Separable expansion fit to the Reid soft-core fully off-shell t matrices: ${}^1S_0$ and ${}^3S_1$-${}^3D_1$ states

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(Received 28 June 1982)

The generalized separable expansion method is used to calculate the fully off-energy-shell t matrices from the ${}^1S_0$ and ${}^3S_1$-${}^3D_1$ Reid soft-core potentials. It is proved that the bound state is reproduced rigorously by the rank-one approximation. For positive energies, on-shell and half on- (off-) shell scattering amplitudes are reproduced exactly by the generalized separable expansion rank-one and/or, at least, rank-two approximations. The off-shell parts of the t matrices are determined by N fitting parameters which are chosen to minimize the Hilbert-Schmidt norm of the nonseparable term in the rank-N approximation. We obtained a very good off-shell fit for rank-two and rank-three approximations.

[NUCLEAR REACTIONS Nucleon-nucleon interactions, new separable expansion method for the ${}^1S_0$, ${}^3S_1$-${}^3D_1$ states via the Reid soft-core potentials.]

I. INTRODUCTION

The Yamaguchi type of separable potential[1] has been used widely in three-body problems as a model of the two-body short range interaction.[2] Since the potential has a very simple form, one can analytically obtain the fully off-(energy-)shell two-body t matrix. Improvements in experimental techniques and in numerical analysis for the three-body problem led to the creation of phenomenological separable potentials which were able to reproduce experimental data such as phase shifts over a wide energy region.[3–8] On the other hand, the history of investigations of the nuclear force by meson theoretical treatments is very long and thorough.[9,10] The potentials obtained by this method, although with some adjustable parameters, are often termed "realistic potentials."[9,11–16] The use of pure meson theoretical and realistic potentials via a separable expansion is an interesting problem, particularly as input for the three-body scattering problem.[6–8,17–21]

The unitary pole expansion (UPE) and the unitary pole approximation (UPA) were proposed by Harms *et al.* to approximate the local potential by a separable one.[22–24] These methods are quite simple, but the convergency of the fit of the on-shell t matrix is very slow, and that of the half-off-shell one is worse than other new separable expansion methods.[26,40]

Ernst, Shakin, and Thaler presented a method which permits the construction of a rank-N separable potential.[25] The resulting t matrix is exact on and half off (the energy) shell at N selected bound-state and/or continuum energies. By this method, Pieper calculated separable potentials of rank one to five for the nucleon-nucleon ${}^3S_1$-${}^3D_1$ channel.[26] The phase shifts of his rank-four and rank-five potentials are quite close to the experimental phase shifts. Adhikari and Sloan[27] proposed a more general approach than the Ernst, Shakin, and Thaler (EST) method, which contains the EST method as a special case, but the Adhikari and Solan (AS) method allows a much wider choice of basis functions.[27,28] The EST method reproduces the phase shifts at certain selected energies exactly and provides a convenient interpolation for other energies. However, they found that oscillations tend to build in the neighborhood of the interpolation points as the order of the interpolating function is increased. The AS method leads to a similar rank as Pieper's potential for the phase shifts and the mixing parameter of the ${}^3S_1$-${}^3D_1$ state.

Recently, McLeod and Ernst[29] examined a technique of approximating local interactions by separable potentials of simple analytic form. Almost all the effort in developing the separable potential approximations mentioned above is concerned with fitting (or getting good convergency for) the exact on-shell variables such as experimental phase shifts, mixing parameters, bound state energies, scattering lengths, effective ranges, (differential) cross sections,

and so on.

On the other hand, Kowalski and Noyes proposed an approach which is exact for on- and half-off-(energy-)shell t matrices for positive energies[30–32] for a one term separable approximation, in contrast to the point-wise EST method or Bateman's method.[25,26,33,34] The Kowalski and Noyes (KN) method is also useful for practical calculations in the three-body scattering problem. Kloet and Tjon[20] calculated the elastic neutron-deuteron scattering with local potentials[19] by using the KN method for positive two-body energies. It was pointed out by Osborn[35] that in the n-p $^1S_0$ state the KN method includes an unphysical singularity at the energy where the phase shift has a zero. In a previous paper,[36] it was found that in the KN method a pinching singularity appears in the negative energy region.

A new rank-N separable approximation method, the so-called "generalized separable expansion" (GSE) method, has been proposed.[37–41] It includes the KN method as its first approximation. Here, a t-matrix formalism is constructed which uses all the properties of the given potential, i.e., not only those of the rank-N separable potential but also those of the nonseparable term of the potential. As a result, our method satisfies explicitly off-(energy-)shell unitarity, either as a rank-N separable approximation of the t matrix, or when the nonseparable residual term is included. Therefore, our separable t matrix is mathematically exact on and half off (or on) the energy shell for all energies. Our method allows the analytic continuation to negative energies needed in the Faddeev equations,[42,36] in contrast to the KN separable approximation, which has a pinching singularity in this region. Furthermore, the unphysical singularity of the KN method for positive energies disappears by using a rank-two approximation generated by our method.[37] Moreover, it is proved that the rank-one approximation exactly reproduces the bound-state pole. Also the unitarity and analyticity of the fully off-shell t matrix are investigated in the complex energy and momentum planes. The properties of the GSE method are investigated in Sec. II. In Sec. III, the GSE method is applied to the Reid soft-core potential.[12] Because our method is exact on shell, our aim is to find a good off-shell fit. Our results are compared with some off-shell values of the AS t matrix for the $^1S_0$ state.[28] The fitting parameters for the $^1S_0$ state and the $^3S_1$-$^3D_1$ state are given. In Sec. IV, the properties of the GSE method are summarized and discussed.

II. THE GENERALIZED SEPARABLE EXPANSION FORMALISM

A. Theory

The GSE (Refs. 36–41 and 44) of the off-energy-shell t matrix for the single channel case may be written as

$$t(p,p';z)=\sum_{i,j=1}^{N}\frac{\eta_{ij}(z)}{\eta(z)}t(p,k_j;z)t(k_i,p';z)+t^{(N+1)}(p,p';z)\ , \tag{1a}$$

with

$$\eta(z)=\det[t(k_i,k_j;z)]$$

and

$$\eta_{ij}(z)=[i-j \text{ cofactor of } \eta(z)]\ . \tag{2a}$$

The partial wave index l is omitted for simplicity. The nonseparable remainder $t^{(N+1)}$ satisfies a Lippmann-Schwinger (LS) type of equation:

$$t^{(N+1)}(p,p';z)=v^{(N+1)}(p,p')+\int_0^\infty v^{(N+1)}(p,p'')G_0(p'';z)t^{(N+1)}(p'',p',z)dp''\ , \tag{3a}$$

with

$$v^{(N+1)}(p,p')=v(p,p')-\sum_{i=1}^{N}\frac{v^{(i)}(p,k_i)v^{(i)}(k_i,p')}{v^{(i)}(k_i,k_i)}\ ,$$

and $v^{(1)}(p,p')\equiv v(p,p')$.

We want to show another useful feature of the GSE formalism which has numerical advantages. Instead of Eq. (1a), we write

$$t(p,p';z)=\sum_{i,j=1}^{N}\frac{\theta_{ij}(z)}{\theta(z)}\varphi(p,k_j;z)\chi(k_i,p';z)+t^{(N+1)}(p,p';z)\ , \tag{1b}$$

where the form factors $\varphi(p,k_j;z)$ and $\chi(k_i,p';z)$ satisfy the nonsingular integral equations

$$\varphi(p,k_j;z)=v(p,k_j)+\int_0^\infty dp''v^{(N+1)}(p,p'')\times G_0(p'',z)\varphi(p'',k_j;z) \tag{3b}$$

and

$$\chi(k_i,p';z)=v(k_i,p') + \int_0^\infty dp''\chi(k_i,p'';z) \times G_0(p'';z)v^{(N+1)}(p'',p') . \quad (3c)$$

The denominator function $\theta(z)$ is a determinant of a matrix which consists of $N\times N$ propagators $A(k_i,k_j;z)$, i.e.,

$$\theta(z)=\det[A(k_i,k_j;z)], \quad (1\le i,j\le N) \quad (2b)$$

with

$$A(k_i,k_j;z)=v(k_i,k_j)-\int_0^\infty dp''v(k_i,p'') \times G_0(p'';z)\varphi(p'',k_j;z) \quad (4a)$$

$$=v(k_i,k_j)-\int_0^\infty dp''\chi(k_i,p'';z) \times G_0(p'';z)v(p'',k_j) . \quad (4b)$$

The function $\theta_{ij}(z)$ is the $i-j$ cofactor of $\theta(z)$. These detailed derivations are given in Refs. 37 and 40. The multichannel t matrices such as the $^3S_1$-$^3D_1$ states are easily treated by including some additional suffixes (see the Appendix). In the following sections, the parameters k_i and k_j will be chosen to satisfy off-energy-shell unitarity and to obtain the off-shell fits of the separable t matrix.

