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Threshold estimates of $(\pi,2\pi)$ on nuclei in the Fermi gas model: One-body mechanism

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The one-nucleon contribution to threshold pion production in pion-nucleus collisions is calculated using a Fermi gas model of the nuclear excitation spectrum in order to carry out the sum over final nuclear states. For the one-body input a threshold approximant to the production amplitude from the phenomenological Lagrangian theory is used. The cross sections are found to have an η^4 dependence near threshold, where $\eta=(\omega_k-2m_\pi)/m_\pi$ for total incident pion energy ω_k. In a simple eikonal approach incident pion attenuation is found to lead to a reduction of threshold estimates by a factor of ~ 10.

[NUCLEAR REACTIONS $^{11}B(\pi^-,2\pi^-)$, $^{14}N(\pi^-,2\pi^-)$, $^{18}O(\pi^-,2\pi^-)$, $^{27}Al(\pi^+,2\pi^+)$, $^{31}P(\pi^+,2\pi^+)$; threshold pion production cross section calculated in the Fermi gas model with phenomenological Lagrangian input.]

Some time ago in anticipation of the experimental study of threshold pion production in pion-nucleus collisions at the various meson factories, I studied again[1,2] the $(\pi,2\pi)$ reaction in nuclei and presented estimates[2] for the total cross section for collective excitation in nuclei based on the simple collective approach afforded by the Goldhaber-Teller model generalized to spin-isospin vibrations.[3] I was then able to conclude that[2] "while the cross sections for the $(\pi,2\pi)$ reaction in nuclei are still expected to be quite small, the prospect for their accessibility seems reasonably improved."

Recently $(\pi,2\pi)$ activation measurements have been proposed at the Clinton P. Anderson Meson Physics Facility (LAMPF),[4] and this and the consequent need for threshold estimates for both $(\pi^-,2\pi^-)$ and $(\pi^+,2\pi^+)$ reactions on a variety of nuclear targets have prompted the present recalculation of the one-nucleon contribution to this process, this time in terms of the Fermi gas model of the nucleus. One recalls that this particular model proved rather efficacious in providing simple cross section estimates in the initial calculations of threshold pion electroproduction from nuclei[5] and the same is no less true of this approach in the present context. It turns out that some of the more tedious calculational[5] aspects of such a simple approach can be done away with if, instead of viewing the necessary phase space calculation conventionally as an integration over a response function,[6] one views it rather as a *Fermi averaging of phase space.* With this useful change in perspective I am able to relate directly the calculation of the Fermi-averaged phase space for the threshold emission of two pions to that for a single pion (which is what is encountered in the case of threshold electroproduction). I have relegated these interesting details (as well as those pertaining to the reconstruction[7] of the threshold behavior of the incoherent cross section of Ref. 6) to the Appendix.

In the threshold region the one-body $(\pi^\pm,2\pi^\pm)$ input is the one-body matrix element (in a nucleon-to-nucleon transition) which I write approximately as[8,9]

$$\mathscr{M}^{(1)}_{N\to N}(\pi^\pm,2\pi^\pm)=\mp i\left[\frac{1}{\sqrt{2F_\pi}}\right]^3\left[\frac{g_A}{g_V}\right]\times\left[\frac{m_\pi}{\omega_0-m_\pi}\right](\tau_\mp)(\vec{\sigma}\cdot\vec{k})\ , \tag{1}$$

where ω_0 is the total incident pion energy at threshold. [Note that this matrix element is perfectly analogous to the one-body matrix element for P-wave $\pi^\mp$ absorption (in a nucleon-to-nucleon transition). Thus the associated matrix element

$$\mathscr{M}^{(1)}_{N\to\Delta}(\pi^\pm,2\pi^\pm)=\mp 2i\left[\frac{1}{\sqrt{2F_\pi}}\right]^3\left[\frac{g_A}{g_V}\right]\times\left[\frac{m_\pi}{\omega_0-m_\pi}\right](T_\mp)(\vec{S}\cdot\vec{k})\ ,$$

with transition spin ($\vec{S}$) and isospin ($\vec{T}$) operators, will figure in a two-body contribution to the threshold (π,2π) cross section analogous to that occurring in P-wave pionic absorption. The estimate of this contribution will be the subject of a later communication.] It is then easy to write the expression for the threshold (π,2π) cross section with $\mathcal{M}^{(1)}_{N\to N}$ as input in the Fermi gas model. One has[10]

