# Induced nuclear fission viewed as a diffusion process: Transients

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Induced nuclear fission is viewed as a diffusion process of the fission degree of freedom over the fission barrier. We describe this process in terms of a Fokker-Planck equation which contains the fission variable and its canonically conjugate momentum. We solve this equation numerically for several energies (temperatures) of the fissioning nucleus neglecting changes of the fission barrier due to the temperature dependence of nuclear shell effects. We pay particular attention to the time $\tau$ needed for the system to build up the quasistationary probability flow over the fission barrier. The rate of the latter is approximated in terms of the Bohr-Wheeler formula or Kramers's transition state expression; the precise value of the quasistationary current depends on the nuclear friction constant $\beta$. Our results for $\tau$ are consistent with those obtained earlier in the framework of a simplified model: As long as $\beta \lesssim \beta_0$, the time $\tau$ is proportional to $\beta^{-1}$. This relationship exhibits the fact that with increasing friction $\beta$, the diffusion process is accelerated, so that it takes the system increasingly less time to attain the quasistationary distribution. The constant $\beta_0$ is roughly given by $2\omega_1$, where $\omega_1$ is the frequency of a harmonic oscillator potential which osculates the potential at the minimum corresponding to the initial configuration of the fissioning nucleus. The condition $\beta \lesssim \beta_0$ is roughly equivalent with the motion in that minimum being underdamped. The converse relationship—$\tau$ increases with $\beta$—is found for $\beta > \beta_0$. We ascribe this to the fact that now the fission variable executes an overdamped motion. Generalizing Kramers's original derivation, we obtain an analytical expression for the time dependence of the probability current over the fission barrier. For $\beta \lesssim \beta_0$, this expression agrees well with our numerical results. We use it to calculate the energy dependence of the fission probability $P_f$ and find that $P_f$ grows much less rapidly with increasing excitation energy than would be predicted by the Bohr-Wheeler formula. This is in qualitative agreement with recent experimental findings and suggests that the energy dependence of $P_f$ deserves further investigation and can be used to determine $\beta$ experimentally. Our analysis does not yet include the additional time delay incurred by the system on its way from the saddle to the scission point: Clearly the time needed to establish the quasistationary situation at the scission point will be larger than $\tau$. This would probably lead to additional modifications of the energy dependence of $P_f$.

[NUCLEAR REACTIONS, FISSION Diffusion over a potential barrier; transients; deviation from Bohr-Wheeler formula.]

## I. INTRODUCTION

Among all nuclear phenomena, nuclear fission is one of the earliest and most thoroughly studied processes. Over the years, the complexity of fission has become more and more evident. It is a remarkable fact that most major advances in our understanding of the atomic nucleus have had significant repercussions in the domain of fission. One fairly recent example is given by the impact of nuclear shell structure on the properties of the fission barrier, and the experimental developments following it.[1] Another example is provided by the evolution of transport theories to describe salient features of deeply inelas-

 

tic heavy-ion collisions[2] which suggested the application of a diffusion mechanism also to the case of induced fission,[3–6] thereby reviving an old suggestion of Kramers.[7] It is this line of approach which we follow in the present paper. We calculate the energy dependence of the probability $P_f$ for induced fission from a diffusion model. This function depends on the nuclear friction constant $\beta$ in a way which hopefully may eventually allow the experimental determination of $\beta$.

The crucial parameters in a diffusion model for fission are the nuclear friction constant $\beta$ [in units of $(\text{time})^{-1}$] (which gives the strength of the coupling between the fission degree of freedom and the rest of the system which is considered as a heat bath) and the diffusion constant. The latter is related to $\beta$ via the nuclear temperature $T$ (in MeV) and the Einstein relation.[2] (We disregard here questions of strong coupling that would modify this relation.[2]) A diffusion model is applicable whenever the internal equilibration time $t_{\text{equ}}$ of the heat bath is short in comparison to the characteristic time of the diffusion process itself (related to $\beta^{-1}$), and to $\hbar/\Gamma_f$ and $\hbar/\Gamma_n$, where $\Gamma_f$ ($\Gamma_n$) are the fission (neutron) widths at the excitation energy under consideration. Simple estimates[2,8] of these time scales (leading to $t_{\text{equ}} \simeq 3\times 10^{-22}$ sec) suggest that a diffusion model is applicable for $\beta \lesssim 3\times 10^{21}$ sec$^{-1}$, and for excitation energies up to 100 MeV or more. In the sequel, we take the applicability of a diffusion model for granted although we occasionally study the results of such a model also outside the domain of parameter values just indicated.

The standard analysis of induced nuclear fission is based on the Bohr-Wheeler formula,[9] an expression involving the height $E_f$ of the fission barrier and the nuclear temperature $T$ as essential parameters, but independent of the nuclear friction constant $\beta$.

A competing formula for the *quasistationary* diffusive current which flows over the fission barrier was derived by Kramers[7] in the limits of small and large values of $\beta$. These results show that this current increases linearly with $\beta$ for very small $\beta$, and that the current decreases essentially as $\beta^{-1}$ for very large values of $\beta$, this latter behavior being a direct consequence of the overdamping of the motion of the fission variable. Extrapolating Kramers's expression for large $\beta$ [our Eq. (7) below] to $\beta=0$, one obtains the value one often refers to as being given by the transition state method. This expression is in essential agreement with the Bohr-Wheeler formula; differences between the two expressions have been attributed to the presence of collective vibrations[10]; in practical applications, the differences are not very significant.

Accepting Kramers's diffusion model as physically meaningful, and disregarding corrections due to the temperature dependence of either the fission barrier or the nuclear friction constant, we investigate in this paper the manner in which the quasistationary flow of probability over the barrier is attained. More specifically, we ask for the time $\tau$ it takes to build up this quasistationary flow, for the dependence of $\tau$ on $\beta$, and for the possibility of measuring $\beta$ experimentally through the effect exerted via $\tau$ on the entire diffusion process. Qualitatively speaking, the time $\tau$ which elapses between the start of the induced fission process and the (approximate) attainment of the stationarity condition is governed by the nuclear friction constant $\beta$, which describes the coupling of the fission degree of freedom with the rest of the system. The larger $\tau$, the more intensely will neutron decay compete with the fission process. This will lead to an effective fission probability *smaller* than predicted by the Bohr-Wheeler formula. Since the neutron decay width increases very strongly with increasing excitation energy of the fissioning nucleus, we expect that the fission probability, plotted versus excitation energy, will at some point fall significantly below the value obtained from the Bohr-Wheeler formula. This is the effect we wish to study quantitatively in the present paper. Our study has partially been motivated by recent experimental findings[11] which seem inconsistent with the Bohr-Wheeler prediction in showing an excess of evaporated neutrons.