B. Unitarity

Unitarity for this expansion can be demonstrated as follows. When we discuss the unitarity relation of the two-body separable expansion of the t matrix, we have two kinds of criteria, one for a t matrix obtained from a given separable potential, and the other one for a t matrix obtained from a given separable plus nonseparable potential. In the former case, the t matrix is automatically separable and satisfies the LS equation, so it satisfies the unitarity relation. The latter case has a different form as the total t matrix consists of a separable t matrix and a nonseparable term. Our interest is in the latter case, since we prefer the most general form of the potential. That is, the potential is

$$v=v(\text{separable})+v(\text{nonseparable})$$

or

$$v=v_{\text{sep}}+v_{\text{non}}$$

and the t matrix is

$$t=t(\text{separable})+t(\text{nonseparable})$$

or

$$t=t_{\text{sep}}+t_{\text{non}} .$$

Since the total t matrix t satisfies the LS equation, t has to satisfy the unitarity condition

$$t-t^\dagger=-2\pi i t^\dagger\delta(z-H_0)t . \quad (5)$$

If one substitutes

$$t=t_{\text{sep}}+t_{\text{non}}$$

into this equation, one gets

$$\begin{aligned}(t_{\text{sep}}-t_{\text{sep}}^\dagger)+(t_{\text{non}}-t_{\text{non}}^\dagger)&=-2\pi i(t_{\text{sep}}^\dagger+t_{\text{non}}^\dagger)\delta(z-H_0)(t_{\text{sep}}+t_{\text{non}})\\ &=-2\pi i t_{\text{sep}}^\dagger\delta(z-H_0)t_{\text{sep}}-2\pi i t_{\text{sep}}^\dagger\delta(z-H_0)t_{\text{non}}\\ &\quad -2\pi i t_{\text{non}}^\dagger\delta(z-H_0)t_{\text{sep}}-2\pi i t_{\text{non}}^\dagger\delta(z-H_0)t_{\text{non}} . \end{aligned} \quad (6)$$

Therefore, one can easily see that the separable t matrix t_{sep} cannot satisfy the unitarity relation without the nonseparable term. However, we find that the following two cases allow us to obtain a unitarity relation for the separable t matrix without the nonseparable term.

(a) t_{non} is a real function and the half on (off) shell t_{sep} is analytically exact.

(b) t_{non} and v_{non} satisfy the LS type of equation:

$$t_{\text{non}}=v_{\text{non}}+\int v_{\text{non}}G_0t_{\text{non}} , \quad (7)$$

and the half on (off) shell t_{sep} is analytically exact.

In case (a) one can see that

$$t_{\text{non}}-t_{\text{non}}^\dagger=0$$

and

$$\delta(z-H_0)t_{\text{non}}=t_{\text{non}}^\dagger\delta(z-H_0)=0 , \quad (8)$$

because the half off shell t_{non} becomes zero for exact half off shell t_{sep}. Therefore Eq. (6) becomes

$$t_{\text{sep}}-t_{\text{sep}}^\dagger=-2\pi i t_{\text{sep}}^\dagger\delta(z-H_0)t_{\text{sep}} . \quad (9)$$

In case (b), because of Eq. (7), t_{non} satisfies

$$t_{\text{non}}-t_{\text{non}}^\dagger=-2\pi i t_{\text{non}}^\dagger\delta(z-H_0)t_{\text{non}} , \quad (10)$$

and for exact half off shell t_{sep}, we obtain

$$\delta(z-H_0)t_{\text{non}}=t_{\text{non}}^\dagger\delta(z-H_0)=0 . \quad (11)$$

By inserting Eqs. (10) and (11) into Eq. (6), it is also

found that

$$t_{\rm sep}-t_{\rm sep}^{\dagger}=-2\pi i t_{\rm sep}^{\dagger}\delta(z-H_0)t_{\rm sep}\ . \tag{12}$$

This is the unitarity relation for the separable t matrix $t_{\rm sep}$. It should be stressed that $t_{\rm sep}$ satisfies the relation without the nonseparable term. In any separable expansion method, there are only two cases (a) and (b) which will satisfy the off-shell unitarity relation for $t_{\rm sep}$. Furthermore, we prefer case (b) rather than case (a), because in case (a) $t_{\rm non}$ is not always a real function, e.g., the three-body calculation by the contour deformation method demands the complex value of $t_{\rm non}$ which is no longer a real function. In our theory $t_{\rm non}$ satisfies a LS type of equation

$$t^{(N+1)}(p,p';z)=v^{(N+1)}(p,p')+\int_0^{\infty}dp''v^{(N+1)}(p,p'')G_0(p'';z)t^{(N+1)}(p'',p';z)\ , \tag{13}$$

and $t_{\rm sep}(p,k;z)$ is exact for the half-off-shell case, where k is the on-shell momentum. Therefore, our separable part of the t matrix satisfies the off-energy-shell unitarity relation under condition (b), if we choose one of the parameters k_i $(i=1,2,\ldots,N)$ equal to the on-shell momentum k.

C. Analyticity

While the Kowalski-Noyes separable approximation also satisfies fully off-energy-shell unitarity, it is useless in the Faddeev equations for most situations. This is because the on-energy-shell t matrix, and hence the half-on-energy-shell t matrix which carries this as a factor, usually have a singularity ("left-hand cut" in the language of dispersion theory). We show, for the case of a Yukawa potential, that the GSE does not suffer from this difficulty if we choose all the k_i to be positive. The Yukawa potential is given for the partial wave l in momentum representation by

$$v_l(p,p')=\frac{C}{pp'}Q_l\left[\frac{p^2+p'^2+\mu^2}{2pp'}\right]\ . \tag{14}$$

On the other hand, using the rank-one GSE formalism the separable form factor has to fall into step with the KN theory, and is given by

$$\varphi(p,k_1;z)=v(p,k_1)+\int_0^{\infty}dp''v^{(2)}(p,p'')G_0(p'';z)\varphi(p'',k_1;z)\ . \tag{15}$$

The analyticity of the form factor $\varphi(p,k_1;z)$ must be investigated not only in the two-body complex energy and momentum planes, but also in the three-body complex energy and momentum planes.

The analyticity of $\varphi(p,k_1;z)$ is evident from the inhomogeneous term $v(p,k_1)$ and the kernel of Eq. (15). The kernel has a pole from the Green's function at $p''=k=\sqrt{2\nu z}$ (ν is the reduced mass) but it must be canceled with a zero of $v^{(2)}(p,p'')$ because $k_1=k$ for positive energies. Therefore, the remaining analytic structure of the form factor comes from $v(p,k_1)$ and $v^{(2)}(p,p'')$ itself, i.e.,

$$v^{(2)}(p,p'')=v(p,p'')-\frac{v(p,k_1)v(k_1,p'')}{v(k_1,k_1)}\ . \tag{16}$$

In other words, all the analytic structure of $\varphi(p,k_1;z)$ is given by $v(k_1,k_1)$, $v(p,k_1)$, $v(k_1,p'')$, and $v(p,p'')$. Furthermore, in the three-body problem, these two-body structures will be mapped onto the complex three-body momentum plane. For example, in the case of three identical particles, an important part of the Faddeev equations is the one particle exchange diagram as in Fig. 1, which is given by the following equation:

$$B(\vec{q},\vec{q}\,';E)=\chi(k_1,p;z)G_0(\vec{q},\vec{q}\,';E)\varphi(p',k_1';z) \tag{17}$$