$$\sigma(\pi,2\pi)\simeq\frac{\omega_k}{k}\left[\frac{1}{\sqrt{2}F_\pi)}\right]^6\left[\frac{g_A}{g_V}\right]^2\left|\frac{m_\pi}{\omega_0-m_\pi}\right|^2 k^2$$
$$\times 2\Omega\int\frac{d\vec{p}\,d\vec{k}'d\vec{q}_1 d\vec{q}_2}{(8\omega_k\omega_1\omega_2)(2\pi)^{12}}\frac{1}{2!}\theta(p_F-p)\theta(\,|\,\vec{p}+\vec{k}'\,|\,-p_F)$$
$$\times 2\pi\delta\left[\frac{(\vec{p}+\vec{k}')^2}{2M^*}+\omega_1+\omega_2-\omega_k-\frac{p^2}{2M^*}\right](2\pi)^3\delta(\vec{k}'+\vec{q}_1+\vec{q}_2-\vec{k})\ , \tag{2}$$

where[11]

$$\Omega=A\left[\frac{4\pi r_0{}^3}{3}\right]$$

with $r_0=1.2$ fm. Expression (2) reduces to

$$\sigma(\pi,2\pi)\simeq\frac{k}{8}\Omega\left[\frac{1}{\sqrt{2}F_\pi}\right]^6\left[\frac{g_A}{g_V}\right]^2\left[\frac{m_\pi}{\omega_0-m_\pi}\right]^2\frac{1}{(2\pi)^8}I(\eta)\ , \tag{3}$$

with

$$I(\eta)\cong\int_{\substack{|\vec{p}+\vec{k}-\vec{q}_1-\vec{q}_2|>p_F\\ p_F>p}}\frac{d\vec{p}\,d\vec{q}_1 d\vec{q}_2}{m_\pi{}^2}\delta\left[\frac{(\vec{p}+\vec{k}-\vec{q}_1-\vec{q}_2)^2}{2M^*}-\frac{\vec{p}^2}{2M^*}-\eta m_\pi+\frac{q_1{}^2}{2m_\pi}+\frac{q_2{}^2}{2m_\pi}\right] \tag{4}$$

and

$$\eta=(\omega_k-2m_\pi)/m_\pi\ . \tag{5}$$

After making the threshold approximation of Ref. 6 in Eq. (4), namely,

$$\vec{p}+\vec{k}-\vec{q}_1-\vec{q}_2\simeq\vec{p}+\vec{k}$$

[I have already set

$$1/\omega_1\omega_2\simeq 1/m_\pi{}^2$$

with

$$\omega_1+\omega_2\simeq 2m_\pi-(q_1{}^2+q_2{}^2)/2m_\pi$$

inside the δ function], I find the threshold behavior,

$$I(\eta)\simeq\frac{2\pi^4}{3k}M^{*2}m_\pi{}^5\eta^4\ , \tag{6}$$

for $k<2p_F$ (with $p_F\simeq 260$ MeV) and for[12]

$$\eta<(k/2M^*m_\pi)(2p_F-k)\ . \tag{7}$$

Finally, for σ conveniently expressed in μb I write

$$\sigma(\pi,2\pi)=\left[\frac{1}{\sqrt{2}F_\pi}\right]^6\left[\frac{g_A}{g_V}\right]^2\left[\frac{m_\pi}{\omega_0-m_\pi}\right]^2\times\frac{2nr_0{}^3M^{*2}}{288(2\pi)^3}m_\pi{}^5\eta^4\left[\frac{10^4}{197.3}\right]\ , \tag{8}$$

where all masses (as well as F_π) are given in MeV and r_0 is given in fermis. The threshold predictions[13] of expression (8) are presented in Fig. 1. For display purposes the curves are extended beyond the region of validity given by (7) which determines the *proper* threshold region.

