Analysis of threshold (p,π^0) reactions

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A procedure is derived for analyzing (p,π^0) reactions near threshold where pions are present in only s- and p-wave states. The pions, which decay into two gammas, have an angular distribution in the center of mass which can be correlated to the angular distribution of its decay gammas in the laboratory. Functions are calculated which give the laboratory distribution of both gammas in terms of parameters describing the angular distribution of pions in the center of mass. By fitting these functions to the observed gamma distribution, the pion differential cross section is determined.

[NUCLEAR REACTIONS Analysis threshold (p,π^0) reactions.]

I. INTRODUCTION

Pion production experiments allow the test of models for pions in nuclear matter. A complete description of pions in nuclei must include the modification of the nuclear force by the nuclear medium, exchange currents, absorption, isobar effects, and relativistic effects. These effects can be studied with pion production experiments to final states with either charged or uncharged pions.[1] The (p,π^0) reaction near threshold allows the study of s- and p-wave pion production to a final state which has no Coulomb distortion.

The observation of (p,π^0) reactions near threshold requires the detection of the pion, except in reactions with light nuclei in which the recoil nucleus can be detected. Since the pion has a lifetime of 10^{-16} sec, it will travel only a short distance before decaying into two gammas. Therefore the energy and angle of the pion must be inferred from its decay gammas. The gammas, which decay back to back in the pion center of mass, have an angular distribution in the laboratory which is related to the momentum and angle of the pion. A measurement of the angle and energy of each gamma allows a reconstruction of the event.

The angular distribution of one of the gammas from pion decay is given by Cocconi and Silverman, who assume the pion production cross section is given by

$$\frac{d\sigma}{d\Omega}=a+b\cos^2\theta^* \ ,$$

where θ^* is the angle of the pion in the center of mass. The distribution function was used to find the differential cross section for pion photoproduction on hydrogen and deuterium.[2]

A system has been developed which detects neutral pions of 40 to 500 MeV energy with a full width at half maximum energy resolution of 2 MeV and a typical angular acceptance of 1 msr.[3] This technique reconstructs the pion event from its decay gammas by restricting the measurement to events which have gamma pairs with nearly equal energies. A good measurement of the opening angle of the decay gammas with only a fair energy measurement of each gamma is then sufficient to determine the pion energy and angle.

The method used here does not require an energy measurement because it considers only reactions near threshold which leave the nucleus in the ground state. In Sec. II we present the angular distribution of the gammas assuming pions are emitted in only s and p states. The method of fitting for the s- and p-wave parameters is given in Sec. III. A test of the fitting procedure is described in Sec. IV, and Sec. V discusses the effect of a finite angular resolution in the measurement of the gamma angles. The angular distribution for the gammas has a singularity which must be integrated in order to normalize the distribution. A technique for the integration is described in the Appendix.

II. ANGULAR DISTRIBUTION OF GAMMAS

For reactions near threshold, the angular distribution of the pion can be determined by measuring the angular distribution of the gammas which are produced when the pion decays. At energies sufficiently close to threshold such that the outgoing nucleus is not excited, an energy measurement of the gammas is not needed to resolve the nuclear final state. The experiment is analyzed by expanding the differential cross section in terms of angle up to the p wave while assuming the proton beam is unpolarized,

$$\frac{d\sigma}{d\Omega}=a+b\cos\theta^*+c\cos^2\theta^* , \tag{1}$$

where θ^* is the center-of-mass polar angle of the pion with respect to the beam.

The angular distribution of the gammas from pion decay can be related to the parameters a, b, and c for the differential cross section given by Eq. (1). By measuring the angular distribution of gammas in the laboratory and fitting their distribution to obtain the parameters a, b, and c, the angular distribution of pions in the center of mass is obtained. Figure 1 indicates the relevant angles for pion production and decay. For a typical spatial resolution, production and decay of the pion can be considered as occurring at the same point since the pion has a low velocity and a very short lifetime. If it were desired to reconstruct each event, an energy measurement would be needed because of the two possible pion directions associated with each pair of gamma angles, as shown in Fig. 2. An energy measurement resolves the ambiguity. Since we are not reconstructing each event to obtain the pion distribution in the center of mass, an energy measurement is not required in the present procedure.