The present study continues and considerably extends an earlier work[5] in which we used a simple one-dimensional diffusion model (for the energy variable) for the fission process, identifying diffusion in energy with a value given by the height of the fission barrier with a fission event. As pointed out in Ref. 5, this simple model is expected to be reasonably accurate for sufficiently small values of $\beta$. In this case, all points of phase space having the same energy are expected to be filled uniformly, and the two-dimensional diffusion process in the fission variable and its canonically conjugate momentum is expected to be modeled properly by a diffusion model in the energy variable only. This expectation is indeed borne out by the present study in which the fully two-dimensional problem is solved numerically. We also find that as $\beta$ increases, the one-dimensional model no longer reflects the situation correctly. This is due to the fact that the equilibration in momentum becomes very rapid, while the position variable of the fission process undergoes a strongly damped diffusion process, this leading to a strong *increase* of $\tau$ with $\beta$.

Our simulation of the fission process in terms of a diffusion model employs but a single collective variable and its canonically conjugate momentum.

Several such variables (which take a fuller account of the collective potential surface) can be identified and have, in fact, been used in a realistic treatment of the fission process.[3] However, in the spirit of the transition state method referred to above, the fission width is solely determined by the probability current across the fission barrier in the direction of steepest descent; hence our simplification.

We solve the two-dimensional diffusion equation numerically for a rather general shape of a potential having a fission barrier. The fission width (as a function of time) is obtained by calculating the probability current through the saddle point. (See Sec. II.) We find (see Sec. III) that the fission width approaches the quasistationary value as given by the method of Kramers over a characteristic time $\tau$ which we extract from the data, and interpret. We observe that the approach to the quasistationary value is not monotonic but displays characteristic oscillations. Generalizing the quasistationary approach by Kramers[7] to a time-dependent one, we derive (see Sec. IV) a simple analytical formula for the time dependence of the fission width which displays both the quasistationary limit and the oscillations during the approach phase. This formula does not correctly reproduce our numerical findings in the case of strongly overdamped motion, which we do not believe applies in realistic cases, and which can be dealt with anyhow in a completely analytic fashion. Using this formula, we evaluate in Sec. V the fission probability (i.e., the probability of first-chance fission) as a function of excitation energy (temperature). For the nucleus $A=226$ studied in our earlier paper,[5] we find good agreement with our earlier results, and a strong reduction of the fission probability as compared to the value predicted on the basis of the Bohr-Wheeler formula at higher energies, which suggests that in heavy nuclei and at several tens of MeV's of excitation energy, the finiteness of $\tau$ might significantly influence the ratio of emitted neutrons versus fission events. Section VI contains the conclusions.

We emphasize that the question of transients is not answered completely by the present study. Indeed, our investigation focuses on the way the quasistationary condition is approached at the *saddle point.* For the dynamics of the fission process and competition against neutron decay, the quantity of interest is probably the time $\tilde{\tau}$ it takes to attain quasistationarity at the *scission point.* Clearly, $\tilde{\tau}>\tau$, so that the effects observed in this paper are expected to be enhanced when the calculations are performed at the scission point. Roughly speaking, we have observed that $|\tilde{\tau}-\tau|$ is of the same order as $\tau$ itself. We have not yet carried out a more precise investigation of the fate of transients between saddle and scission points, for two reasons: (i) the shape of the potential is probably rather more crucial in this domain than in the one we have investigated; and (ii) the time and accuracy required to perform the numerical calculations are considerably greater. We hope that we can tackle this problem in a future communication. It would be of interest to compare numerical results with simple analytical estimates, the latter combining our results for $\tau$ with an estimate of the time it takes the system to pass from saddle point to scission point.

## II. THE DIFFUSION MODEL: NUMERICAL PROCEDURE

### A. Basic definitions

As done in Refs. 3-7, we consider the fission variable $x$ as a classical variable, with canonically conjugate momentum $\tilde{p}$ and time variable $t$. We consider the time evolution of the distribution function $P(x,\tilde{p};t)$ in the two-dimensional phase space $(x,\tilde{p})$. This function is assumed to obey a Fokker-Planck equation (FPE). Assuming that the reduced mass $\mu$ of the fissioning system is independent of $x$, and introducing the velocity $p=\tilde{p}/\mu$, we write this equation as a FPE for the distribution function $P(x,p;t)$,

$$\frac{\partial}{\partial t}P(x,p;t)+p\frac{\partial}{\partial x}P(x,p;t)-K(x)\frac{\partial}{\partial p}P(x,p;t)=\beta\frac{\partial}{\partial p}\{pP(x,p;t)\}+\epsilon\frac{\partial^2}{\partial p^2}P(x,p;t)\ . \tag{1}$$

Here,

$$K(x)=-\mu^{-1}\frac{\partial}{\partial x}U(x)$$

relates to the potential $U(x)$ of the fissioning system, and $\beta$ is the friction constant which describes the coupling of the fission degree of freedom to the remainder of the system. The diffusion constant $\epsilon$ is given by

$$\epsilon=\beta T/\mu\ , \tag{2}$$

where $T=(E^*/a)^{1/2}$ is the nuclear temperature (in units of energy) as given by the excitation energy $E^*$ and the level density parameter $a$. The phenomena studied in this paper occur at excitation energies of several tens of MeV's; we therefore neglect the change of $E^*$, and of $T$, with $x$ due to the fission barrier. We likewise assume $a$ to be independent of

$x$, which is probably an oversimplification.

Let $x_0$ denote the saddle point (the local maximum of the potential $U(x)$]. The diffusion current across $x_0$ is given by

$$J(x_0;t)=\int_{-\infty}^{+\infty} dp\, pP(x_0,p;t) . \tag{3}$$

Let

$$\Pi(x_0;t)=\int_{-\infty}^{x_0} dx \int_{-\infty}^{\infty} dp\, P(x,p;t) \tag{4}$$

be the probability that the system is to the left of the saddle point. Assuming $P(x,p;t)$ to be normalized to unity, and using Eq. (1) and the vanishing of probability and current for $p\to\pm\infty, x\to-\infty$, we obtain the conservation law

$$J(x_0;t)=-\frac{d}{dt}\Pi(x_0;t) . \tag{5}$$

We define the time-dependent fission width $\Gamma_f(t)$ by

$$\Gamma_f(t)=\hbar\lambda_f(t)=+\hbar J(x_0;t)/\Pi(x_0;t) . \tag{6}$$

For sufficiently large $t$, and for values of $\beta$ which are not unreasonably small, $\Gamma_f$ attains the quasistationary value given by Kramers[7]

$$\Gamma_k=\hbar\lambda_k=\hbar\frac{\omega_1}{2\pi\omega_0}\{[\omega_0^2+(\beta/2)^2]^{1/2}-(\beta/2)\}\exp(-E_f/T) . \tag{7}$$

Here, $E_f$ is the height of the fission barrier, and $\omega_0$ and $\omega_1$ are the oscillator frequencies of the two parabolas osculating the potential $U(x)$ in the first minimum at $x=x_1$ and at the saddle point $x=x_0$, respectively.