The effect of incident pion attenuation on the one-body cross section $\sigma(\pi^\pm\to 2\pi^\pm)$ may be semiquantitatively treated in a simple eikonal approach. I use for the distorted incident pion wave function the eikonal function[14]

$$\psi_{\vec{k}}(\vec{r};\pi^\pm)=e^{i\vec{k}\cdot\vec{r}}\exp\left[\frac{i2\pi A}{k}\bar{f}_0^{\pm}\int_{-\infty}^{z}\rho(\vec{b},\xi)d\xi\right]\ , \tag{9}$$

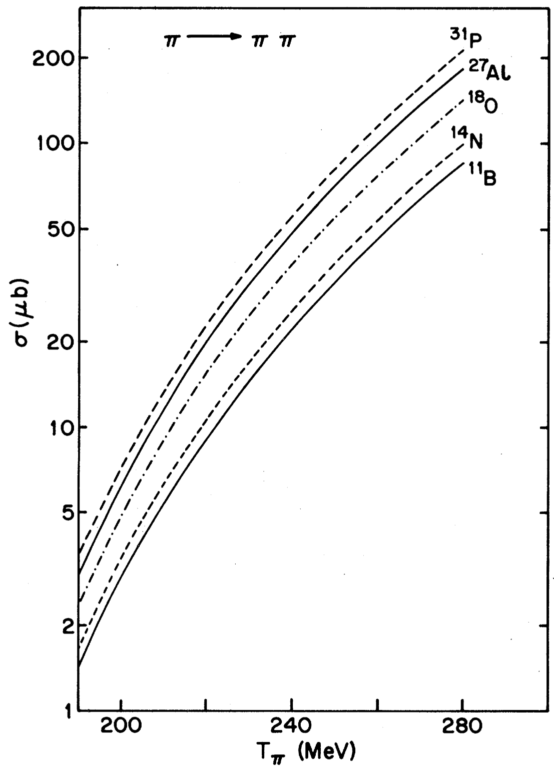

FIG. 1. Theoretical predictions for threshold $(\pi^-,2\pi^-)$ cross sections on ^{11}B, ^{14}N, and ^{18}O targets and for threshold $(\pi^+,2\pi^+)$ cross sections on ^{27}Al and ^{31}P targets versus incident pion kinetic energy T_π in the one-nucleon model.

where the integration is to be performed along the direction of the incident pion momentum $\vec{k}$ and $\vec{b}$ is the customary impact parameter variable perpendicular to that direction. I take the nuclear density ρ to be the same for protons and neutrons and uniform within the nuclear radius R,

$$\rho(r)=\frac{3}{4\pi R^3}\theta(R-(b^2+\xi^2)^{1/2}) \ . \quad (10)$$

$\bar{f}_0^{\pm}$ is the averaged forward $\pi^{\pm}N$ scattering amplitude, with[14]

$$\bar{f}_0^{\pm}=\frac{ik}{4\pi A}[N\sigma_{\pi^\pm n}(1-i\alpha_{\pi^\pm n}) + Z\sigma_{\pi^\pm p}(1-i\alpha_{\pi^\pm p})] \quad (11)$$

for N neutrons (Z protons) with total cross sections $\sigma_{\pi n}$ ($\sigma_{\pi p}$) and real-to-imaginary amplitude ratios $\alpha_{\pi n}$ ($\alpha_{\pi p}$). (Note that for $(\pi^{\pm},2\pi^{\pm})$ processes, there is only the *single* relation

$$\bar{f}_0^{\pm}\equiv\bar{f}_0(n)=\frac{ik}{4\pi A}[(A-n)\sigma_{\pi^+ n}(1-i\alpha_{\pi^+ n}) + n\sigma_{\pi^+ p}(1-i\alpha_{\pi^+ p})] \ , \quad (12)$$

where $n=Z$ for a $(\pi^+,2\pi^+)$ process and $n=N$ for a $(\pi^-,2\pi^-)$ process.)

In the *absence* of initial-state interaction one has formally for σ, using the discrete notation for simplicity,[15]

$$\sigma\propto\frac{1}{\Omega}\sum_{\vec{l}}\vec{k}^2(\delta_{\vec{l},\vec{k}})^2\langle\delta(\Delta E(\vec{l}))\rangle_{\text{Fermi av}} \ , \quad (13)$$

where Ω is the volume of the interaction "box." The usual result,

$$\sum_{\vec{l}}(\delta_{\vec{l},\vec{k}})^2\langle\delta(\Delta E(\vec{l}))\rangle_{\text{Fermi av}} =\Omega\sum_{\vec{l}}\delta_{\vec{l},\vec{k}}\langle\delta(\Delta E(\vec{l}))\rangle_{\text{Fermi av}} \ , \quad (14)$$