$$\begin{aligned}&=v(k_1,p)G_0(\vec{q},\vec{q}\,';z)v(p',k_1')+v(k_1,p)G_0(\vec{q},\vec{q}\,';E)\int_0^{\infty}d\bar{p}\,''v^{(2)}(p',\bar{p}\,'')G_0(\bar{p}\,'';z')v(\bar{p}\,'',k_1')\\&\quad+\int_0^{\infty}dp''v(k_1,p'')G_0(p'';z)v^{(2)}(p'',p)G_0(\vec{q},\vec{q}\,';E)v(p',k_1')\\&\quad+\int_0^{\infty}dp''\int_0^{\infty}d\bar{p}\,''v(k_1,p'')G_0(p'';z)v^{(2)}(p'',p)G_0(\vec{q},\vec{q}\,';z)\,v^{(2)}(p',\bar{p}\,'')G_0(\bar{p}\,'',z')v(\bar{p}\,'',k_1')+\cdots\ ,\end{aligned} \tag{18}$$

where

$$\vec{p}=\vec{q}\,'+\vec{q}/2,\quad \vec{p}\,'=\vec{q}+\vec{q}\,'/2\ ,$$

and E is the three-body center of mass energy which is related to the two-body energy z by

$$E=z+3\vec{q}^{\,2}/4m=z'+3\vec{q}\,'^2/4m.$$

Here, we must investigate the analytic structures of

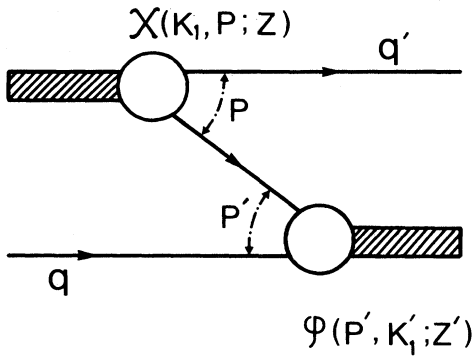

FIG. 1. The Born term in the three-nucleon scattering problem.

$v(k_1,k_1)$, $v(p,k_1)$, $v(k_1,p')$, and $v(p,p')$ in Eq. (14). In general the Legendre function $Q_l(t)$ of the second kind has a branch cut for $-1\leq t\leq 1$. For the on-shell case $p=p'=k$, it has a branch cut along the real axis from $-\mu^2/4$ to $-\infty$ in the two-body complex energy plane. Or, there are two cuts, one is from $i\mu/2$ to $i\infty$, and the other is from $-i\mu/2$ to $-i\infty$ in the complex k plane. The general case $p\neq p'\neq k\neq p$ is rather complicated, but if one assumes p is a parameter, then the cuts are given in the complex p' plane by

$$p'^2+p^2+\mu^2=2pp't \quad (|t|\leq 1)\,. \tag{19}$$

This equation includes two branch cuts with the branch points

$$p'=i\mu\pm p \text{ and } p'=-i\mu\pm p\,. \tag{20}$$

These cuts are between $i\mu+p$ and $i\mu-p$ and between $-i\mu+p$ and $-i\mu-p$ in the p' plane. This is obtained from Eq. (19):

$$\begin{aligned} p' &= pt\pm[p^2(t^2-1)-\mu^2]^{1/2} \\ &= pt\pm i[\mu^2+(1-t^2)p^2]^{1/2}\,. \end{aligned} \tag{21}$$

Therefore, if we give p a real positive value, we have

$$\begin{aligned} &\mathrm{Re}p'=pt\,, \\ &\mathrm{Im}p'=\pm[\mu^2+(1-t^2)p^2]^{1/2}\,, \end{aligned} \tag{22}$$

and a set of circles in the (complex) p' plane

$$(\mathrm{Re}p')^2+(\mathrm{Im}p')^2=p^2+\mu^2\,. \tag{23}$$

With a couple of restrictions, we get

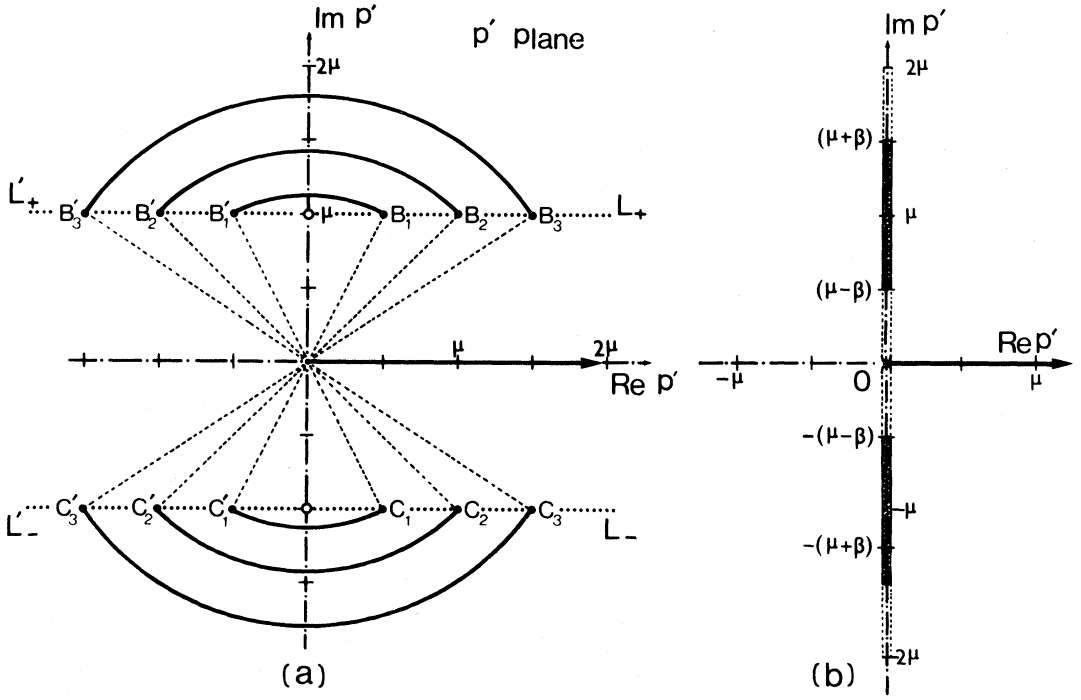

FIG. 2. (a) Off energy-shell potential cuts of the Yukawa potential $v(p,p')$ are illustrated in the p' plane for the parameter $0\leq p<\infty$. Solid semicircles B_1B_1', B_2B_2', and B_3B_3' are cuts which correspond to the parameters, e.g., $p=\mu/2$, $p=\mu$, and $p=3\mu/2$, respectively. C_1C_1', C_2C_2', and C_3C_3' are conjugate cuts to B_1B_1', B_2B_2', and B_3B_3', respectively. Dotted lined L_+L_+' and L_-L_-' are the loci of these branch points. The integral for p' is carried out along the real axis from 0 to ∞, as usually done in the two-body integral equation. (b) In the direct extension of the KN theory to the negative energy region, usual off-shell cuts B_1B_1', C_1C_1', B_2B_2', and so on [shown in (a)], are concentrated onto the imaginary axis and these cuts pinch the integral contour at $p'=0$, corresponding to the energy where $p=k=i\beta$ ($\beta>0$).

$$\tan(\arg p')\,|_{\text{at branch points}} = \frac{\mathrm{Im}p'}{\mathrm{Re}p'} = \pm\frac{\mu}{p} . \quad (24)$$