Let $\Gamma_n$ be the neutron decay width. Then,

$$\eta(t)=(\Gamma_n/\hbar)\exp(-\Gamma_n t/\hbar) \tag{8}$$

is the probability that a neutron is emitted at time $t$. Since

$$[1-\Pi(x_0;t)]$$

is the probability that fission has occurred in the interval $(0,t)$, the total fission probability $P_f$ (i.e., the probability for first-chance fission) is given by

$$P_f(E^*,\beta)=\int_0^{\infty} dt\,\eta(t)[1-\Pi(x_0;t)] . \tag{9}$$

In the limit in which $\lambda_f$ is independent of time and equal to $\lambda_K$, Eq. (9) yields

$$P_f=\frac{\Gamma_K}{\Gamma_K+\Gamma_N} ,$$

as it should. Deviations from this value are expected to occur whenever the time $\tau$ it takes $\lambda_f$ to assume the value $\lambda_K$ is of the order of or larger than $\hbar/\Gamma_n$. Since $\Gamma_n$ increases steeply with $E^*$, while $\tau$ is almost independent of $E^*$, this happens at high values of $E^*$ (typically several tens of MeV's).

### B. Choice of the potential $U(x)$

We choose a fourth-order polynomial,

$$U(x)=\mu g x^2(x-c)(x+b) , \tag{10}$$

with $g>0$, $c>0$, $b>0$. This function goes to $+\infty$ for $|x|\to\infty$, and has two minima and a central peak located at $x=0$. We identify this peak with the saddle point, the minimum at $x_1<0$ with the initial configuration of the fissioning nucleus, and the (much deeper) minimum at $x_2>0$ with the completed fission process. Our choice of $U(x)$ is obviously only semirealistic; in reality, there is no barrier at large positive values of $x$. This choice was dictated by the desire to keep the domain of integration of the differential Eq. (1) effectively bounded in $x$.

Most of the calculations in this paper were carried out using the following parameters, where $m$ is the nucleon mass:

$$\begin{aligned} &c=19.688 \text{ fm} , \quad b=5 \text{ fm}; \quad \frac{1}{m}\mu=\tfrac{1}{4}A=62 ; \\ &g=0.013287\times10^{42} \text{ fm}^{-2}\text{sec}^{-2} . \end{aligned} \tag{11}$$

This choice yields the shallow minimum of 4 MeV depth (this is then the height of the fission barrier) at $x_1=-3.41$ fm. It will become apparent later that the precise form of $U(x)$ is not very important for the questions raised in this paper. What really matters besides $x_1$ and $E_f=4$ MeV is the depth of the second minimum, $-199.2$ MeV at $x_2=14.43$ fm, to ensure that at the temperature considered here, no backflow into the shallow minimum at $x_1$ is possible, and the frequencies $\omega_1$ and $\omega_0$ of the harmonic oscillator potentials osculating $U(x)$ at the points $x=x_1$ and $x=0$, respectively. These frequencies are given by

$$\begin{aligned} &\omega_0=(2gcb)^{1/2}=1.65\times10^{21} \text{ sec}^{-1} , \\ &\omega_1=g^{1/2}[12x_1^2-6x_1(c-b)-2cb]^{1/2} \\ &\quad=1.83\times10^{21} \text{ sec}^{-1} , \end{aligned} \tag{12}$$

respectively. The main features of $U(x)$ are quite similar to those of the potential used in Ref. 6.

The set of parameters (11) was used throughout unless indicated otherwise. To convert excitation energy into temperature and vice versa, we used a level density parameter

$$\frac{A}{10}=\frac{\mu/m}{40} .$$

We use the FPE (1) for all values of $x$, although we should obviously set $\beta=0$ for $x>x_{sc}$. This simplification may affect the probability flow from the saddle point to the scission point. However, the analysis of Kramers[7] suggests that it does not affect the rate of escape over the barrier. This is intuitively obvious if one recalls the values of $U(x_2)$ and $U(x_{sc})$, and is borne out by the results of our calculations.

### C. Numerical procedure

It is our aim to integrate Eq. (1) for several values of the parameters $\beta$ and $T$, with the purpose of eventually using the resulting function $P_f(E^*,\beta)$ defined in Eq. (9) to determine $\beta$ from a comparison with experimental data. We begin at time $t=0$ with a Gaussian type of distribution located entirely in the minimum at $x_1$. We solve Eq. (1) by taking advantage of the fact that this equation contains only first order derivatives with respect to $x$ and use a suitable variant of the alternate difference implicit (ADI) method.[12] Equation (1) is integrated alternatingly in the $p$ direction and in the $x$ direction, keeping the other variable fixed. For the $p$ integration we used the method of Ref. 5, which leads to an error of order $(\Delta p)^6$ where $\Delta p$ is the step size in the $p$ direction. The ADI method works well even for not too small increments $\Delta x$ and $\Delta p$ and thus leads to a grid in the $(x,p)$ plane which is of manageable size. (We used about 150 grid points for $x$ and about 70 grid points for $p$.) The integration in the $x$ direction, executed in the subsequent time step $\Delta t/2$, is simply accomplished by interpolation of $P(x,p;t)$ generated in the previous time step. Indeed, the equation

$$\left[\frac{1}{2}\frac{\partial}{\partial t}+p\frac{\partial}{\partial x}\right]P=0$$

is solved by any function which has the argument $(x-2pt)$. The initial condition being given by $P(x,p;t)$, we see that the solution has the form $P(x-p\Delta t,p;t)$. We have used a fifth order Aitken interpolation to construct $P(x-p\Delta t,p;t)$ in terms of $P(x,p;t)$. When compared with standard ADI techniques used in a different context,[13] the gain in precision in both the $x$ and $p$ directions resulting from our method yields a mesh size that can be handled by a computer of standard capacity, while the accuracy in time increases as $(\Delta t)^2$, i.e., as in any ADI technique.

We have tested our program by comparing with analytical results for two cases: (i) diffusion in a harmonic oscillator potential and (ii) diffusion starting at the top of an inverted harmonic oscillator. Since we are interested in calculating the flux over the saddle point (a point of mechanical instability), the test of the latter situation is quite important.[4] In both cases we found very satisfactory agreement. (We reproduced the analytical values for the variances to the fourth significant figure.)