is, of course, understood. In the *presence* of initial-state interaction, this result is altered to

$$\sigma\propto\frac{1}{\Omega}\sum_{\vec{l}}\vec{l}^2\,|\tilde{\psi}_{\vec{k}}(\vec{l};\pi^{\pm})|^2\langle\delta(\Delta E(\vec{l}))\rangle_{\text{Fermi av}} \ , \quad (15)$$

with the transform $\tilde{\psi}_{\vec{k}}(l;\pi^{\pm})$ given by

$$\begin{aligned}\tilde{\psi}_{\vec{k}}(\vec{l};\pi^{\pm})&=\int_\Omega d\vec{r}\exp\left[i(\vec{k}-\vec{l})\cdot\vec{r}+ik\lambda\int_{-\infty}^{z}d\xi\,\theta(R^2-\vec{b}^2)^{1/2}-|\xi|\,\right]\\ &=\int d\vec{b}\,dz\,\theta((R^2-\vec{b}^2)^{1/2}-|z|)e^{-i\vec{l}_\perp\cdot\vec{b}+i(k-l_{\parallel})z}\\ &\quad\times\exp\left[ik\lambda\int_{-\infty}^{z}d\xi\,\theta((R^2-\vec{b}^2)^{1/2}-|\xi|)\right]\\ &=\int d\vec{b}\,e^{-i\vec{l}_\perp\cdot\vec{b}}\frac{2e^{ik\lambda(R^2-\vec{b}^2)^{1/2}}}{(k-l_{\parallel}+k\lambda)}\sin((k-l_{\parallel}+k\lambda)(R^2-\vec{b}^2)^{1/2})\\ &\simeq2\pi\int_0^\infty b\,db\,J_0(l_\perp b)\frac{2e^{ik\lambda(R^2-\vec{b}^2)^{1/2}}}{(k-l_{\parallel}+k\lambda)}\sin((k-l_{\parallel}+k\lambda)(R^2-\vec{b}^2)^{1/2}) \ ,\end{aligned} \quad (16)$$

where

$$\lambda \equiv \lambda_r + i\lambda_i = \frac{3}{2k^2 r_0^3}\bar{f}_0(n) \ . \tag{17}$$

In the limit of large R,

$$\begin{aligned}
&\sum_{\vec{l}} \vec{l}^2 |\tilde{\psi}_{\vec{k}}(\vec{l};\pi^\pm)|^2 \langle \delta(\Delta E(\vec{l}))\rangle_{\text{Fermi av}} \\
&\quad\simeq \sum_{\vec{l}} \vec{l}^2 \Bigg[\left[2\pi \int_0^\infty b\,db\,J_0(l_\perp b) 2\pi \int_0^\infty b'db' J_0(l_\perp b') \exp[i(k-l_{\parallel})((R^2-\vec{b}^2)^{1/2}-(R^2-\vec{b}'^2)^{1/2})] \right] \\
&\qquad \times \frac{1}{((l_{\parallel}-k+k\lambda_r)^2+(k\lambda_i)^2)} \\
&\qquad - \left[2\pi \int_0^\infty b\,db\,J_0(l_\perp b) 2\pi \int_0^R b'db' J_0(l_\perp b') \exp[i(k-l_{\parallel})((R^2-\vec{b}^2)^{1/2}-(R^2-\vec{b}'^2)^{1/2})] \right. \\
&\qquad\qquad \left. \times e^{2ik\lambda(R^2-\vec{b}'^2)^{1/2}} + \text{c.c.} \right] \\
&\qquad \times \frac{1}{[(l_{\parallel}-k+k\lambda_r)^2+(k\lambda_i)^2]} \Bigg] \langle \delta(\Delta E(\vec{l}))\rangle_{\text{Fermi av}} \ .
\end{aligned} \tag{18}$$