Transforming Eq. (1) for the pion distribution to find the resulting gamma distribution in the laboratory, we obtain $d^4N(\theta,\theta',\phi,\phi')$, the number of coincident gammas emitted into solid angles $d\Omega$ and $d\Omega'$,[4]

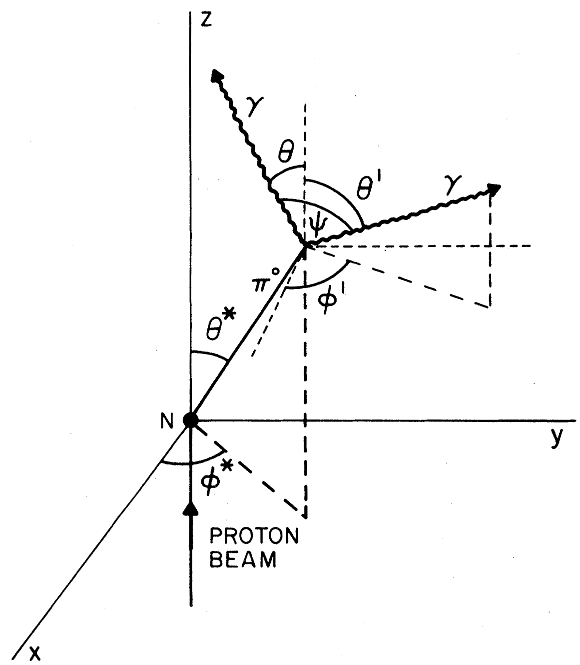

FIG. 1. Definition of angles which give the pion and gamma directions. θ^*,ϕ^* are the polar and azimuthal angles for the pion in the center of mass with respect to the incident proton beam. θ,ϕ (ϕ is not shown) and θ',ϕ' give the angle of the two gammas in the laboratory. ψ is the opening angle between the two gammas.

$$\begin{aligned}d^4N(\theta,\theta',\phi,\phi')=&AK(\theta,\theta',\phi,\phi')\\&\times[a+bT_1(\theta,\theta',\phi,\phi')\\&+cT_2(\theta,\theta',\phi,\phi')]d\Omega\, d\Omega' ,\end{aligned} \tag{2}$$

where

$$K(\theta,\theta',\phi,\phi')=\frac{m_\pi{}^2\Theta(\Delta)}{4\pi p_\pi\gamma_c\sqrt{\Delta}(1-\cos\psi)^2} , \tag{3}$$

$$T_1(\theta,\theta',\phi,\phi')=\frac{\frac{1}{2}(1+\beta_c{}^2)(\cos\theta+\cos\theta')-\beta_c(1+\cos\theta\cos\theta')}{p_\pi(1-\beta_c\cos\theta)(1-\beta_c\cos\theta')}E_\pi , \tag{4}$$

$$\begin{aligned}T_2(\theta,\theta',\phi,\phi')=&\{\Delta(1-\beta_c{}^2)(\cos\theta-\cos\theta')^2\\&+E_\pi{}^2[\tfrac{1}{2}(1+\beta_c{}^2)(\cos\theta+\cos\theta')-\beta_c(1+\cos\theta\cos\theta')]^2\}/[p_\pi(1-\beta_c\cos\theta)(1-\beta_c\cos\theta')]^2 ,\end{aligned} \tag{5}$$

$$\Delta=(E_\pi/2\gamma_c)^2-\tfrac{1}{2}(1-\beta_c\cos\theta)(1-\beta_c\cos\theta')m_\pi{}^2/(1-\cos\psi) , \tag{6}$$

$$\begin{aligned}\Theta(\Delta)&=0 \text{ for } \Delta<0 ,\\&=1 \text{ for } \Delta>0 .\end{aligned} \tag{7}$$

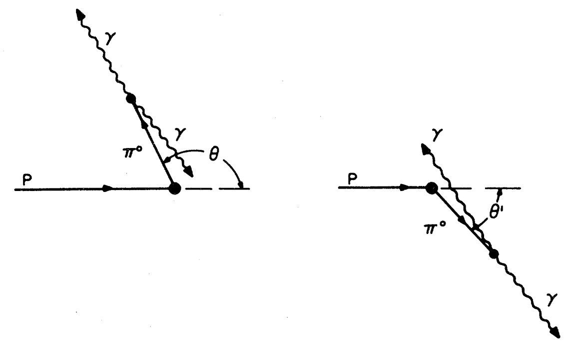

FIG. 2. Angle ambiguity in determining the direction of the pion from the direction of the decay gammas. The pions, emitted at different angles relative to the incident proton, produce gammas in the same directions. Consequently a measurement of the gamma angles cannot distinguish between the two events. An energy measurement is required.