In all cases investigated, we started with an initial distribution function $P(x,p;0)$ given by $T_0^{-1}\exp\{-E/T_0\}$, where

$$E=\tfrac{1}{2}\mu[p^2+\omega_1^2(x-x_1)^2]$$

is the classical oscillator energy and $\omega_1$ is the frequency of the parabolic potential osculating $U(x)$ in the minimum at $x_1$. Throughout the calculations, we have chosen $T_0=0.3$ MeV. This is slightly less than the value of $\frac{1}{2}\hbar\omega_1\simeq 0.50$ MeV which would correspond to the zero point energy of the lowest quantum state of a harmonic oscillator osculating $U(x)$ at $x=x_1$. The difference in the root mean square value of $(x-x_1)$ is, however, only about 0.25 fm, and thus small compared to the distance $|x_1|=3.41$ fm between minimum and saddle. The time $\tau$ it takes to build up the quasistationary flow over the saddle is not seriously (less than 10%) affected by this difference between $\hbar\omega_1$ and $T_0$. We remark that for the choice of parameters just given, we have $E_f\gg T_0$. This is obviously an essential condition for the applicability of the classical diffusion equation.

## III. DIFFUSION OVER THE BARRIER: NUMERICAL RESULTS

The examples given in the present section have all been obtained for the potential parameters specified in Sec. II B. The reduced mass $\mu=(A/4)m$ with $m$ the nucleon mass, $A=248$ was that of the symmetric fission configuration. The calculations were carried through for a range of $\beta$ values (henceforth always expressed in units of $10^{21}$ sec$^{-1}$) suggested both from the study of deeply inelastic collisions,[2] and by earlier investigations of the fission process.[3-6]

Figure 1 shows the projection of a three-dimensional plot of the probability density, calculated for $\beta=0.5$ and $T=1.0$ MeV, for $x$ values larger than zero (we recall that $x=0$ denotes the location of the saddle point) and for a range of $k$ values, with $k=\mu p/\hbar$ expressed in units of fm$^{-1}$. The figure shows $P$ evaluated at a time $t=9.8\times 10^{-21}$ sec, i.e., after the quasistationary flow over the saddle point is established. We note how strongly the distribution is tilted towards higher momenta $\hbar k$ away from the saddle point. This tilting is a direct consequence of the steepness of the minimum at $x_2$, and is much more pronounced than the widening of the distribution in the $k$ direction caused by diffusion.[4] The type of pattern shown in Fig. 1 prevails for the en-

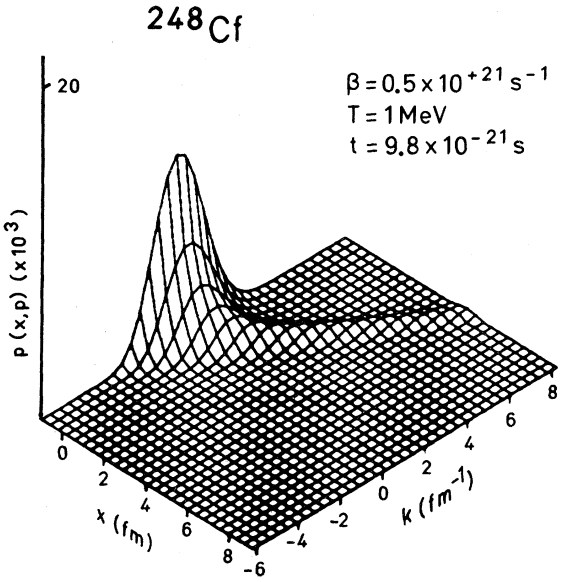


FIG. 1. The phase-space distribution beyond the saddle point. In this and Figs. 2–5, the potential parameters are those specified in Sec. II B.

tire range of $\beta$ values and temperatures considered in this paper. It justifies our not setting $\beta=0$ beyond the scission point.

Figure 2 shows the variances $\sigma^2$ of $x$ and $p$ as functions of time $t$ for two values of $\beta$ and $T$ each. The right-hand scales refer to $\sigma_x{}^2$ in units of $\mathrm{fm}^{+2}$, and the left-hand scales to $\sigma_p{}^2$ in units of $\mathrm{fm}^2\times 10^{42}$ $\mathrm{sec}^{-2}$. The dashed curves are the result of our numerical calculations, while the solid curves give the analytical results obtained[14] for a diffusion process in a harmonic oscillator potential which osculates $U(x)$ at $x_1$. We observe that the numerical results for $\sigma_p{}^2$ very closely follow the analytical formulae and reach a plateau corresponding to equilibration in the momentum variable. In contrast to this behavior, the numerical values for $\sigma_x{}^2$ show an increase beyond the plateau predicted by the analytical results. This increase is caused by diffusion over the barrier, and by the tendency of the entire distribution to shift to the right ($x > x_1$) of the initial center at $x_1$. Calculations extending in time beyond the values displayed on the abscissae show that the variance $\sigma_x{}^2$ do attain a quasistationary plateau. This