Performing the integration over $l_{\parallel}$ *first*, in the approximation of small width, where

$$\frac{1}{(l_{\parallel}-k+k\lambda_r)^2+(k\lambda_i)^2} \underset{(k\lambda_i)\to 0}{\longrightarrow} \frac{1}{k\lambda_i}\pi\delta(l_{\parallel}-k+k\lambda_r) \ , \tag{19}$$

and neglecting the small shift $k \to \bar{k} = k - \lambda k_r$ in the phases of the integrals over the impact parameter, one finds

$$\begin{aligned}
&\sum_{\vec{l}} \vec{l}^2 |\tilde{\psi}_{\vec{k}}(\vec{l};\pi^\pm)|^2 \langle \delta(\Delta E(\vec{l}))\rangle_{\text{Fermi av}} \\
&\quad\simeq \sum_{\vec{l}} \vec{l}^2 \left[\pi R^2 \frac{2\pi\delta(l_\perp)}{l_\perp} \frac{\pi}{k\lambda_i}\delta(l_{\parallel}-\bar{k}) - \frac{2\pi\delta(l_\perp)}{l_\perp} 4\pi \int_0^R b\,db\,e^{-2k\lambda(R^2-\vec{b}^2)^{1/2}} \frac{\pi}{k\lambda_i}\delta(l_{\parallel}-\bar{k}) \right] \\
&\qquad \times \langle \delta(\Delta E(\vec{l}))\rangle_{\text{Fermi av}} \\
&\quad\simeq \sum_{\vec{l}} \vec{l}^2 \frac{\pi R^2}{2k\lambda_i}(2\pi)^2 \frac{\delta(l_\perp)}{l_\perp}\delta(l_{\parallel}-\bar{k}) \langle \delta(\Delta E(\vec{l}))\rangle_{\text{Fermi av}} \left[1 - \frac{4}{R^2}\int_0^R b\,db\,e^{-2k\lambda_i(R^2-\vec{b}^2)^{1/2}} \right] \\
&\quad\simeq \Omega \bar{k}^2 \langle \delta(\Delta E(\bar{k}))\rangle_{\text{Fermi av}} \frac{1}{4k\lambda_i R} \left[1 + \frac{2}{k\lambda_i R} e^{-2k\lambda_i R} + \frac{1}{(k\lambda_i R)^2}(e^{-2k\lambda_i R}-1) \right] \ .
\end{aligned} \tag{20}$$

Thus the cross section corrected for incident pion attenuation is given to a good approximation in terms of the uncorrected cross section σ_0 by

$$\sigma = \sigma_0 \frac{\bar{k}^2}{k^2} \frac{\rho(\bar{k})}{\rho(k)} \frac{1}{4k\lambda_i R} \left[1 + \frac{2}{k\lambda_i R} e^{-2k\lambda_i R} + \frac{1}{(k\lambda_i R)^2}(e^{-2k\lambda_i R}-1) \right] \ , \tag{21}$$

where

$$\frac{\rho(\bar{k})}{\rho(k)}=\frac{\langle\delta(\Delta E(\bar{k}))\rangle_{\text{Fermi av}}}{\langle\delta(\Delta E(k))\rangle_{\text{Fermi av}}}\ . \tag{22}$$

The corrected threshold cross sections are displayed in Fig. 2. The input values of $\bar{f}_0(n)$ have been taken from the tabulation of Höhler *et al.*[16] One finds the uncorrected threshold estimates of Fig. 1 reduced by a factor of ~ 10.

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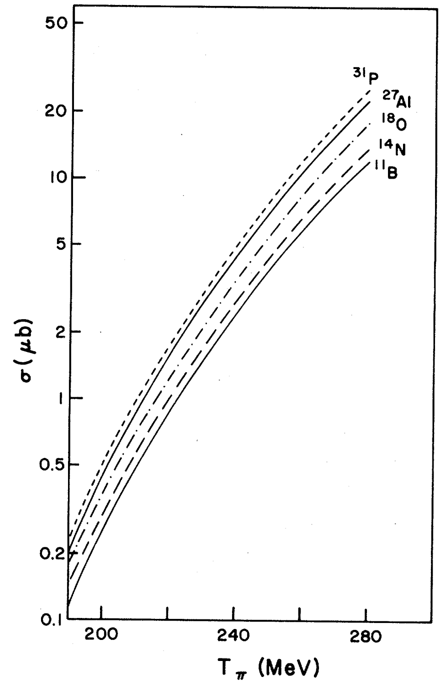

FIG. 2. Theoretical predictions corrected for incident pion attenuation for threshold $(\pi^-,2\pi^-)$ cross sections on ^{11}B, ^{14}N, and ^{18}O targets and for threshold $(\pi^+,2\pi^+)$ cross sections on ^{27}Al and ^{31}P targets versus incident pion kinetic energy T_π in the one-nucleon model.