ψ is the opening angle between the two gammas,

$$\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') , \qquad (8)$$

m_π is the pion mass, p_π and E_π are the momentum and total energy of the pion in the center of mass, β_c is the velocity of the center of mass,

$$\gamma_c = 1/(1-\beta_c{}^2)^{1/2} ,$$

and A is a constant which depends on target thickness, beam intensity, and counter efficiencies.

The kinematically allowed region in $\theta,\theta',\phi,\phi'$ is defined by Eqs. (6) and (7), the surface of which corresponds to $\Delta=0$. The angles for all coincident gamma events must fall in the volume bounded by this surface. Figure 3 illustrates the shape of the $\Delta=0$ surface for a ^{12}C$(p,\pi^0)^{13}$N reaction producing pions near threshold with a fixed relative azimuthal angle Φ, where $\Phi=\phi-\phi'$. As the center-of-mass energy of the pion increases, the volume of the kinematically allowed region increases.

As can be seen from Eq. (3), the angular distribution has a singularity when $\Delta=0$. A plot of K as a function of θ,θ' with $\phi=90°$ and $\phi'=-90°$ is shown in Fig. 4 for the ^{12}C$(p,\pi^0)^{13}$N reaction with an incident proton laboratory energy of 147 MeV. Plots of the functions T_1, $K\cdot T_1$, T_2, and $K\cdot T_2$ for fixed ϕ and ϕ' are shown in Figs. 5 and 6. Most of the events fall near the $\Delta=0$ surface.

The placement of counters for measuring the angular distribution parameters a, b, and c of Eq. (1) can be optimized by examining Figs. 5 and 6. For example, Fig. 6 indicates that the term containing the c parameter has maxima at $\theta=0°$, $\theta'=180°$ and $\theta=180°$, $\theta'=0°$. Consequently counters set near either of these angle pairs would be sensitive to a measurement of the c parameter.

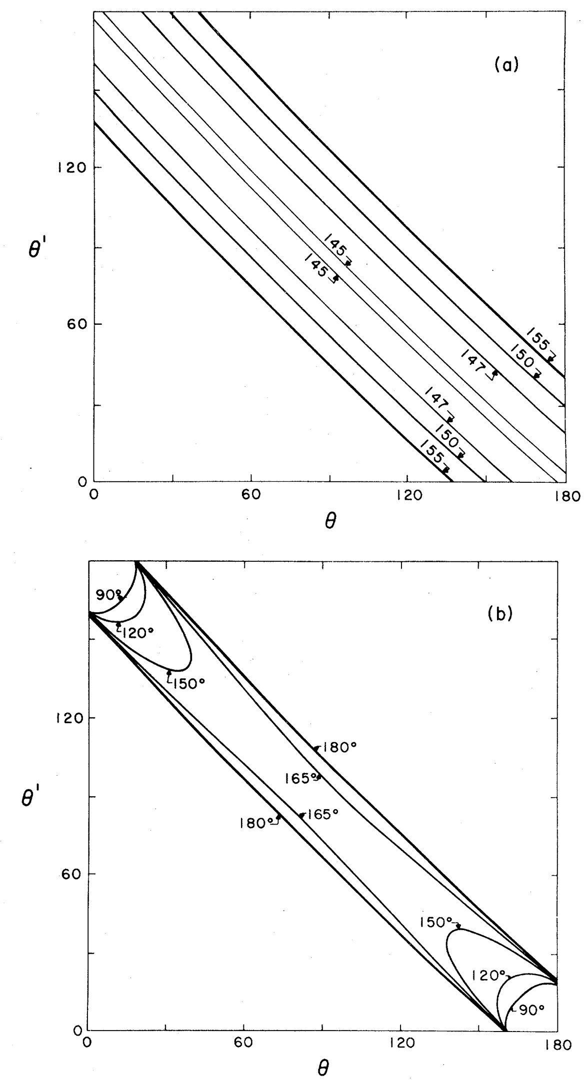

FIG. 3. Boundary for kinematically allowed $\pi^0\rightarrow 2\gamma$ decay events for the ^{12}C$(p,\pi^0)^{13}$N reaction. (a) Φ, the relative azimuthal angle, equals 180°. The boundary is shown for several different incident proton laboratory energies. Decay events must have polar angles θ,θ' which fall in the region between the lines for the proton laboratory energy producing the event. (b) Boundary for events with a proton kinetic energy of 147 MeV and different relative azimuthal angles Φ.