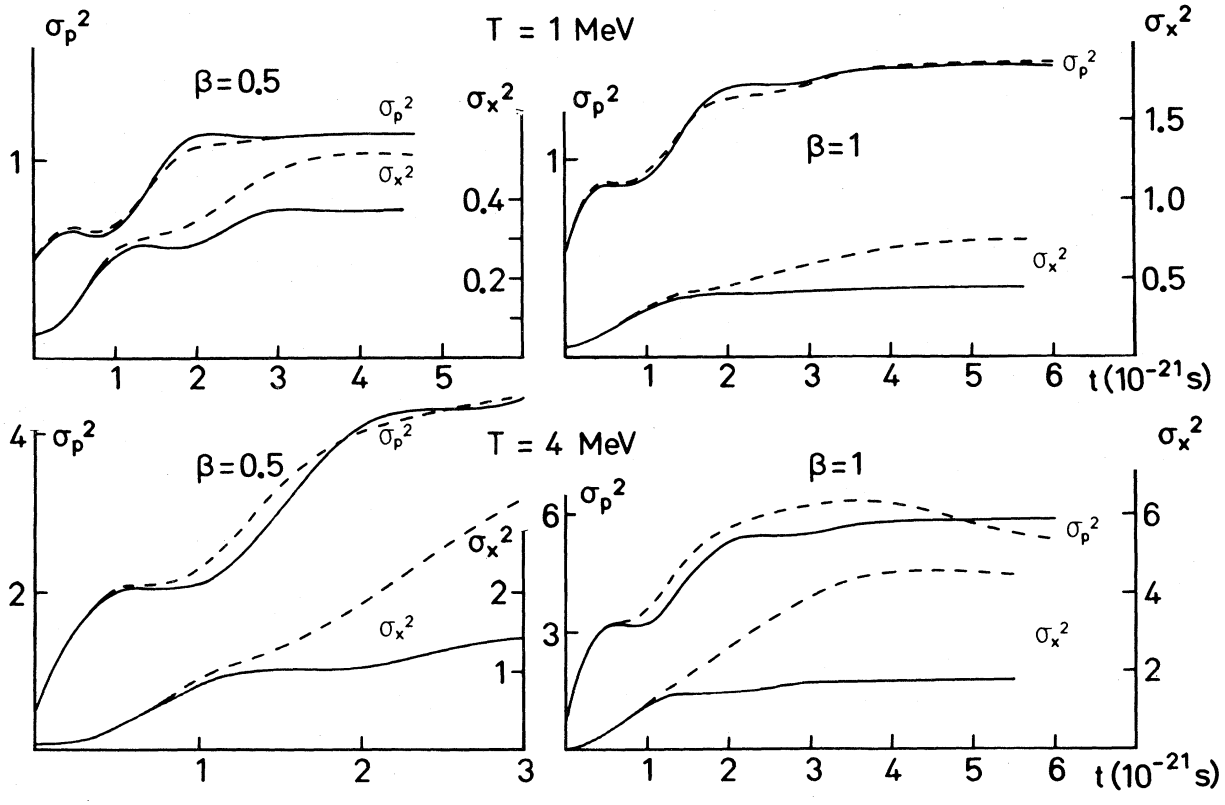


FIG. 2. Variances $\sigma_p{}^2$ (left ordinate) and $\sigma_x{}^2$ (right ordinate) of velocity and position, respectively, versus time, $t$ (in $10^{-21}$ sec). The continuous curves correspond to analytical results for the harmonic-oscillator potential osculating $U(x)$ at $x_1$; the dashed curves are obtained for the full potential $U(x)$ for various values of $\beta$ and $T$ as indicated.

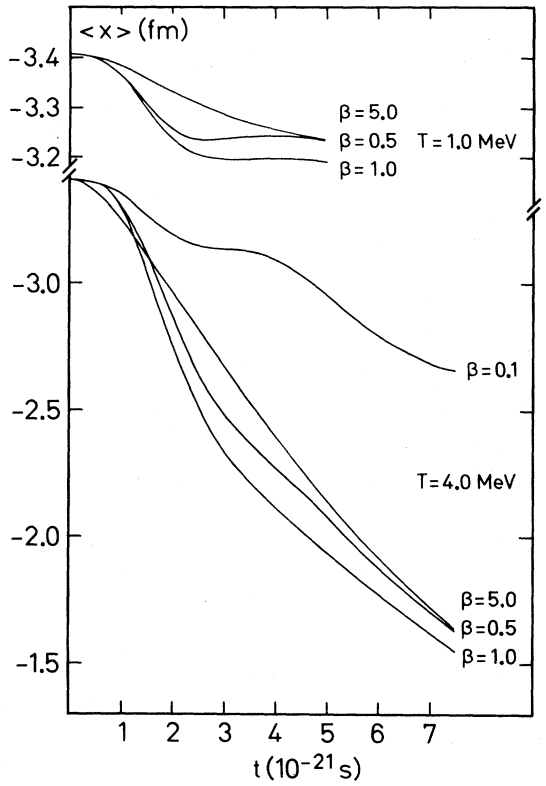


FIG. 3. The mean value $\langle x \rangle$ (in fm) versus time (in $10^{-21}$ sec) for various values of $\beta$ and $T$ as indicated.

plateau is reached when the quasistationary probability current over the barrier is established.

The mean value of $p$, equal to zero at $t=0$, remains very close to zero throughout the diffusion process and is therefore not displayed. In Fig. 3 we show the mean value $\langle x \rangle$ of $x$ as a function of time for various values of $\beta$ and $T$. Note that $\langle x \rangle = -3.41$ fm for $t=0$, and that $\langle x \rangle$ *increases* with time. The results for $T=1.0$ MeV ($E^*=24.8$ MeV) in the upper part of the figure show that the center shifts towards larger $x$ values, and that a quasistationary situation is reached on about the same time scale as is characteristic for equilibration in the first well. The situation is very different for $T=4$ MeV ($E^* \simeq 400$ MeV), a temperature equal to the height of the fission barrier. Now equilibration in the momentum variable is again governed by the dynamics in the first well, as was the case before. However, in the position variable, the mean value $\langle x \rangle$ moves with almost constant velocity. Closer inspection of the full probability distribution not shown in the figures reveals that for $T=1$ MeV, the shape of the distribution in the first well and up to the saddle point relaxes towards an equilibrium form. This form remains nearly constant but is reduced in size as more and more probability leaks out over the barrier. For $T=4$ MeV, on the other hand, the shape of the distribution also relaxes towards an equilibrium form in the first minimum at $x_1$, but the entire distribution moves towards the barrier. We ascribe this to the large temperature which is almost equal to the height of the barrier. We expect that, once the equilibrium shape is attained, this motion should affect the current over the saddle point, and that the additional contribution due to this motion cannot be accounted for in terms of the quasistationary ansatz of Kramers.[7] Figure 3 shows how this motion depends upon $\beta$. We notice that the slope of $\langle x \rangle$ first increases and then again decreases as $\beta$ increases. We attribute this dependence to the fact that increasing $\beta$ first enhances the mobility of the system by speeding up the diffusion process. As $\beta$ increases beyond 2.0 or so, the overall motion in the $x$ direction becomes damped more and more effectively, which in turn *reduces* the mobility of the system. This interpretation is borne out by the following figures. We conclude that increasing $T$ enhances the flux over the barrier, and that for fixed $T$, there is a value of $\beta$ which yields maximum flux. Our result is, of course, completely in keeping with Kramers's early analytical results,[7] in the domain of applicability of the quasistationary regime.

In Figs. 4 and 5 we show the escape rate $\lambda_f(t)$ calculated via Eq. (6) as a function of time. We display $\lambda_f$ for a series of $\beta$ values and the same two temperatures shown before. The time and $\lambda_f(t)$ are given in units of $10^{-21}$ sec and $10^{21}$ sec$^{-1}$, respectively. The dashed curves are the results of our calculations. The full curves correspond to the model of Ref. 5, with an asymptotic decay rate determined from the Bohr-Wheeler formula. This does not give the correct asymptotic value for all values of $\beta$; the upper solid curve in Fig. 4 for $\beta=0.5$ gives the result of the model of Ref. 5 if Eq. (7) rather than the Bohr-Wheeler formula is used to define the asymptotic rate.

We note that the present numerical calculations yield an oscillatory time dependence rather than the monotonic increase predicted by our earlier model. These time oscillations are less pronounced for $T=4$ MeV than for $T=1$ MeV. They can be understood and simulated analytically; this is done in the next section and yields the dashed-dotted curves.