APPENDIX: FERMI-AVERAGED PHASE SPACE FOR ONE- AND TWO-PION EMISSION NEAR THRESHOLD

The gas model calculation of Czyz and Walecka[5] of threshold pion electroproduction by nuclei proceeds straightforwardly via the integration of the product of the appropriate one-nucleon cross section and the familiar response function[6] $R(\epsilon_1-\epsilon_2-\omega_q,\vec{k}-\vec{q})$,[17]

$$R(\epsilon_1-\epsilon_2-\omega_q,\vec{k}-\vec{q})=\frac{\Omega}{2\pi^3}\int d\vec{p}\,\theta(p_F-p)\theta(|\vec{p}+\vec{k}-\vec{q}|-p_F)\delta\left[\epsilon_1-\epsilon_2-\omega_q-\frac{(\vec{p}+\vec{k}-\vec{q})^2}{2M^*}+\frac{p^2}{2M^*}\right], \tag{A1}$$

$$\begin{aligned}=\frac{\Omega M^*}{\pi^2|\vec{k}-\vec{q}|}\Bigg\{&M^*(\epsilon_1-\epsilon_2-\omega_q)-\frac{1}{2}\left[\left[\frac{M^*(\epsilon_1-\epsilon_2-\omega_q)}{|\vec{k}-\vec{q}|}+\frac{|\vec{k}-\vec{q}|}{2}\right]^2-p_F{}^2\right]\\&\times\theta\left[\frac{M^*(\epsilon_1-\epsilon_2-\omega_q)}{|\vec{k}-\vec{q}|}+\frac{|\vec{k}-\vec{q}|}{2}-p_F\right]\\&+\frac{1}{2}\left[\left[\frac{M^*(\epsilon_1-\epsilon_2-\omega_q)}{|\vec{k}-\vec{q}|}-\frac{|\vec{k}-\vec{q}|}{2}\right]^2-p_F{}^2\right]\\&\times\theta\left[\left|\frac{M^*(\epsilon_1-\epsilon_2-\omega_q)}{|\vec{k}-\vec{q}|}-\frac{|\vec{k}-\vec{q}|}{2}\right|-p_F\right]\Bigg\}\ ,\end{aligned} \tag{A2}$$

over the outgoing pion variables q and Ω_q. It is remarked[5] that the ensuing integrations are "elementary but quite tedious," although for $\epsilon_1-\epsilon_2$ close to m_π they simplify to some extent. In terms of an energy-defect variable, $\eta\equiv(\epsilon_1-\epsilon_2-m_\pi)/m_\pi$, they find the interesting threshold result[18]

$$\int \frac{d\vec{q}}{\omega_q} \int d\vec{p}\, \theta(p_F-p)\theta(|\vec{p}+\vec{k}-\vec{q}|-p_F)\delta\left[\epsilon_1-\epsilon_2-\omega_q-\frac{(\vec{p}+\vec{k}-\vec{q})^2}{2M^*}+\frac{p^2}{2M^*}\right]$$

$$=\frac{32\pi^2}{5}\sqrt{2}\frac{M^{*2}m_\pi{}^3}{|\vec{k}|}\eta^{5/2}\theta(2p_F-|\vec{k}|)\ , \qquad (A3)$$

where the region of validity of this formula is $\eta<(|\vec{k}|/2M^*m_\pi)(2p_F-|\vec{k}|)$.

On the other hand, it proves useful for determining the threshold behavior of the suitably weighted integrated response function, and *especially in the present context of two-particle emission,* to *defer* the Fermi averaging implied by the integration in (A1) over $\vec{p}$, i.e., to integrate the threshold approximant over the outgoing meson momentum $\vec{q}$ *first.* Thus I write

$$\int \frac{d\vec{q}}{\omega_q} R(\epsilon_1-\epsilon_2-\omega_q,\vec{k}-\vec{q})$$

$$\simeq\frac{\Omega}{2\pi^3}\int d\vec{p}\,\theta(p_F-p)\theta(|\vec{p}+\vec{k}|-p_F)\int\frac{d\vec{q}}{m_\pi}\delta\left[\frac{p^2}{2M^*}-\frac{(\vec{p}+\vec{k})^2}{2M^*}+\eta m_\pi-\frac{q^2}{2m_\pi}\right]\ , \qquad (A4)$$

where the threshold approximations,[19]