III. PARAMETER FITTING

The probability of a single event j with one gamma at angles θ, ϕ and the second gamma at θ', ϕ' is

$$P^j = N^j(\theta,\theta',\phi,\phi')/N_S , \qquad (9)$$

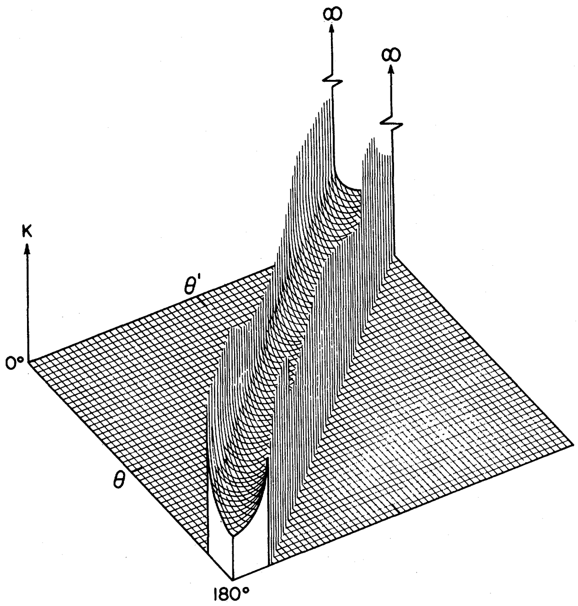

FIG. 4. Angular distribution function K given in Eq. (3). The relative azimuthal angle Φ equals 180°.

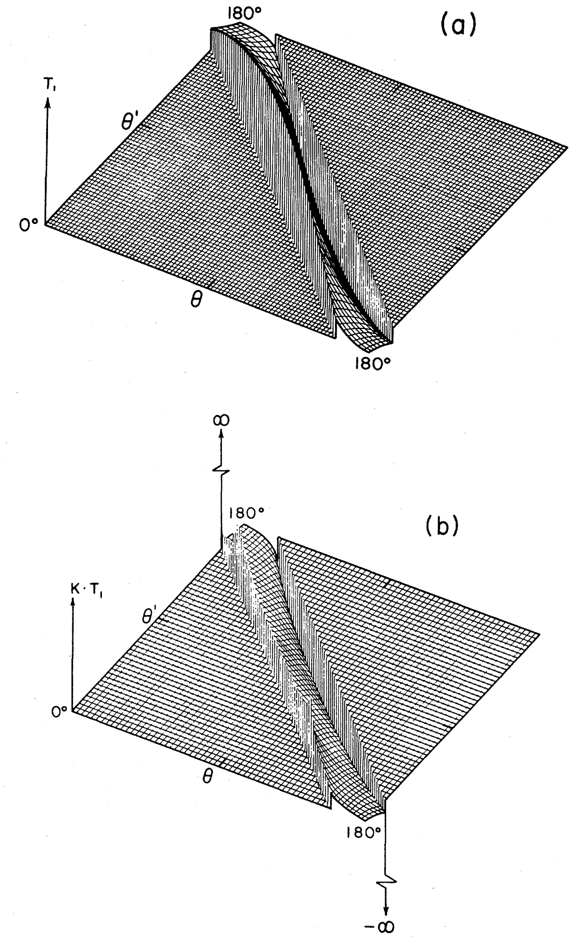

FIG. 5. (a) Angular distribution function T_1 given in Eq. (4). (b) Angular distribution of the product $K\cdot T_1$ which appears in the distribution function of Eq. (2). The relative polar angle Φ equals 180°.

where N^j is the un-normalized distribution function given by Eq. (2) and

$$N_S = \int_S N\, d\Omega\, d\Omega' \ . \tag{10}$$

The integral extends over the region S bounded by the kinematic constraint and the solid angle subtended by the gamma counters. The probability of m events is proportional to the product of the probabilities of each event,

$$L = \prod_{j=1}^{m} P^j(\theta,\theta',\phi,\phi') \ . \tag{11}$$

The parameters a, b, and c are determined by finding the set which maximize L, the likelihood function, while holding N_S constant.