We see that for $T=1$ MeV, the model of Ref. 5 gives a good overall approximation to our numerical results for the escape rate so long as $\beta \lesssim 1.0$. The reduction of the Fokker-Planck equation in $(x,p)$ to a diffusion equation in only one variable leads to a smoothing of the escape rate. The one-dimensional model obviously breaks down completely at $\beta=5.0$. This is not surprising and was anticipated in Ref. 5:

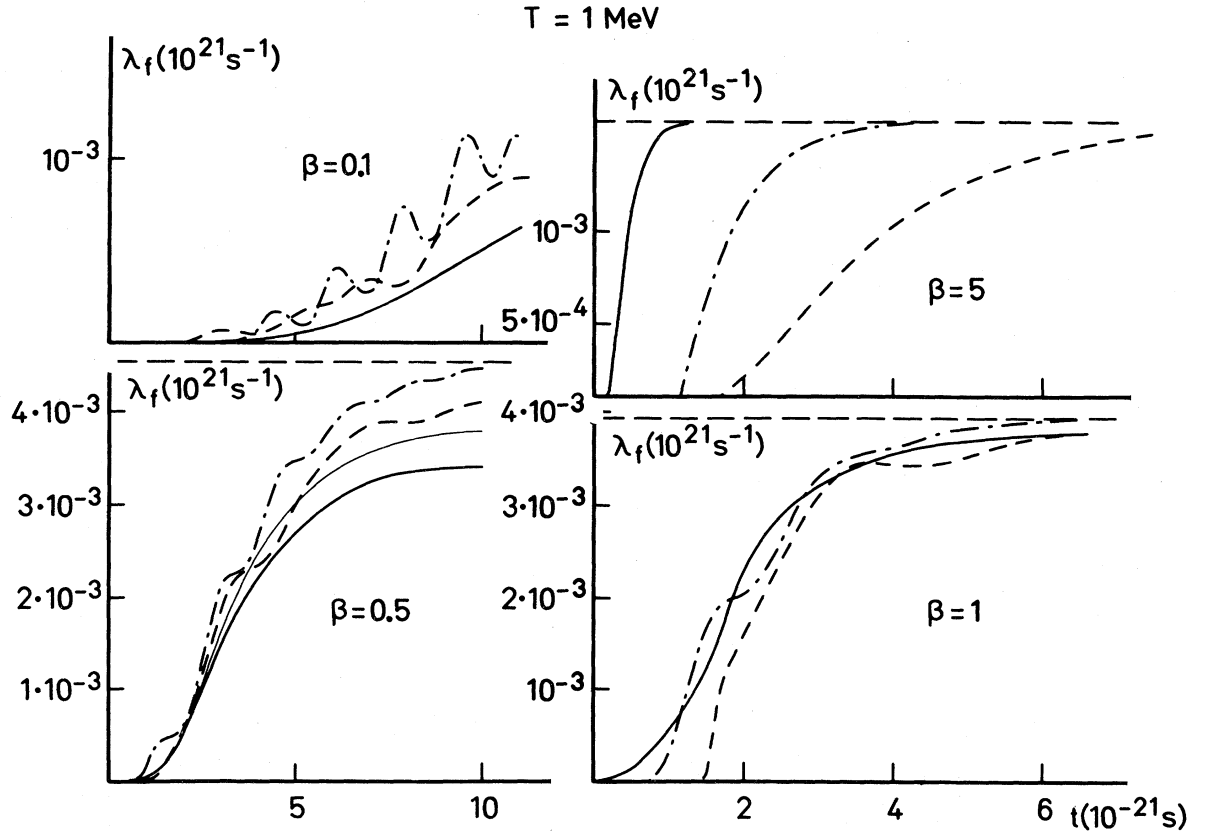


FIG. 4. The fission rate $\lambda_f(t)$ (in units of $10^{21}$ sec$^{-1}$) defined in Eq. (6) for various values of $\beta$ as indicated and for $T=1$ MeV versus time $t$ (in units $10^{-21}$ sec). The dashed curves are the results of the numerical calculations, the full curves are the result of the model of Ref. 5, and the dashed-dotted curves are the result of the analytical approximation of Eq. (23). Here and in the following figures, the dashed straight lines give the quasistationary value of Eq. (7).

The overdamping of the motion in $x$ of the probability distribution limits the current available for the fission process in a manner that is not contained in the one-dimensional model with its equal treatment of $p$ and $x$.

It is reassuring that for a range of values of $\beta$, the same overall behavior of $\lambda_f(t)$ as in the one-dimensional model is obtained. This shows that in the equilibration phase and for this range of $\beta$ values, the limiting factor in the probability current over the barrier is the feeding mechanism from the minimum at $x_1$. For $\beta=5.0$, the overdamping of the motion in the $x$ direction, on the other hand, is the decisive factor, leading to a much later onset of the fission process than the diffusion mechanism of Ref. 5 would suggest. We see that the time $\tau$ which elapses until the onset of the quasistationary flow over the barrier, which obviously tends to infinity as $\beta\to0$, decreases with increasing $\beta$, attains a minimum near $\beta=1.0$ or so, and then increases again as a consequence of overdamping.

The temperature of $T=4$ MeV (Fig. 5) is equal to the barrier height; one may wonder whether in such a case the quasistationary approach by Kramers[7] still yields meaningful results. For all $\beta$ values shown, $\lambda_f(t)$ closely approaches the value (dashed straight line) predicted by Eq. (7), which shows that the answer can be given in the affirmative. Regarding the manner in which the stationary distribution is built up, the comparison between our results and the model of Ref. 5 leads for $\beta=0.1$ and $\beta=0.5$ to the same conclusion as for $T=1$ MeV. For $\beta=1.0$ and $\beta=5.0$ the calculated rate overshoots the value calculated from Eq. (7); the overshooting is more pronounced for the smaller $\beta$ value. Taking into account in addition the fact that for $\beta=1.0$ the total fission lifetime is only $12\times10^{-21}$ sec (so that the time $\tau$ needed to build up the probability flow is roughly one sixth of the total lifetime), we are led to view this overshooting as the first indication of a tendency of the entire probability distribution to pass the barrier in a single swoop. We expect this tendency to increase with temperature. For $\beta=5.0$, the overshoot is less and the fission lifetime is about $30\times10^{-21}$ sec, showing the effect of overdamping.