$$\omega_q\simeq m_\pi+\frac{q_2}{2m_\pi}\ ,\quad \vec{p}+\vec{k}-\vec{q}\simeq\vec{p}+\vec{k}\ , \qquad (A5)$$

enable just such a reversal of the order of integration over $\vec{p}$ and $\vec{q}$. Since the integrand of the threshold approximant (A4) is a function of the *magnitude* of q, there remains only the "Fermi averaging" of the result of a trivial q integration,

$$\int\frac{d\vec{q}}{\omega_q}R\simeq\frac{\Omega}{\pi^2}2\sqrt{2m_\pi}\int d\vec{p}\,\theta(p_F-p)\theta(|\vec{p}+\vec{k}|-p_F)\theta\left[\frac{p^2}{2M^*}-\frac{(\vec{p}+\vec{k})^2}{2M^*}+\eta m_\pi\right]$$

$$\times\left[\frac{p^2}{2M^*}-\frac{(\vec{p}+\vec{k})^2}{2M^*}+\eta m_\pi\right]^{1/2}\ . \qquad (A6)$$

This is routinely carried out using the formal device[20] introduced a long time ago by Dubois[20] for dealing with such calculations in the regime $k\leq 2p_F$. Thus

$$\int\frac{d\vec{q}}{\omega_q}R\simeq\Omega\frac{4}{\pi}\sqrt{2m_\pi}p_F{}^2k\int_0^1 d\alpha\int_{k\alpha/2p_F}^1 x\,dx\,\theta\left[\eta m_\pi+(2\alpha-1)\frac{k^2}{2M^*}-p_F\frac{kx}{M^*}\right]$$

$$\times\left[\eta m_\pi+(2\alpha-1)\frac{k^2}{2M^*}-p_F\frac{kx}{M^*}\right]^{1/2}\theta(2p_F-k)$$

$$\to\frac{16\Omega}{5\pi}\sqrt{2}\frac{M^{*2}}{|\vec{k}|}m_\pi{}^3\eta^{5/2}\theta[k/(2M^*m_\pi)(2p_F-k)-\eta]\theta(2p_F-k)\ . \qquad (A7)$$

In the case of near threshold pion production by pions one has to deal with the analogous Fermi-averaged phase space calculation involving two-particle emission. Conforming to the notation of the text, I write

$$\int \frac{d\vec{q}_1 d\vec{q}_2}{\omega_1\omega_2} R(\omega_k-\omega_1-\omega_2, \vec{k}-\vec{q}_1-q_2)$$

$$= \frac{\Omega}{2\pi^3}\int \frac{d\vec{q}_1 d\vec{q}_2}{\omega_1\omega_2}\int d\vec{p}\,\theta(p_F-p)\theta(|\vec{p}+\vec{k}-\vec{q}_1-\vec{q}_2|-p_F)\delta\left[\frac{(\vec{p}+\vec{k}-\vec{q}_1-\vec{q}_2)^2}{2M^*}-\frac{p^2}{2M^*} - (\omega_k-\omega_1-\omega_2)\right]$$

$$\underset{q_1,q_2\to 0}{\longrightarrow} \frac{\Omega}{2\pi^3}\int d\vec{p}\,\theta(p_F-p)\theta(|\vec{p}+\vec{k}|-p_F)$$

$$\times\int \frac{d\vec{q}_1 d\vec{q}_2}{m_\pi^2}\delta\left[\frac{(\vec{p}+\vec{k})^2}{2M^*}-\frac{p^2}{2M^*}-(\omega_k-2m_\pi)+\frac{1}{2m_\pi}(q_1^2+q_2^2)\right], \quad \text{(A8)}$$

where in its last incarnation the threshold approximations

$$\omega_1+\omega_2\to 2m_\pi+\frac{1}{2m_\pi}(q_1^2+q_2^2), \quad \vec{p}+\vec{k}-\vec{q}_1-\vec{q}_2\to\vec{p}+\vec{k}, \quad \text{(A9)}$$

have been made along with the explicit deferral of the integration over $\vec{p}$. The dependence of the threshold approximant on $Q^2=q_1^2+q_2^2$ suggests we work in the six-dimensional momentum space of the emitted pions with $d\vec{q}_1 d\vec{q}_2\to\pi^3 Q^5 dQ$. The result

$$\int \frac{d\vec{q}_1 d\vec{q}_2}{\omega_1\omega_2} R \to \frac{\pi\Omega}{3k} M^{*2} m_\pi^5 \eta^4 \theta[(k/2M^* m_\pi)(2p_F-k)-\eta]\theta(2p_F-k), \quad \text{(A10)}$$

with

$$\eta m_\pi = \omega_k - 2m_\pi, \quad \text{(A11)}$$

then follows routinely as in the case of one-particle emission discussed earlier.