These are shown in Fig. 2(a). These cuts, which come from the $Q_l(t)$ function, are all for the parameter $0 \le p < \infty$. Another zero such as $p=0$ does not produce any singularity in Eq. (14), because

$$v(p,p') \underset{p \text{ or } p' \to 0}{\to} \text{finite} .$$

Since

$$p = (q'^2 + q^2/4 + qq'\cos\theta)^{1/2} \ge 0$$

and

$$p' = (q^2 + q'^2/4 + qq'\cos\theta)^{1/2} \ge 0 ,$$

$v(p,p')$ has no problem on the real axis of the three-body momentum plane for the variables q and q'. Similar results hold for the cases $v(p,k_1)$, $v(k_1,p')$, and $v(k_1,k_1)$ for the parameter k_1 (≥ 0). However, if one takes

$$k_1 = k = (mE - 3q'^2/4)^{1/2}$$

by the direct extension of the KN theory, then k becomes a pure imaginary value for $q' > (4mE/3)^{1/2}$ (or $k^2 < 0$). Thus, putting in Eq. (20) $p = k = i\beta$ ($\beta > 0$), one gets

$$p' = i\mu \pm k = i(\mu \pm \beta) \quad (25)$$

and

$$p' = -i\mu \pm k = -i(\mu \mp \beta) . \quad (26)$$

Then one has two cuts on the imaginary axis, one is from $i(\mu-\beta)$ to $i(\mu+\beta)$ and the other is from $-i(\mu-\beta)$ to $-i(\mu+\beta)$. Therefore, when $\beta=\mu$, cuts from 0 to $2\mu i$ and from 0 to $-2\mu i$ pinch the integral contour p', at $p'=0$, as illustrated in Fig. 2(b). This seems to be painless, because $p'=0$ and $k=i\mu$ do not cause any singularities in the potential $v(p,k_1)$ or $v(k_1,p')$. However, these cuts also pinch the real axis of the three-body momentum plane at

$$q' = [\tfrac{4}{3}(mE + \mu^2)]^{1/2}$$

and

$$q^2 + q'^2/4 + qq'\cos\theta = 0$$

or

$$\vec{q} = \pm\vec{q}\,'/2 .$$

This is also made clearer by the structure of $v(p',k)$ in the k plane (Fig. 3) by choosing $p' \ge 0$. Figure 3 is very similar to the three-body structure of the kernel of the Faddeev equations except for the poles of the two-body propagator. Here, the three-body integral contours are illustrated by $s \to O \to a$ and $s \to O \to b$, corresponding to the integral contours of the real axis and of the contour deformation methods, respectively. The latter contour crosses a cut and creeps into the other Riemann sheet, and so it can never be used in three-body calculations. Furthermore, one also has cuts for the on-shell potential $v(k,k)$ from $i\mu/2$ to $i\infty$ and $-i\mu/2$ to $-i\infty$. Therefore, at a point P which is defined by $p'=0$ and $k=i\mu$, a very narrow loophole is bunged with the on-shell potential cut for the other contour $s \to O \to a$. In other words, one can never draw the three-body integral contour without pain; hence, the direct extension of the KN theory to $k^2 < 0$ is not valid. The GSE theory, however, chooses $k_1 \ge 0$ for $k^2 < 0$; then the potentials $v(p,k_1)$ and $v(k_1,p')$ have no pinching singularity in the three-body momentum space. Also the cuts of $v(k_1,k_1)$ are safe and similar to the Fig. 2(a) case if the parameter k_1 is chosen greater than zero instead of having P greater than zero. Therefore, the analytic structure of the GSE form factors $\varphi(p,k_1;z)$ and $\chi(k_1,p';z)$ is a

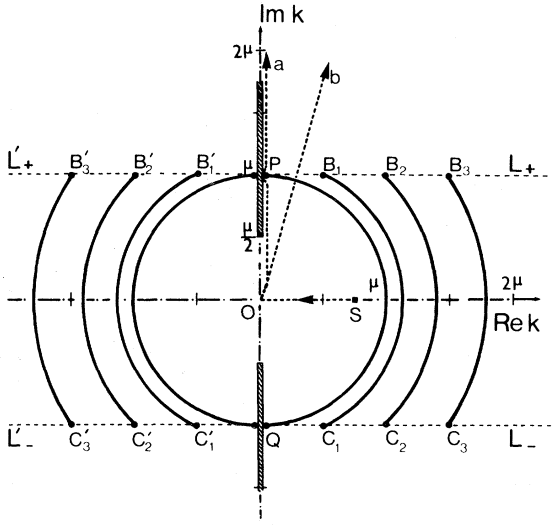

FIG. 3. k-plane cuts of $v(p',k)$ and $v(k,k)$ are illustrated for the parameter $0 \le p' < \infty$. The cuts PQ on both sides ($\mathrm{Re}k > 0$ and $\mathrm{Re}k < 0$) pinch the imaginary axis at P and Q. These cuts are given by the parameter value $p'=0$ corresponding to Fig. 2(b). In the same way, solid lines B_1C_1 and $B'_1C'_1$ are cuts which are given by $p'=\mu/2$; cuts B_2C_2 and $B'_2C'_2$ are for $p'=\mu$; cuts B_3C_3 and $B'_3C'_3$ are for $p'=3\mu/2$; and so on. Dashed lines $L_+L'_+$ and $L_-L'_-$ are the loci of these branch points. Shadowy lines from $i\mu/2$ to $i\infty$ and from $-i\mu/2$ to $-i\infty$ are well-known branch cuts given by $v(k,k)$. Dotted lines $s \to O \to a$ and $s \to O \to b$ correspond to different three-body integral contours.

favorable one in all energy regions for two- and three-body problems. Furthermore, in the rank-N case, the cuts of $\varphi(p,k_j;z)$ and $\chi(k_i,p';z)$ for $k_2,k_3,\ldots,k_N$ are trivial and safe in two- and three-body calculations.

D. Bound states

We already mentioned that the GSE formalism is exact for the on- and half-off-(the-energy-)shell t matrices.[37,40] In this subsection, we want to prove that the GSE equation is mathematically exact in the rank-one approximation for the bound-state case and any positive value of the fitting parameter. It is well known that the bound states are represented by the poles of the D function, which is the denominator of the N/D separation of the t matrix in the terminology of dispersion theory. Therefore, one can easily imagine that the separation is contained in $\theta_{ij}(z)/\theta(z)$ in the first term of the right-hand side of Eq. (1b), although $\theta(z)$ is in general not identical to the D function.

In order to simplify our discussion, let us take the rank-one approximation; then $\theta(z)$ is given by $A(k_1,k_1;z)$. In Eq. (4a), the condition that there be a bound state at $z=-E_B$ should be written as

$$A(k_1,k_1;-E_B)=0\ . \tag{27}$$

Substituting this into Eq. (4a), we obtain

$$\begin{aligned}
A(k_1,k_1;z)&=v(k_1,k_1)-\frac{1}{2\pi^2}\int_0^\infty dp''\cdot p''^2\frac{v(k_1,p'')\varphi(p'',k_1;z)}{z-p''^2/m}\\
&\quad-\frac{1}{2\pi^2}\int_0^\infty dp''\cdot p''^2\frac{v(k_1,p'')\varphi(p'',k_1;-E_B)}{-E_B-p''^2/m}+\frac{1}{2\pi^2}\int_0^\infty dp''\cdot p''^2\frac{v(k_1,p'')\varphi(p'',k_1;-E_B)}{-E_B-p''^2/m}\\
&=\frac{-1}{2\pi^2}\int_0^\infty dp''\cdot p''^2 v(k_1,p'')\left\{\frac{\varphi(p'',k_1;z)}{z-p''^2/m}+\frac{\varphi(p'',k_1;-E_B)}{E_B+p''^2/m}\right\}\\
&=\frac{-(E_B+z)}{2\pi^2}\int_0^\infty dp''\cdot p''^2\frac{v(k_1,p'')}{(E_B+p''^2/m)}\left\{\frac{\varphi(p'',k_1;z)}{z-p''^2/m}+\frac{\varphi(p'',k_1;-E_B)-\varphi(p'',k_1;z)}{E_B+z}\right\}\\
&\equiv-(E_B+z)F(k_1,k_1;z,E_B)\ ,
\end{aligned} \tag{28}$$

where $F(k_1,k_1;z,E_B)$ is a regular function at $z=-E_B$. Thus the rank-one approximation for the t matrix is represented by

$$t(p,p';z)=-\frac{\varphi(p,k_1;z)\chi(k_1,p';z)}{(z+E_B)F(k_1,k_1;z,E_B)}+t^{(2)}(p,p';z)\ , \tag{29}$$

where the t matrix diverges at the bound-state energy $z=-E_B$ for any value of the fitting parameter k_1. The general form $A(k_i,k_j;z)$ also fulfills the conditions