The search for the parameters which maximize L is simplified by defining a function χ^2,

$$\chi^2 = -2\ln L = -2\sum_{j=1}^{m} \ln\frac{N^j}{N_S} \ . \tag{12}$$

The function L is then maximized by minimizing χ^2 with standard fitting techniques.[5] The error in the parameters is the change in the parameters which increases χ^2 by one.

The gamma angles fall in the small region allowed by reaction kinematics and the gamma counter solid angles. Since a fit to the data in this small region can lead to unphysical (negative) cross sections outside the region of fit, a new parametrization is introduced which constrains the cross section to be positive for all angles. We let

$$\frac{d\sigma}{d\Omega} = |\alpha+\beta e^{i\eta}\cos\theta^*|^2 + \gamma^2\sin^2\theta^* = (\alpha^2+\gamma^2) + 2\alpha\beta\cos\eta\cos\theta^* + (\beta^2-\gamma^2)\cos^2\theta^* \ , \tag{13}$$

where α is the s-wave amplitude, β the p-wave amplitude, η the phase between the s- and p-wave amplitudes, and γ the p-wave, spin-flip amplitude. Since the procedure determines only three parameters, the four new parameters are not determined uniquely. Comparing Eq. (13) to Eq. (1), we find

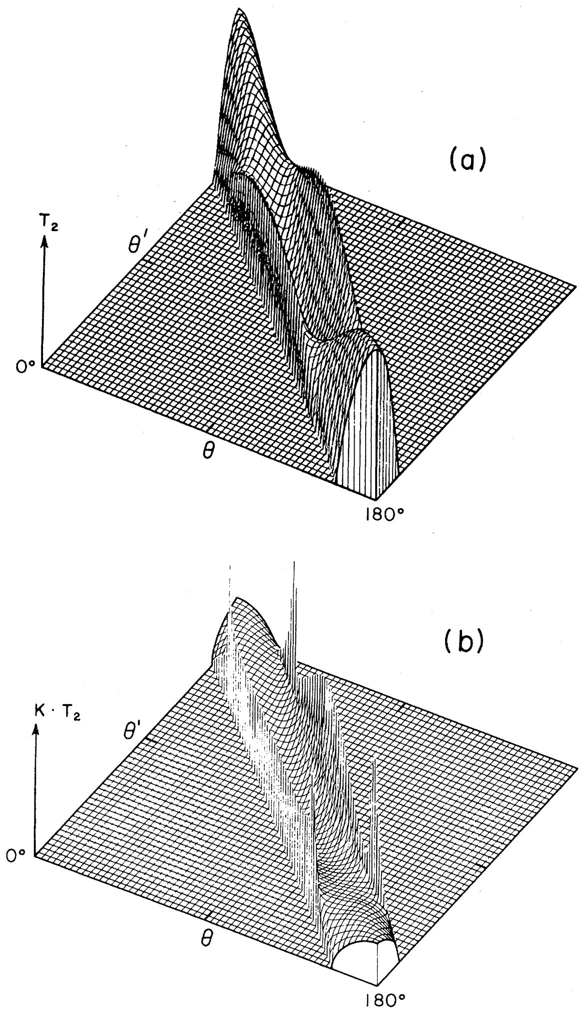

FIG. 6. (a) Angular distribution function T_2 given in Eq. (5). (b) Angular distribution of the product $K \cdot T_2$ which appears in the gamma angular distribution of Eq. (2). The relative polar angle Φ equals 180°.

$$\begin{aligned} a &= \alpha^2+\gamma^2 , \\ b &= 2\alpha\beta\cos\eta , \\ c &= \beta^2-\gamma^2 . \end{aligned} \tag{14}$$

By varying α, β, γ, and η to fit the data, we guarantee that the parameters a, b, and c will produce a positive cross section for all values of θ^*.

The distribution function N depends linearly on the parameters a, b, and c which are determined to an arbitrary constant by maximizing the likelihood function as described above. Once the relative angular distribution is known, the absolute distribution can be found by scaling the parameters so that, when integrated over the solid angle of the counters and corrected for counter efficiency, they give the observed number of events.