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[6] W. Czyz and K. Gottfried, Ann. Phys. (N.Y.) 21, 47 (1963).

[7] Except for the explicit writing out of the response function $R(\Delta, \vec{t})$, where Δ and $\vec{t}$ are the energy transfer and momentum transfer, respectively, no further details of the calculation are given in Ref. 6.

[8] M. G. Olsson and L. Turner, Phys. Rev. Lett. 20, 1127 (1968).

[9] I take $F_\pi \simeq 82$ MeV and $g_A/g_V=1.17$ and work in the Weinberg model ($\xi=0$). To simplify matters I neglect the contribution to $\mathscr{M}^{(1)}$ from the contact terms which is insignificant for $(\pi^\pm, 2\pi^\pm)$ as well as the constructive recoil corrections $O(m_\pi/M)$.

[10] The factor 1/2! in Eq. (2) takes into account the identity of the final pions in the calculations presented here. The step function $\theta(x)$ is conventionally given by $\theta(x)=1$, $x>0$, $\theta(x)=0$, $x<0$. M^* is taken to be $(\frac{3}{5})M$ as in Ref. 5.

[11] Later we replace A by $2n$, where $n=Z$ for a $(\pi^+, 2\pi^+)$ process and $n=N$ for a $(\pi^-, 2\pi^-)$ process.

[12] Note that the region of validity of this formula is identical to that of Ref. 6. (See the Appendix for details.)

[13] I remind the reader that as in the earlier work (Refs. 1 and 2) initial- and final-state pion interactions are *not* taken into account so that cross sections for the $(\pi, 2\pi)$ process may be reduced by as much as a factor of 8 from the soft-pion limit prediction (8).

[14] D. Tow and J. M. Eisenberg, Nucl. Phys. A237, 441 (1975).

[15] The factor $\vec{k}^2$ is a consequence of gradient coupling; in the presence of initial-state interaction (see below) one has $\vec{k}e^{i\vec{k}\cdot\vec{r}}\to -i\nabla\psi_{\vec{k}}(\vec{r};\pi^\pm)$.

[16] G. Höhler, G. Ebel, and J. Giesecke, Z. Phys. 180, 430 (1964).

[17] As in Ref. 5, ϵ_1 (ϵ_2) is the initial (final) electron energy and M^* is the effective nucleon mass; ω_q is the total en-

ergy of the outgoing pion with momentum $\vec{q}$.

[18]This result is obtained by dissecting the one nucleon electroproduction cross section $\propto |b|^2$,

$$\left[\frac{d^3\sigma}{d\epsilon_2 d\Omega_{s_2} d\Omega_q}\right]_{\text{nucleon}} = \frac{2\alpha^2}{\pi^2}\left[\frac{f}{m_\pi}\right]^2 \epsilon_2^2 \frac{|\vec{q}|}{k^4} \times |b|^2 \cos^2\frac{\theta}{2} \times \left[\frac{k^2}{\omega_k^2} + 2\tan^2\frac{\theta}{2}\right],$$

from the incoherent cross section for s-wave production near threshold [Eq. (12) of Ref. 5]. I do not discuss the essentially negligible corrections to this threshold behavior $O(\eta^{7/2})$.

[19]*No* correction term $O(\eta^{7/2})$ is obtained in setting $1/\omega_q \simeq 1/m_\pi$ and taking $\omega_q \simeq m_\pi + (q^2/2m_\pi)$ (inside the δ function); however, such a term *does* result when the respective expansions in powers of q^2 are carried one step further, i.e.,

$$\frac{1}{\omega_q} \simeq \frac{1}{m_\pi} - \frac{q^2}{2m_\pi^3},$$

$$\omega_q \simeq m_\pi + \frac{q^2}{2m_\pi} - \frac{q^4}{8m_\pi^3}.$$

[20]D. F. DuBois, Ann. Phys. (N.Y.) 8, 24 (1959).