16}$$

where

$$\tilde{z}=e^{\tilde{u}/kT}=e^{(\mu-\epsilon_0)/kT}\ , \tag{A17}$$

$$\tilde{\lambda}_T=\left[\frac{2\pi\hbar^2}{m^*kT}\right]^{1/2}\ .$$

The density is given by

$$\rho=[g_{S,I}/(\tilde{\lambda}_T(m^*))^3]f_{3/2}(\tilde{z})\ , \tag{A18}$$

and although Eq. (A6) is no longer valid, one can still use the virial series [Eq. (A9)] to eliminate $\tilde{z}$ and write

$$\frac{g_{S,I}}{\tilde{\lambda}_T{}^3}f_{5/2}(\tilde{z})=\sum_{n=1}^{\infty}\tilde{B}_n\rho^n$$

(except that this is not equal to P/kT) and

$$\epsilon_0=\frac{g_{S,I}-1}{g_{S,I}}v(0)\rho-\frac{3}{2}\frac{kT}{m^*}\frac{dm^*}{d\rho}\sum_{n=1}^{\infty}\tilde{B}_n\rho^n\ , \tag{A19}$$

where $\tilde{B}_n$ is the same as B_n except that $\lambda_T(m)$ is replaced by $\tilde{\lambda}_T(m^*)$.

It is also possible to invert Eq. (A18) to get z as a power series in ρ,

$$\begin{aligned}\tilde{z}&=e^{(\mu-\epsilon_0)/kT}\\&=\frac{(\lambda T)^3}{g_{S,I}}\rho+\frac{1}{2^{3/2}}\frac{(\lambda_T{}^3\rho)^2}{(g_{S,I})^2}+\left[\frac{1}{4}-\frac{1}{3^{3/2}}\right]\left[\frac{\lambda_T{}^3\rho}{g_{S,I}}\right]^3+O(\rho^4)\ ,\end{aligned} \tag{A20}$$

which allows the determination of the chemical potential in terms of ρ and T:

$$\mu=\epsilon_0+kT\ln\tilde{z}\ ,$$

$$\mu(\rho,T)=\frac{g_{S,I}-1}{g_{S,I}}v(0)\rho-\tfrac{3}{2}kT\frac{1}{m^*}\frac{dm^*}{d\rho}\sum_{n=1}^{\infty}\tilde{B}_n\rho^n+kT\left[\ln\frac{\tilde{\lambda}_T{}^3\rho}{g_{S,I}}+\frac{1}{2^{3/2}}\frac{\lambda_T{}^3\rho}{g_{S,I}}+\left[\frac{3}{16}-\frac{1}{3^{3/2}}\right]\left[\frac{\lambda_T{}^3\rho}{g_{S,I}}\right]^2\cdots\right]\ . \tag{A21}$$

Note that for an ideal gas, the chemical potential is given by

$$\mu_{\text{id}}=kT\left[\ln\frac{\lambda_T{}^3\rho}{g}+\frac{1}{2^{3/2}}\frac{\lambda_T{}^3\rho}{g_{S,I}}+\left[\frac{3}{16}-\frac{1}{3^{3/2}}\right]\left[\frac{\lambda_T{}^3\rho}{g_{S,I}}\right]^2+\cdots\right]\ . \tag{A22}$$

Since

$$\mu=-(\partial F/\partial N)_{T,V},\quad P=-(\partial F/\partial V)_{T,N},$$

where F is the Helmholtz free energy, one can calculate P from μ using the fact that

$$\mu=-\left[\frac{\partial f}{\partial\rho}\right]_T,\quad P=\rho\mu-f(\rho,T)\,, \tag{A23}$$

where

$$f(\rho,T)=F(N,V,T)/V\,.$$

using (A23) and (A9) it can be shown that

$$\mu_{\rm id}=kT\left[\ln\frac{{\lambda_T}^3\rho}{g_{S,I}}+\sum_{n=2}^{\infty}\frac{n}{n-1}B_n\rho^{n-1}\right]. \tag{A24}$$

By comparing (A21) with (A22) and (A24), it can be inferred that

$$\mu(\rho,T)=\frac{g_{S,I}-1}{g_{S,I}}v(0)\rho-\tfrac{3}{2}\frac{kT}{m^*}\frac{dm^*}{d\rho}\left[\sum_{n=1}^{\infty}\widetilde{B}_n\rho^n\right]+kT\left[\ln\frac{{\widetilde{\lambda}_T}^3\rho}{g_{S,I}}+\sum_{n=2}^{\infty}\frac{n}{n-1}\widetilde{B}_n\rho^{n-1}\right], \tag{A25}$$

from which it follows that

$$f(\rho,T)=\frac{g_{S,I}-1}{2g_{S,I}}v(0)\rho^2+kT\left[\rho\ln\frac{{\widetilde{\lambda}_T}^3\rho}{g_{S,I}}-\rho\right]+kT\sum_{n=2}^{\infty}\frac{\widetilde{B}_n}{n-1}\rho^n \tag{A26}$$

and

$$P=\frac{g_{S,I}-1}{2g_{S,I}}v(0)\rho^2+kT\left[1-\frac{3}{2}\frac{\rho}{m^*}\frac{dm^*}{d\rho}\right]\sum_{n=1}^{\infty}\widetilde{B}_n\rho^n \tag{A27}$$

or

$$P=\frac{g_{S,I}-1}{2g_{S,I}}v(0)\rho^2+\left[1-\frac{3}{2}\frac{\rho}{m^*}\frac{dm^*}{d\rho}\right]P_{\rm id}(m^*)\,, \tag{A28}$$

where $P_{\rm id}(m^*)$ is the pressure for an ideal Fermi gas with particles having a mass m^*.

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