The fitting analysis proceeds by integrating the distribution function,

$$\begin{aligned} N_S &= \int_S N\, d\Omega\, d\Omega' \\ &= A \int_S K(a+bT_1+cT_2)d\Omega\, d\Omega' \\ &= A\,[(\alpha^2+\gamma^2)C_1+2\alpha\beta C_2\cos\eta+(\beta^2-\gamma^2)C_3] , \end{aligned} \tag{15}$$

where

$$\begin{aligned} C_1 &= \int_S K\, d\Omega\, d\Omega' , \\ C_2 &= \int_S KT_1 d\Omega\, d\Omega' , \\ C_3 &= \int_S KT_2 d\Omega\, d\Omega' , \end{aligned} \tag{16}$$

and the integration extends over the surface S of the gamma counters. Solving for γ^2,

$$\gamma^2 = \frac{\dfrac{N_S}{A} - \alpha^2 C_1 - 2\alpha\beta C_2\cos\eta - \beta^2 C_3}{C_1 - C_3} . \tag{17}$$

Since the distribution is un-normalized, we can choose $N_S/A=1$ and designate the corresponding parameters as α_0, β_0, and γ_0. The fitting procedure uses a gradient search technique to minimize χ^2 by varying α_0 and β_0. η is held fixed because only two parameters are needed to fit the data when the normalization is not specified. The χ^2 fitting process yields values of α_0, β_0, and γ_0 with associated errors and correlation coefficients computed from elements of the inverse of the second derivative matrix of χ^2. Parameters a, b, and c with associated errors can then be calculated.

The parameters obtained from the fitting procedure can be used to find the total cross section σ,

$$\sigma = \frac{n_T}{It} , \tag{18}$$

where n_T is the total number of pions produced by a beam intensity I of protons on a target with t atoms per unit area. n_T is given by

$$n_T = \frac{n}{\epsilon_1\epsilon_2}\frac{N_T}{N_S} , \tag{19}$$

where n is the total number of observed events, ϵ_1 and ϵ_2 are the efficiencies of the two gamma counters, and

$$\begin{aligned} N_T &= \int_{4\pi} N\, d\Omega\, d\Omega' \\ &= A\,[(\alpha^2+\gamma^2)C_{T1}+(\beta^2-\gamma^2)C_{T3}] , \end{aligned} \tag{20}$$

where

$$C_{T1}=\int_{4\pi}K\,d\Omega\,d\Omega' ,$$
$$C_{T3}=\int_{4\pi}KT_2 d\Omega\,d\Omega' , \tag{21}$$

and the integration extends over all angles. Because of symmetry, $C_{T2}=0$. Then

$$\frac{N_T}{N_S}=\frac{(\alpha_0{}^2+\gamma_0{}^2)C_{T1}+(\beta_0{}^2-\gamma_0{}^2)C_{T3}}{(\alpha_0{}^2+\gamma_0{}^2)C_1+2\alpha_0\beta_0C_2\cos\eta+(\beta_0{}^2-\gamma_0{}^2)C_3} . \tag{22}$$

Once σ is known, the normalization factors for the parameters a, b, and c can be found from the relation

$$\sigma=\int\frac{d\sigma}{d\Omega}$$
$$=a+\tfrac{1}{3}c \tag{23}$$

and a and c given by Eq. (14).

IV. MONTE CARLO TEST

To check the fitting process and the assignment of errors, a Monte Carlo calculation was used to generate a set of gamma events to which the fitting procedure could be applied. As a test case, the (p,π^0) reaction on ^{12}C at 147 MeV was chosen with a gamma-counter array which placed two arrays opposite each other with respect to the target and on a line perpendicular to the beam. The arrays measured 10 cm by 10 cm and were 15 cm from the target. The calculation assumed that the array would allow a precise measurement of the angle of any gamma falling within the array.

The angular distribution of pions in the center of mass was randomly generated by utilizing Eq. (1) with the following parameters:

$$\alpha=0.75\ ,\quad \beta=0.50\ ,\quad \gamma=0.10\ ,\quad \cos\eta=0.5\ .$$

The decay-gamma angles were calculated by generating angles in the pion rest frame, assuming a uniform angular distribution, and then transforming the gamma angles into the laboratory system. The event was accepted if each of the gamma counters was intersected by a gamma trajectory. The procedure was repeated 154 830 times to generate 5000 acceptable events.