$$A(k_i,k_j;-E_B)=0\ , \tag{30}$$

in analogy to Eq. (27). Finally, we obtain for the general case of Eq. (28)

$$A(k_i,k_j;z)=-\frac{(E_B+z)}{2\pi^2}\int_0^\infty dp''\cdot p''^2\frac{v(k_i,p'')}{(E_B+p''^2/m)}\left\{\frac{\varphi(p'',k_j;z)}{z-p''^2/m}+\frac{\varphi(p'',k_j;-E_B)-\varphi(p'',k_j;z)}{E_B+z}\right\}, \tag{31}$$

$$=-(E_B+z)F(k_i,k_j;z,E_B)\ , \tag{32}$$

where the function $F(k_i,k_j;z,E_B)$ is also regular at $z=-E_B$. Therefore, it is easily seen that in Eq. (1b) the separable term diverges as $1/(z+E_B)$, although every matrix element of $\theta(z)$ and $\theta_{ij}(z)$ is zero in the vicinity of the bound state. Furthermore, the pole is independent of the rank and the number of fitting parameters. This result will be confirmed numerically in Sec. III by calculating the deuteron binding energy for the $^3S_1$-$^3D_1$ state.

III. NUMERICAL RESULTS

In this section, we would like to obtain the separable form of the Reid soft-core (RSC) nucleon-

nucleon t matrix.[12] First of all, we take for the rank-one parameter k_1 the on-shell momentum k to normalize the exact on- and half-off-shell t matrices as proposed by Kowalski and Noyes for positive energies.[30] Their procedure, however, is not sufficient in the whole energy region; the parameters cause unphysical singularities of the term $t^{(2)}(p,p';z)$. For instance, the $^1S_0$ state of the nucleon-nucleon interaction at 125 MeV corresponds to a zero of the phase shift.[35,37,40] In order to avoid such a difficulty and also to get a good fit of the off-shell elements, we propose to determine the second fitting parameter k_2 by minimizing the norm $\bar{\chi}_2(k_1,k_2;z)$ over the entire energy region for fixed k_1. This norm is defined in the general rank case by

$$\bar{\chi}_N{}^2(k_1,k_2,\ldots,k_N;z)=\sum_{\alpha,\beta=1}^{M}\int_0^\infty dp\int_0^\infty dp' F^*_{\alpha\beta}(p,p';k_1,k_2,\ldots,k_N;z)F_{\beta\alpha}(p',p;k_1,k_2,\ldots,k_N;z)$$
$$=\sum_{\alpha,\beta=1}^{M}||t^{(N+1)}_{\alpha\beta}(p,p';z)G_0(p';z)||^2 . \tag{33}$$

Here α and β denote the channels. Using the rank-N nonseparable term of the t matrix given in Eqs. (1a), (1b), (3a), and also (A10), the integrand is given by

$$F_{\alpha\beta}(p,p';k_1,k_2,\ldots,k_N;z)=t^{(N+1)}_{\alpha\beta}(p,p';z)G_0(p';z) ,$$

with

$$G_0(p';z)=\frac{1}{2\pi^2}\frac{p'^2}{z-p'^2/m+i\epsilon} .$$

In the same way, we can obtain a set of suitable parameters $k_3,k_4,\ldots,$ and k_N, step by step. For negative energies, one has to take a positive parameter k_1, and also minimize the norm $\bar{\chi}_1(k_1,z)$ of Eq. (33), because of the pinching singularity just discussed above.

On the other hand, we can also define the Hilbert-Schmidt norm for the total t matrix and the separable term, similar to Eq. (33). Therefore, the convergence of the separable approximation is treated by a ratio which is (by using both norms) defined by

$$R_{\rm sep}(N)=\frac{\sum_{\alpha,\beta=1}^{M}||t^{\rm sep}_{\alpha\beta}(\text{rank-}N)G_0||}{\sum_{\alpha,\beta=1}^{M}||t_{\alpha\beta}(\text{total})G_0||}100(\%) . \tag{34}$$

Each element of the numerator and the denominator functions represents the norm of the rank-N separable t matrix and that of the total t matrix, respectively.

A. $^1S_0$ state fit

In this subsection, we would like to apply our GSE method to the low energy nucleon-nucleon interaction which is mainly given by the $^1S_0$ and $^3S_1$-$^3D_1$ states. Many historical separable fits were reviewed in Sec. I. We remind the reader that the on- and half-off-shell t matrices for positive energies and the bound-state poles are exact in our method, which has been shown in Sec. II and Ref. 40. Therefore, we are only interested in the off-shell fit of the separable expansion. Consequently, we obtain a set of fitting parameters which are shown in Table I, and the results of the convergence are given in Table II for the $^1S_0$ state of the RSC potential. The fits are dramatically improved by the rank-two and rank-three approximations. Furthermore, the rank-two approximation can exactly reproduce the phase shift in the whole energy region defined by the RSC potential. Here, we also show in Figs. 4(a) and (b) the real parts of some individual off-shell t matrices to compare them with the results of AS.[28]

On the other hand, the imaginary part of the t matrix is mathematically exact in our rank-one approximation, because the GSE satisfies the unitarity relation. So we do not need a comparison with the results of AS which is almost exact in their rank-three approximation. In Figs. 5(a) and (b), the amplitude at zero energy which is given only by a real function is compared with the corresponding one of AS. The zero energy case is very important because the absolute value of the t matrix is largely enhanced by a virtual $^1S_0$ bound state, and this enhancement

TABLE I. GSE parameters for the $^1S_0$ state of the RSC nucleon-nucleon interaction.

	$E_{\rm c.m.}\geq 0$	$E_{\rm c.m.}<0$
k_1	k	0.0
k_2	$\dfrac{4320.0\alpha}{(k/\alpha)^2+216.0}$	$30.5\alpha-\dfrac{46.25\alpha}{(k/10.0\alpha)^4+2.5}$
k_3	45.0α	45.0α
k_4	10.0α	10.0α
	$\alpha=0.2316(\text{fm}^{-1})$,	$k=\sqrt{mE_{\rm c.m.}}=\sqrt{mz}$

TABLE II. The convergence of the Hilbert-Schmidt norm of the fully off-shell separable t matrix in the RSC $^1S_0$ state, corresponding to the rank-one approximation using k_1, the rank-two approximation using k_1 and k_2, and the rank-three approximation using k_1, k_2, and k_3, respectively.

$E_{c.m.}$ (MeV)	$R_{sep}(1)$ (%)	$R_{sep}(2)$ (%)	$R_{sep}(3)$ (%)	$R_{sep}(4)$ (%)
500.0	90.7	88.1	107.0	104.9
250.0	123.3	82.0	101.4	102.5
200.0	157.8	81.2	99.9	100.0
150.0	307.6	83.0	100.1	100.0
125.0	1648.6	85.1	101.0	99.8
100.0	282.8	88.3	102.6	99.6
50.0	64.1	97.6	107.5	99.2
30.0	64.2	102.0	109.5	99.1
10.0	79.4	104.5	108.6	99.0
0.0	99.6	99.8	100.0	100.0
−10.0	72.5	87.4	97.5	100.5
−30.0	54.0	84.0	100.3	102.5
−50.0	46.3	82.9	101.5	103.0
−100.0	38.6	82.0	102.5	102.5
−125.0	36.9	82.3	102.9	102.0
−150.0	35.9	83.1	103.4	101.5
−200.0	35.1	86.1	105.2	100.5
−250.0	34.9	90.1	107.1	100.0
−500.0	37.8	99.9	108.9	101.5

causes a final state interaction in few-body scattering problems. In both cases, one can see that our results are better than each of the other individual t matrices. Compare also the total averages given in Table II. For the negative energy case, one has no

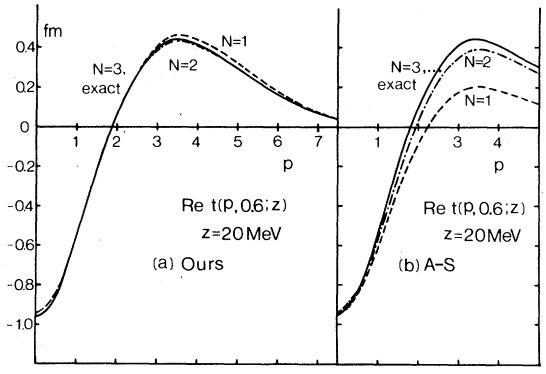

FIG. 4. Comparison of the convergence between the real parts of off-shell t matrices given by Adhikari and Sloan (b) and ours (a). Dashed lines are rank-one approximations. Dotted-dashed lines are rank-two approximations. The solid lines show exact off-shell t matrices (in fm), where the unit of p is fm^{-1}, and the two-body center of mass energy is $z=20$ MeV with a fixed momentum $p'=0.6$ fm^{-1}. The imaginary part of the t matrices is exactly reproduced by our method.