In reactions involving two complex nuclei, a sig-

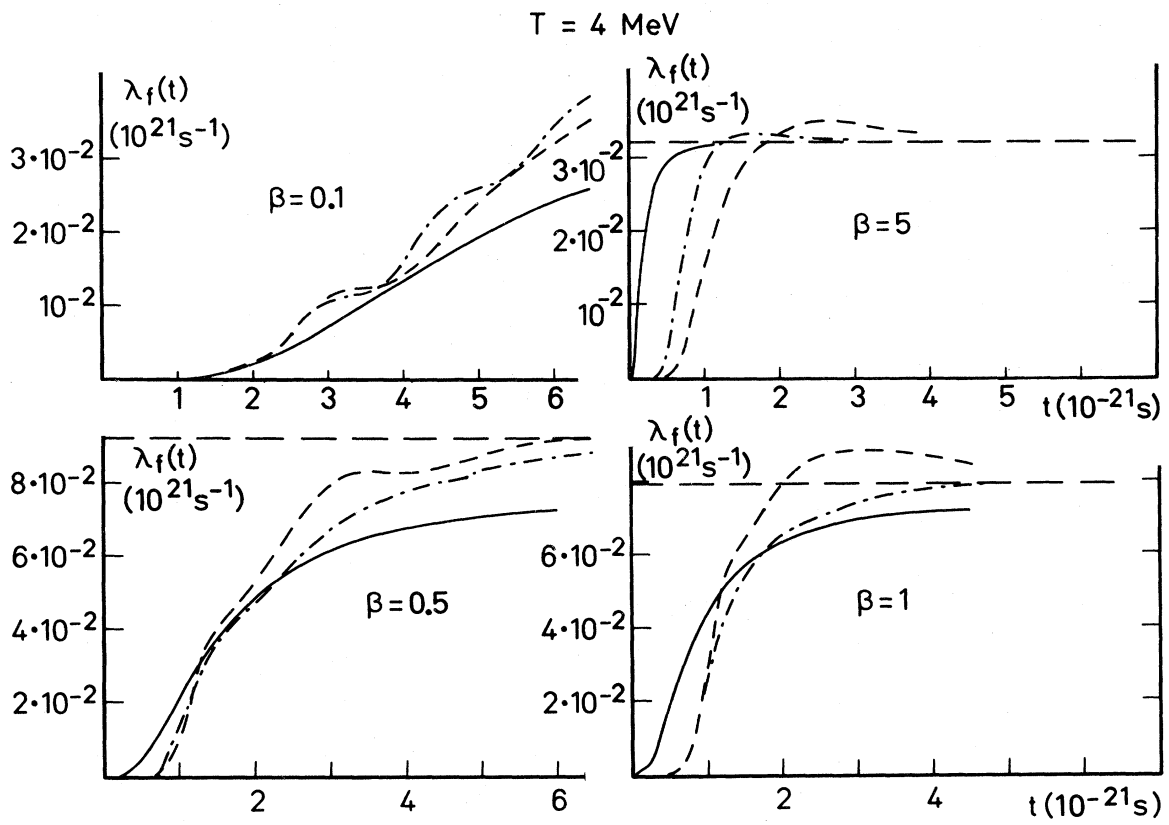


FIG. 5. The same as Fig. 4 for $T=4$ MeV.

nificant amount of angular momentum is brought into the fissioning nucleus, and the fission barrier is therefore lowered. Figure 6 displays results calculated for such a case, with the following parameters: $\mu=42$ u, $b=7$ fm, $c=10.7$ fm, $g=0.00465\times10^{42}$ fm$^{-2}$ sec.$^{-2}$ The height of the fission barrier is now 1.6 MeV, $|x_1|=4.89$ fm, the level density parameter has been chosen equal to $A/8=21$, and the oscillator energies are $\hbar\omega_1=0.70$ MeV and $\hbar\omega_0=0.54$ MeV, respectively. The friction constant is $\beta=1.0$ in all cases shown in Fig. 6. The constant $\beta_0=2\omega_1=2.12$ so that the motion in the potential minimum at $x_1$ is yet underdamped. The dashed curves again display the numerical results, and the dashed-dotted curves give the results of the analytical formula of Sec. IV. The dashed horizontal lines give the prediction of Kramers's formula (7). The full curves shown for $T=2$, 3, and 4 MeV give the value of $J(t)$ as defined in Eq. (5), i.e., the current without the normalization factor. We note that considerable overshooting happens whenever $T>E_f$. For $T=4$ MeV, we have $\lambda_k=0.062\times10^{21}$ sec$^{-1}$ or $(\lambda_k)^{-1}=16\times10^{-21}$ sec, which shows that overshooting happens during a substantial fraction (one third) of the total decay time: We are now close to the situation that the entire fission process is a "transient."

We summarize our findings. For $\beta\lesssim\beta_0=2\omega_1$ $(=3.7\times10^{21}$ sec$^{-1})$, i.e., for the case when the motion in the first minimum is underdamped, and for $E_f\gg\frac{1}{2}\hbar\omega_1$, we find approximately $\tau\propto\beta^{-1}$: The diffusion process is governed by the feeding mechanism, and the transient time increases with decreasing $\beta$. This relation can even be understood semiquantitatively in terms of an approximate analytical solution of the diffusion equation. The resulting formula for $\tau$ is given in Eq. (24) in the following section. For $\beta\gtrsim\beta_0$, $\tau$ increases with $\beta$; this increase is due to the overdamping of the motion in the first minimum. When $E_f\approx2\hbar\omega_1$ and $T>E_f$, the pattern changes; the fission rate overshoots the quasistationary value; the overshoot already occurs at fairly small times $\tau\approx\beta^{-1}$.

## IV. ANALYTICAL EXPRESSION FOR THE TIME DEPENDENCE OF THE ESCAPE RATE; EXCITATION ENERGY DEPENDENCE OF THE FISSION PROBABILITY

We attempt to find an analytical, if approximate, expression for the time dependence of $\lambda_f(t)$. This would help in understanding the qualitative features of $\lambda_f(t)$ including the oscillations displayed in Figs.

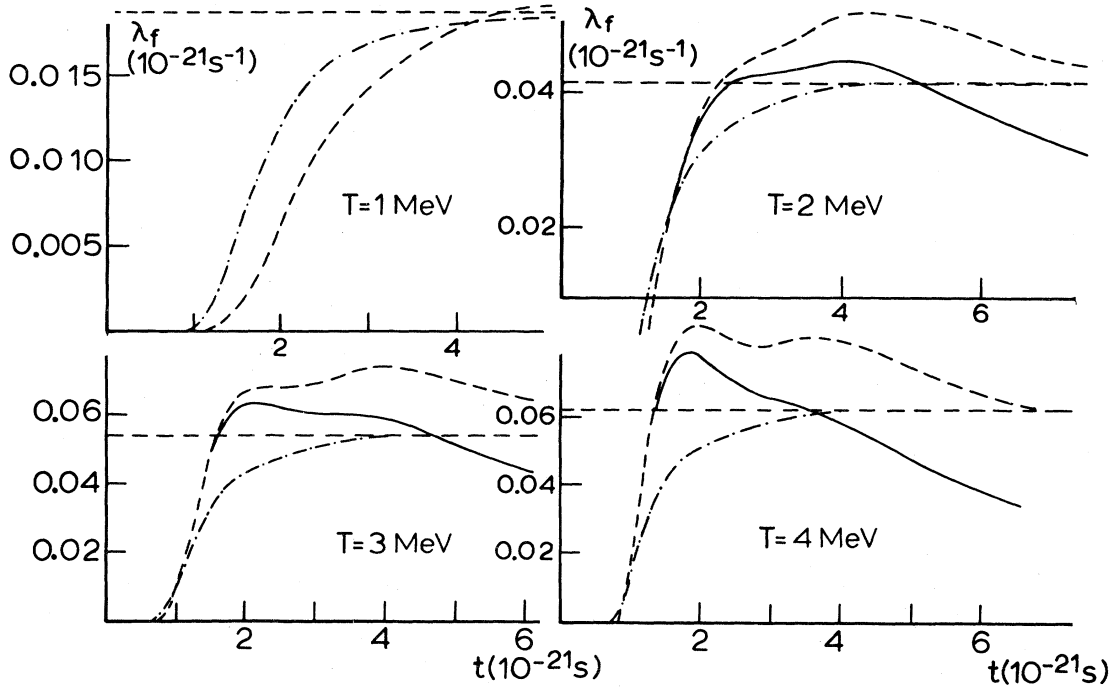


FIG. 6. The fission rate $\lambda_f$ (in units $10^{21}$ sec$^{-1}$) as a function of time for different temperatures as indicated, and for the compound nucleus $^{168}$Yb with a fission barrier of 1.6 MeV, with parameter values as given in the text. The dashed curves give the numerical results, the full curves the current calculated from Eq. (5), and the dotted-dashed curves the prediction of Eq. (23).