The C_i coefficients of Eq. (16) were calculated for the gamma counter geometry,

$$C_1=0.1066\ ,\quad C_2=-0.0100\ ,\quad C_3=0.0335\ .$$

The parameter $\cos\eta=0.5$ was held constant while parameters α and β were varied to minimize χ^2. The values of the parameters which minimized χ^2 are

$$\alpha_m=2.13\pm0.19\ ,\quad \beta_m=2.82\pm0.16\ .$$

These parameters differ from those used to generate the events because of the arbitrary normalization in the maximum-likelihood-fitting procedure. However, the ratios b/a and c/a are independent of the normalization and can be used to compare the two sets of parameters. The comparison is shown in Table I. n_T was determined from the fitted parameters by using Eqs. (19) and (21)

$$n_T=\frac{n}{\epsilon_1\epsilon_2}\frac{C_{T1}+\frac{c}{a}C_{T3}}{C_1+\frac{b}{a}C_2+\frac{c}{a}C_3} \tag{24}$$

with $\epsilon_1=1$, $\epsilon_2=1$, and $n=5000$, the number of events which satisfied the gamma counter geometry requirement. There is good agreement between the parameters used to generate the data and those from the fitting analysis.

The results presented in Table I are for a fixed value of $\cos\eta$ since only two parameters can be determined from the angular distribution. As a result $\cos\eta$ is a free parameter subject to a constraint derived from Eq. (14),

$$|\cos\eta|\geq\frac{B}{A+C}\ .$$

If the fitting procedure used a value of $\cos\eta$ that did not satisfy this constraint, results are obtained for b/a and c/a which are different from those presented in Table I but with a larger value of χ^2. All values of $\cos\eta$ satisfying the constraint produce the same results for b/a, c/a, and χ^2.

TABLE I. Results from test of fitting procedures which derive angular distribution parameters for pions in the center of mass from the angular distribution of the decay gammas in the laboratory. The input parameters were used to generate a set of data to which the fitting program was applied to give the fitted parameters.

Parameters	Input	Fitted
α	0.75	2.13 ± 0.19
β	0.50	2.82 ± 0.16
b/a	0.65	0.68 ± 0.03
c/a	0.42	0.42 ± 0.06
n_T	154 830	155 658

V. EFFECT OF FINITE ANGULAR RESOLUTION

The gamma distribution given by Eq. (2) with separate terms shown in Figs. 4—6 has a maximum at the boundary defining the region of $\theta,\theta',\phi,\phi'$ allowed by kinematics. Therefore most of the events will fall on the kinematic boundary. Because of the finite angular resolution of the gamma counters, a point near the boundary can be detected as falling outside the boundary. These events must be excluded from the analysis since the angular distribution is not defined outside the kinematic region. However, the remaining events may have a different angular distribution which could distort the fitting analysis for the pion angle parameters.

The measured angular distribution can be corrected by making use of the distribution's singularity, which places most of the events near the boundary. To make the correction, one can define a length l, the distance between the event point outside the kinematic boundary and the boundary. The value of $\theta,\theta',\phi,\phi'$ on the boundary which minimizes l is then used as the corrected coordinate for the excluded event.

VI. CONCLUSIONS

A method of analysis has been described and tested which allows the determination of the (p,π^0) differential cross section from a measurement of the angular distribution of the pion's decay gammas. The method is restricted to pion production reactions very near threshold which cannot excite the nucleus and which can be represented by an s- and p-wave parametrization.

ACKNOWLEDGMENTS

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APPENDIX: NORMALIZATION INTEGRAL

The gamma distribution function given by Eq. (2) was integrated numerically to obtain a normalization for the distribution. The integral has a singularity at $\Delta=0$. This appendix presents a description of the numerical techniques which were used to evaluate the integral

$$\int_{\Omega_1}\int_{\Omega_2} f(\theta,\theta',\phi,\phi')d\Omega\,d\Omega' ,$$

where f is one of the basis functions K, $K\cdot T_1$, or $K\cdot T_2$ in Eq. (2) which describe the gamma distribution; $d\Omega=d\cos\theta\,d\phi$; and Ω_1 and Ω_2 are the solid angles subtended by the gamma counters. The Monte Carlo integration technique described by Lepage was used to evaluate the integral.[6]