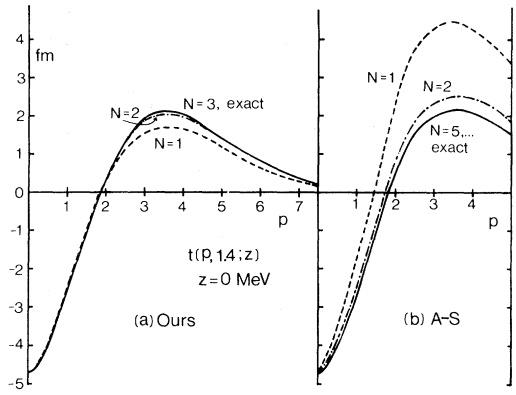

FIG. 5. Comparison of the convergency between off-shell t matrices given by Adhikari and Sloan (b) and our results (a) for $p'=1.4$ fm^{-1} and zero energy. Description of the lines is the same as in Figs. 4(a) and 4(b).

such guide comparable to renormalization in the rank-one positive energy case. However, we obtained a very good fit to the exact off-shell t matrix by the rank-two or rank-three approximation using Eq. (33). We are also interested in some characteristic GSE structures in higher energy regions. Figure 6 illustrates the off-shell t matrix at the critical ener-

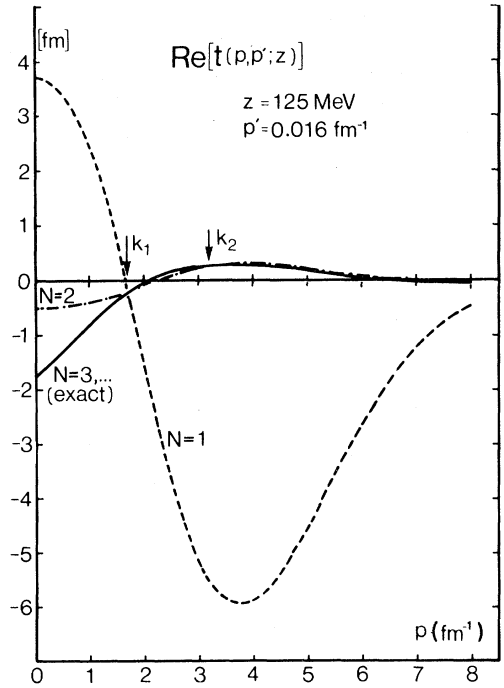

FIG. 6. Off-shell t matrix (real part) for $p'=0.016$ fm^{-1} and $z=125$ MeV. The dashed line shows the rank-one, the dotted-dashed line the rank-two, and the solid line the rank-three and higher approximations. The solid line overlaps with the exact value within drawing accuracy.

gy of 125 MeV mentioned above. Figure 7 shows the higher energy case. The rank-one curve in Fig. 6 is an example for an unphysical enhancement of the separable amplitude by an $^1S_0$ state, where

$$\eta(z)=t(k,k;k^2+i\epsilon)=0$$

in the whole off-energy-shell momentum region. But the rank-two and rank-three calculations completely erase such a singularity automatically. These properties of the GSE method were already mentioned in Ref. 37. Thus we obtain a good convergence.

B. $^3S_1$-$^3D_1$ state fit

The $^3S_1$-$^3D_1$ state is one of the most interesting objects in the nucleon-nucleon interaction, because it is very rare to find a fit consistent with the experimental results, among them the phase shifts $\delta(^3S_1)$ and $\delta(^3D_1)$ and the mixing parameter ϵ_1, by using any separable expansion method.[6,26]

The GSE method is applied straightforwardly to the problem and gets exact on- and half-off-shell values. Thus, we do not repeat them as they are given by Reid.[12] The fitting parameters for the fully off-shell separable amplitudes are shown in Table III. The fitting ratios are given in Table IV. For the individual results of the off-shell t matrices, AS showed a good fit in their rank-nine approximation for t_{SS} (0.06 fm^{-1}, p'; 48 MeV), and in rank-five approximation for t_{DD} (1.46 fm^{-1}, p'; 100 MeV).[27] However, the GSE reproduces them exactly in its rank-three approximation and well even in its rank-two approximation. The unphysical enhancement of the separable amplitude never appears, because the elements of the denominator function $\eta(z)$ in Eq. (1a) do not vanish simultaneously. That means $\eta(z)\neq 0$ for

$$t_{SS}(k,k;z)=0\ ,$$

$$t_{SD}(k,k;z)=t_{DS}(k,k;z)\neq 0\ ,$$

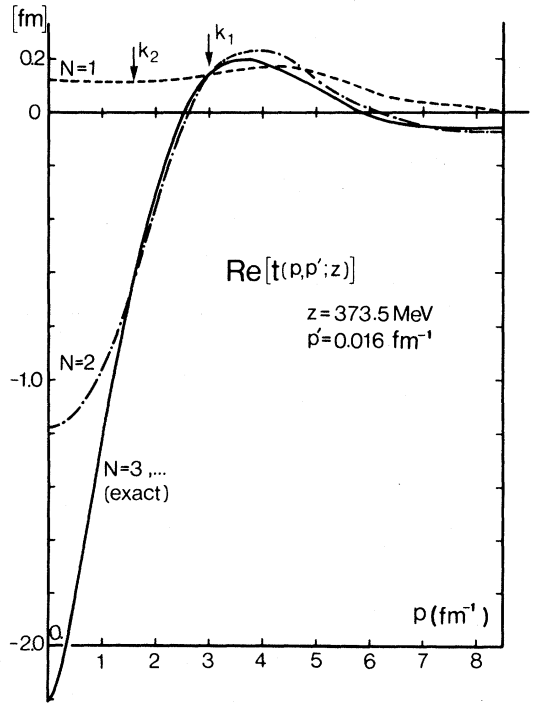

FIG. 7. Off-shell t matrix (real part) for $p'=0.016$ fm^{-1} and $z=373.5$ MeV. The description of the lines is the same as in Fig. 6.

TABLE III. GSE parameters for the $^3S_1$-$^3D_1$ state of the RSC nucleon-nucleon interaction.

		$E_{\text{c.m.}}\geq 0$	$E_{\text{c.m.}}<0$
k_1		k	0.55α
k_2		$0.85k+10.0\alpha$	28.55α
k_3	Set I	45.0α	2.5α
	Set II	$40.5\alpha+6.0\alpha\exp(-k^4/\alpha^4)$	2.5α
	$\alpha=0.2316$ (fm^{-1}), $k=\sqrt{mE_{\text{c.m.}}}=\sqrt{mz}$		

TABLE IV. The convergence of the Hilbert-Schmidt norm of the fully off-shell separable t matrices for the $^3S_1$-$^3D_1$ state, corresponding to the rank-one approximation using k_1, the rank-two approximation using k_1 and k_2, and the rank-three approximation using k_1, k_2, and k_3, respectively.

$E_{\text{c.m.}}$ (MeV)	$R_{\text{sep}}(1)$ (%)	$R_{\text{sep}}(2)$ (%)	$R_{\text{sep}}(3)$ (%) Set I	$R_{\text{sep}}(3)$ (%) Set II
500.0	87.6	90.9	99.5	100.0
250.0	143.2	88.3	101.0	100.0
200.0	331.7	86.9	100.1	100.0
150.0	110.0	85.0	98.2	100.0
125.0	29.8	88.2	97.8	100.0
100.0	7.0	83.5	96.0	100.0
50.0	11.8	83.6	94.7	100.0
30.0	65.1	90.8	97.3	100.0
10.0	86.3	105.5	109.1	100.0
0.0	99.0	100.0	100.0	100.0
−10.0	82.0	85.6	102.4	
−30.0	59.6	79.7	102.1	
−50.0	48.6	81.1	101.1	
−100.0	36.3	84.9	100.0	
−125.0	33.3	86.1	100.1	
−150.0	31.3	87.1	99.5	
−200.0	28.6	88.4	101.7	
−250.0	27.4	89.2	101.2	
−500.0	25.9	91.0	102.2	

and

$$t_{DD}(k,k;z)\neq 0$$

for the typical energy $z=150$ MeV at which $\delta(^3S_1)$ changes sign. Therefore, the rank-one approximation is mathematically exact for the on- and half-off-shell amplitudes in all positive energies for which the RSC potential is used. Furthermore, by increasing the rank such a situation as the unphysical enhancement becomes very rare, since we will have a lot of nonvanishing elements in $\eta(z)$. Our parameters are chosen to be free from such a difficulty.

For the negative energy case the deuteron binding energy is given by the rank-one approximation, in which

$$t_{SS}(k_1,k_1;-E_B)=t_{SD}(k_1,k_1;-E_B)=t_{DS}(k_1,k_1;-E_B)=t_{DD}(k_1,k_1;-E_B)=\infty \ .$$

Consequently, it is seen that the rank-one or the rank-two approximation is mathematically exact for the on- and half-off-energy-shell t matrices in the whole energy region for which the RSC potential is used. Furthermore, the rank-two and the rank-three approximation can dramatically improve the convergence of the separable expansion in the fully off-energy-shell region.

Finally, it should be mentioned that the GSE parameters are chosen as energy-dependent functions. Hence, if the parameters $k_i(z)$ coincide, e.g., $k_1(z_c)=k_2(z_c)$, a division like 0/0 occurs in the first and second terms of the right-hand side of Eq. (1a). This is harmless, but $k_i(z_c)$ $(i\geq 2)$ should be shifted by a small amount to $k_i(z_c)+\Delta$ as a numerical trick.

IV. CONCLUSION AND DISCUSSION

From the above discussion, one can say that the GSE theory can reproduce the off-energy-shell t matrix of a given potential by a very low rank separable expansion formalism. Furthermore, since our formalism has no restriction on the potential, except for the necessity of obtaining its momentum representation, one can automatically use it for a short ranged local as well as for a nonlocal potential. Moreover, one should notice that one can also use the GSE formalism for a separable t matrix which has already been obtained by another method. In that case, where the momentum representation of the potential cannot be obtained, e.g., a potential with a hard core, the GSE method can be applied in the rank-one approximation, because the half off-(on-)the energy shell wave function is given.

Here, we review the outstanding features of the GSE theory.

(1) The off-shell two-body t matrix of a given potential consists of two parts, a separable term and a nonseparable residual term, which are related by a simple algebraic equation.

(2) The form factor which appears in the separable term is obtained by solving a nonsingular LS type of integral equation. The nonseparable residual term also satisfies a similar nonsingular LS type of integral equation.

(3) The separable term is mathematically exact on the energy shell and half off (on) the energy shell.

(4) The lowest rank approximation, except for the energy where $\eta(z)=0$ in Eqs. (1a) and (2a), gives the exact separable on- and half-off- (on-)shell t matrices.

(5) The separable term itself satisfies the off-energy-shell unitarity relation. The residual term does not violate the unitarity relation.

(6) The GSE parameters are chosen to be energy-dependent functions. If the parameters k_i coincide, their values should be shifted by a small amount Δ to $k_i+\Delta$ $(i\geq 2)$.

(7) If the off-shell momentum coincides with the parameters, the separable amplitude is mathematically exact. In the other case, the nonseparable term should be decreased by increasing the rank. One finds that $N=2$ produces good accuracy and $N=3$ high accuracy for the $^1S_0$ and $^3S_1$-$^3D_1$ states.

(8) By substituting a given separable amplitude of rank n into the GSE formalism of Eq. (1a), the rank-one GSE approximation is mathematically exact on and half off the energy shell. The fully off-shell amplitude can be exactly reproduced by the same rank as that of the original amplitude. Moreover, the GSE reduces the required rank and consequently rank N ($\ll n$) is sufficient to reproduce the given amplitude. One can obtain a rank economized separable amplitude by substituting such a given amplitude.

(9) The bound state is exactly reproduced by the rank-one separable term. The binding energy is independent of the rank and the value of the fitting parameters.

(10) Those parameters (Tables I and III) are not available for the potentials and the off-shell t matrices themselves,[43] but they are available for the kernels with the Green's function which were discussed in Ref. 37.

Finally, it should be stressed that the method of

how to obtain parameters depends on the numerical technique used to solve Eq. (33) for the $^1S_0$ and $^3S_1$-$^3D_1$ states. However, one can also choose the parameters by seeking enhancements of the kernels vG_0 and $v^{(N+1)}G_0$ as proposed in Ref. 37. These enhancements depend on the typical physical properties, such as the singularity of the Green's function and the poles of the given potential or the residual potential $v^{(N+1)}$. Consequently, the two-body GSE amplitudes can be used analogously to reproduce the three-body Faddeev equation for the numerical calculation. Furthermore, the method may be applied to obtain the separable amplitude of the three-body t matrix in four-body problems,[44] because the GSE is not restricted to a particular potential, but is generally applicable to almost all potentials and t matrices which can be represented in momentum space. Although our form factor is calculated numerically, the analytic properties are clear, as proved above. Therefore, the method is useful for obtaining precise numerical results in few-body calculations.

ACKNOWLEDGMENTS

The author would like to thank Professor Erich Schmid for his kind hospitality and useful discussions when he was staying in the Institute of Theoretical Physics of Tuebingen University, and also all members of the Institute for their warm hospitality. He is also indebted to Professor H. Pierre Noyes, Professor Marcelle l'Huillier, and Dr. Roland Kircher for encouraging interest and for valuable comments. The author would like to thank Mr. Hiroshi Yamada, Mr. Katsuhiko Ishitani, and Mr. Kazuharu Nakamura for their help with numerical calculations on the HITAC 8800 and M200H computers of the Tokyo University Computer Center.

APPENDIX: THE SEPARABLE EXPANSION FOR THE MULTICHANNEL CASE

The multichannel GSE formalism is given by

$$t_{\alpha\beta}(p,p';z)=\sum_{\gamma=1}^{M}\sum_{i=1}^{N}\frac{\varphi_{\alpha\gamma}^{(i)}(p,k_i;z)\chi_{\gamma\beta}^{(i)}(k_i,p';z)}{A_{\gamma\gamma}^{(i)}(k_i,k_i;z)}+t_{\alpha\beta}^{(N+1)}(p,p';z)\ , \tag{A1}$$

where α, β, and γ denote the channels ($\alpha,\beta,\gamma=1,2,\ldots,M$). The form factors satisfy the following integral equations:

$$\varphi_{\alpha\gamma}^{(i)}(p,k_i;z)=v_{\alpha\gamma}^{(i)}(p,k_i)+\sum_{\delta=1}^{M}\int_0^\infty dp''v_{\alpha\delta}^{(i+1)}(p,p'')G_0(p'';z)\varphi_{\delta\gamma}^{(i)}(p'',k_i;z) \tag{A2}$$

and

$$\chi_{\gamma\beta}^{(i)}(k_i,p';z)=v_{\gamma\beta}^{(i)}(k_i,p')+\sum_{\delta=1}^{M}\int_0^\infty dp''\chi_{\gamma\delta}^{(i)}(k_i,p'';z)G_0(p'';z)v_{\delta\beta}^{(i+1)}(p'',p')\ . \tag{A3}$$