The region of integration for the variables θ,θ' for fixed ϕ,ϕ' is shown in Fig. 7. To improve the efficiency of the Monte Carlo generation of points in θ,θ', the θ,θ' variables are transformed to θ_1,θ_2

$$\theta_1=\tfrac{1}{2}(\theta-\theta') ,$$
$$\theta_2=\tfrac{1}{2}(\theta+\theta'-\pi) .$$

The range in θ_1 and θ_2 can be constrained to fall in a rectangle which includes the boundaries shown in Fig. 7, thereby minimizing the points generated in θ,θ' space which fall outside the bounds defined by $\Delta=0$. The bounds on θ_1 and θ_2 are determined by a numerical investigation of the $\Delta=0$ surface in θ_1 , θ_2, ϕ, and ϕ'.

The integration in ϕ, ϕ' can be simplified by using the invariance of the reaction with respect to a rotation about the z axis. A new variable Φ is defined:

$$\Phi=\phi-\phi' .$$

The region of integration can be further confined by transforming the variable Φ to Δ^*, where

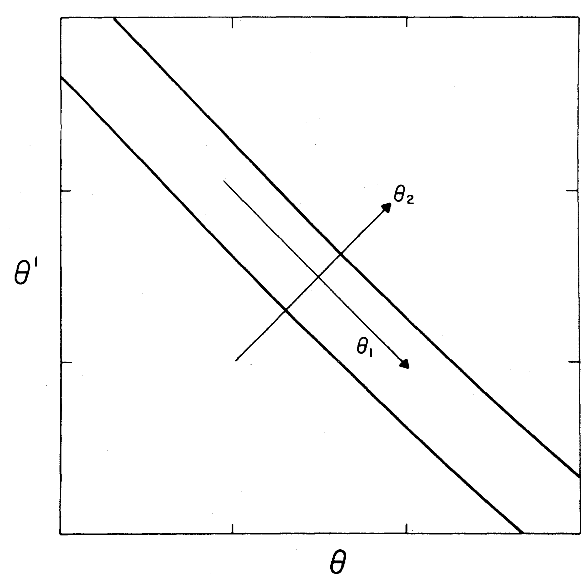

FIG. 7. Definition of the angles θ_1 and θ_2 for the normalization integral.

$$\Delta^* = \left[\frac{E_\pi}{2\gamma_c m_\pi}\right]^2 - \frac{\Delta}{m_\pi^{\,2}}$$

$$= \frac{(1-\beta_c\cos\theta)(1-\beta_c\cos\theta')}{2(1-\cos\theta\cos\theta'-\sin\theta\sin\theta'\cos\Phi)}$$

and Δ is defined in Eq. (6). The constraint on Δ^* is

$$(1-\beta_c^{\,2})/4 < \Delta^* < \left[\frac{E_\pi}{2\gamma_c m_\pi}\right]^2 .$$

The integral is now

$$\int_{\Omega_1\Omega_2} f(\theta_1,\theta_2,\Delta^*,\phi')\frac{\partial\Phi}{\partial\Delta^*}d\cos\theta_1 d\cos\theta_2 d\Delta^* d\phi' ,$$

where $\partial\Phi/\partial\Delta^*$ is the Jacobian for the transformation

$$\frac{\partial\Phi}{\partial\Delta^*} = [(1-\beta_c\cos\theta)(1-\beta_c\cos\theta')](\Delta^*)^{-1}/\{(2\Delta^*\sin\theta\sin\theta')^2 - [2\Delta^*(1-\cos\theta\cos\theta')-(1-\beta_c\cos\theta)(1-\beta_c\cos\theta')]^2\}^{1/2} .$$

The integration can now be performed with Monte Carlo techniques by evaluating the integrand in the following steps:

(1) Generate θ_1, θ_2, Δ^*, ϕ' in the intervals

$$-1.57 < \theta_1 < 1.57 ,$$

$$0.21 < \theta_2 < 0.16 ,$$

$$0.24 \le \Delta^* \le 0.26 ,$$

$$0 \le \phi' \le 1.287 ,$$

where the bounds on θ_1,θ_2 have been determined numerically for the (p,π^0) reaction on ^{12}C at 147 MeV.

(2) Calculate θ, θ', ϕ, and ϕ' to check constraints imposed by kinematics and by the solid angle of the gamma counters.

(3) Evaluate the integrand function and the Jacobian.

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