The denominator function is given by

$$A_{\gamma\gamma}^{(i)}(k_i,k_i;z)=v_{\gamma\gamma}^{(i)}(k_i,k_i)-\sum_{\delta=1}^{M}\int_0^\infty dp''v_{\gamma\delta}^{(i)}(k_i,p'')G_0(p'';z)\varphi_{\delta\gamma}^{(i)}(p'',k_i;z) \tag{A4}$$

$$=v_{\gamma\gamma}^{(i)}(k_i,k_i)-\sum_{\delta=1}^{M}\int_0^\infty dp''\chi_{\gamma\delta}^{(i)}(k_i,p'';z)G_0(p'';z)v_{\delta\gamma}^{(i)}(p'',k_i)\ . \tag{A5}$$

These functions satisfy the recurrence formulae:

$$\varphi_{\alpha\gamma}^{(i+1)}(p,k_j;z)=\varphi_{\alpha\gamma}^{(i)}(p,k_j;z)-\sum_{\delta=1}^{M}\frac{\varphi_{\alpha\delta}^{(i)}(p,k_i;z)A_{\delta\gamma}^{(i)}(k_i,k_j;z)}{A_{\delta\delta}^{(i)}(k_i,k_i;z)}\ , \tag{A6}$$

$$\chi_{\gamma\beta}^{(i+1)}(k_j,p';z)=\chi_{\gamma\beta}^{(i)}(k_j,p';z)-\sum_{\delta=1}^{M}\frac{A_{\gamma\delta}^{(i)}(k_j,k_i;z)\chi_{\delta\beta}^{(i)}(k_i,p';z)}{A_{\delta\delta}^{(i)}(k_i,k_i;z)}\ , \tag{A7}$$

$$A_{\gamma\gamma}^{(i+1)}(k_{i+1},k_j;z)=A_{\gamma\gamma}^{(i)}(k_{i+1},k_j;z)-\sum_{\delta=1}^{M}\frac{A_{\gamma\delta}^{(i)}(k_{i+1},k_i;z)A_{\delta\gamma}^{(i)}(k_i,k_j;z)}{A_{\delta\delta}^{(i)}(k_i,k_i;z)}\ , \tag{A8}$$

and

$$A_{\gamma\gamma}^{(i+1)}(k_j,k_{i+1};z)=A_{\gamma\gamma}^{(i)}(k_j,k_{i+1};z)-\sum_{\delta=1}^{M}\frac{A_{\gamma\delta}^{(i)}(k_j,k_i;z)A_{\delta\gamma}^{(i)}(k_i,k_{i+1};z)}{A_{\delta\delta}^{(i)}(k_i,k_i;z)} . \tag{A9}$$

The remaining term of Eq. (A1) is the nonseparable amplitude which satisfies an integral equation of the LS type,

$$t_{\alpha\beta}^{(N+1)}(p,p';z)=v_{\alpha\beta}^{(N+1)}(p,p')+\sum_{\delta=1}^{M}\int_0^\infty dp''v_{\alpha\delta}^{(N+1)}(p,p'')G_0(p'';z)t_{\delta\beta}^{(N+1)}(p'',p';z) . \tag{A10}$$

Here, the boundary conditions are given by

$$v_{\alpha\beta}^{(N+1)}(p,k_j)=v_{\alpha\beta}^{(N+1)}(k_i,p')=v_{\alpha\beta}^{(N+1)}(k_i,k_j)=0 . \tag{A11}$$

Using Eq. (A10), we obtain

$$t_{\alpha\beta}^{(N+1)}(p,k_j;z)=t_{\alpha\beta}^{(N+1)}(k_i,p';z)=t_{\alpha\beta}^{(N+1)}(k_i,k_j;z)=0 . \tag{A12}$$

On the other hand, recurrence formulae (A6)—(A9) lead to the following equations:

$$\varphi_{\alpha\gamma}(p,k_j;z)=v_{\alpha\gamma}(p,k_j)+\sum_{\delta=1}^{M}\int_0^\infty dp''v_{\alpha\delta}^{(N+1)}(p,p'')G_0(p'';z)\varphi_{\delta\gamma}(p'',k_j;z) , \tag{A13}$$

$$\chi_{\gamma\beta}(k_i,p';z)=v_{\gamma\beta}(k_i,p')+\sum_{\delta=1}^{M}\int_0^\infty dp''\chi_{\gamma\delta}(k_i,p'';z)G_0(p'';z)v_{\delta\beta}^{(N+1)}(p'',p') , \tag{A14}$$

and

$$A_{\alpha\beta}(k_i,k_j;z)=v_{\alpha\beta}(k_i,k_j)-\sum_{\delta=1}^{M}\int_0^\infty dp''v_{\alpha\delta}(k_i,p'')G_0(p'';z)\varphi_{\delta\beta}(p'',k_j;z) , \tag{A15}$$

or

$$A_{\alpha\beta}(k_i,k_j;z)=v_{\alpha\beta}(k_i,k_j)-\sum_{\delta=1}^{M}\int_0^\infty dp''\chi_{\alpha\delta}(k_i,p'';z)G_0(p'';z)v_{\delta\beta}(p'',k_j) . \tag{A16}$$

To simplify notation, $\varphi_{\alpha\gamma}(p,k_j;z)$, $\chi_{\gamma\beta}(k_i,p';z)$, $v_{\alpha\beta}(p,p')$, and $A_{\alpha\beta}(k_i,k_j;z)$ are used instead of $\varphi_{\alpha\beta}^{(1)}(p,k_j;z)$, $\chi_{\gamma\beta}^{(1)}(k_i,p';z)$, $v_{\alpha\beta}^{(1)}(p,p')$, and $A_{\alpha\beta}^{(1)}(k_i,k_j;z)$, respectively. Therefore, an alternative expansion formula for the t matrix is

$$t_{\alpha\beta}(p,p';z)=\sum_{\gamma,\delta}^{M}\sum_{i,j}^{N}\frac{\theta_{ij}^{\gamma\delta}(z)}{\theta(z)}\varphi_{\alpha\delta}(p,k_j;z)\chi_{\gamma\beta}(k_i,p';z)+t_{\alpha\beta}^{(N+1)}(p,p';z) , \tag{A17}$$

where $\theta(z)$ is given by $\det\{A_{\alpha\beta}(k_i,k_j;z)\}$, and $\theta_{ij}^{\gamma\delta}(z)$ is the $i-j$ and $\gamma-\delta$ cofactor of $\theta(z)$. Therefore, Eq. (A17) becomes

$$t_{\alpha\beta}(p,p';z)=-\begin{vmatrix} \underline{A}_{11} & \underline{A}_{12} & \cdots & \underline{A}_{1N} & \chi_{1p'} \\ \underline{A}_{21} & \underline{A}_{22} & \cdots & \underline{A}_{2N} & \chi_{2p'} \\ \vdots & \vdots & & \vdots & \vdots \\ \underline{A}_{N1} & \underline{A}_{N2} & \cdots & \underline{A}_{NN} & \chi_{Np'} \\ \varphi_{p1} & \varphi_{p2} & \cdots & \varphi_{pN} & 0 \end{vmatrix} \Bigg/ \begin{vmatrix} \underline{A}_{11} & \underline{A}_{12} & \cdots & \underline{A}_{1N} \\ \underline{A}_{21} & \underline{A}_{22} & \cdots & \underline{A}_{2N} \\ \vdots & \vdots & & \vdots \\ \underline{A}_{N1} & \underline{A}_{N1} & \cdots & \underline{A}_{NN} \end{vmatrix} + t_{\alpha\beta}^{(N+1)}(p,p'z) . \tag{A18}$$

Here, we used the following algebraical notations:

$$\underline{A}_{ij}\equiv\begin{bmatrix} A_{11}(k_i,k_j;z) & \cdots & A_{1M}(k_i,k_j;z) \\ \vdots & & \vdots \\ A_{M1}(k_i,k_j;z) & \cdots & A_{MM}(k_i,k_j;z) \end{bmatrix} , \tag{A19}$$

$$\varphi_{pj}\equiv[\varphi_{\alpha1}(p,k_j;z),\varphi_{\alpha2}(p,k_j;z),\ldots,\varphi_{\alpha M}(p,k_j;z)] , \tag{A20}$$

and

$$\chi^T_{ip'} \equiv [\chi_{1\beta}(k_i,p';z),\chi_{2\beta}(k_i,p';z),\ldots,\chi_{M\beta}(k_i,p';z)] \ . \tag{A21}$$

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