3 and 4, and would, moreover, yield a useful starting point for analytically calculating the energy dependence of the fission probability $P_f$ via Eqs. (9) and (6). For a cascade type of calculation allowing for the emission of several neutrons and the associated cooling of the compound nucleus, an analytical formula for $\lambda_f(t)$ is obviously indispensable.

From the outset we emphasize that our approach is heuristic and, although apparently successful, not of a stringency comparable to Kramers's derivation of the decay rate given in Eq. (7). Our approach is modeled after Kramers's procedure, and it is therefore useful to recall the latter.[7] Kramers observes that the stationary equilibrium distribution which solves the FPE is given by

$$P_0(x,p)=C_0\exp\{-H(x,p)/T\}\ ,$$

where $H(x,p)$ is the Hamiltonian of the system. For a potential like the one given by Eq. (10), this solution does not describe diffusion over the barrier since the overwhelming part of $P_0$ would be located in the deep minimum at $x=x_2$. Therefore, the solution of the *stationary* FPE (1) [with $(\partial/\partial t)P=0$] is written as

$$P(x,p)=CF(x,p)\exp\{-H(x,p)/T\}\ , \tag{13}$$

and $F(x,p)$ is subject to the boundary conditions that $F(x_1,p)=1$, $F(x_2,p)=0$. The FPE is then considered in a neighborhood of the saddle point, $x=0$, and a quadratic approximation in $x$ is used for the potential near this point. Inserting the ansatz (13) into the stationary FPE, one obtains a partial differential for $F(x,p)$ which, together with the boundary conditions just mentioned, determines $F$ uniquely. Equation (13) is then used to calculate both the probability current over the saddle, and the normalization in the first minimum. The ratio of the two yields expression (7).

We apply basically the same procedure to the *time-dependent* FPE (1). This, however, involves further approximations, as we now demonstrate.

If we approximate the potential near the minimum at $x=x_1$ by the osculating harmonic oscillator $V(x)$ with frequency $\omega_1$, then the solution to the FPE (1) for this problem is known[14] and given by

$$P_{\rm osc}(x,p;t)=C_{\rm osc}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[p^2/\sigma_p{}^2(t)+2V(x)/(\sigma_x{}^2(t)\mu\omega_1{}^2)-\rho(t)p\frac{\partial}{\partial x}V(x)/(\sigma_p(t)\sigma_x(t)\mu\omega_1{}^2)\right]\right\}. \tag{14}$$

Here, $\sigma_p(t)$ and $\sigma_x(t)$ are the time-dependent variances given explicitly in Ref. 14, $\rho(t)$ is the correlation function,[14] and $C_{\rm osc}$ is a normalization constant.

The findings displayed in Figs. 2 and 3—that for $T \ll E_f$ and $\beta < 1.0$ the variances $\sigma_p{}^2$ and $\sigma_x{}^2$ of the full problem are reasonably close to those of the oscillator problem of Eq. (14), and that $\langle x \rangle$ and $\langle p \rangle$ remain reasonably close to their starting values—encourage us to make for the time dependence of the full problem an ansatz patterned after Eq. (14):

$$P_0(x,p;t)=C_0\exp\left\{-\frac{1}{2(1-\rho^2)}\left[p^2/\sigma_p{}^2(t)+2U(x)/(\sigma_x{}^2\mu\omega_1{}^2)-\rho(t)p\frac{\partial U(x)}{\partial x}\Big/(\sigma_p(t)\sigma_x(t)\mu\omega_1{}^2)\right]\right\} . \tag{15}$$

Here, $\sigma_p(t)$, $\sigma_x(t)$, and $\rho(t)$ are the expressions derived in the context of Eq. (14), while $U(x)$ is the full potential (10). We view $P_0(x,p;t)$ as being analogous to the stationary solution $P_0(x,p)$ used by Kramers and defined above. Our ansatz (15) has the same drawback as Kramers's $P_0(x,p)$: As soon as the variances $\sigma_p,\sigma_x$ are finite, the enormous depth of $U(x)$ at $x=x_2$ yields an unrealistically large probability density near this point. We accordingly write, in analogy to Eq. (13), the full solution in the form

$$P(x,p;t)=CF(x,p;t)P_0(x,p;t) . \tag{16}$$

We now determine $F(x,p;t)$ by inserting $P(x,p;t)$ into the FPE (1) *and by assuming that $P_0(x,p;t)$ is a solution to this equation.* This is the decisive assumption for which we can offer no analytical argument, only the numerical findings of Figs. 2 and 3. We stress, however, that $P_0(x,p;t)$ should (approximately) solve the FPE (1) only in a range of $x$ values $x \lesssim 0+\delta$ up to and including the saddle, with $\delta$ small and positive, since we determine $F(x,p;t)$ again from a linearization procedure near $x=0$. We are confident that this (weaker) condition is sufficiently realistic to yield meaningful results.

Inserting the ansatz (16) into Eq. (1), using the assumption just stated, and confining our attention to a small neighborhood of $x=0$ where we replace $U(x)$ by the locally osculating harmonic oscillator with frequency $\omega_0$, we find for $F$ the equation

$$\frac{\partial F}{\partial t}+p\frac{\partial F}{\partial x}+\omega_0{}^2x\frac{\partial F}{\partial p}=\beta p\frac{\partial F}{\partial p}-\frac{2\epsilon}{1-\rho^2}[p/\sigma_p{}^2+xp\omega_0{}^2/(\sigma_x\sigma_p\omega_1{}^2)]\frac{\partial F}{\partial p}+\epsilon\frac{\partial^2 F}{\partial p^2} . \tag{17}$$

We look for a solution which for $x\to-\infty$ $(+\infty)$ tends towards 1 (0). We construct this solution in the manner of Refs. 7 and 14 by writing $F$ as a function of the variable $\xi=p-a(t)x$ only. This ansatz is consistent with Eq. (17) if $a(t)$ fulfills the differential equation

$$\frac{da}{dt}+a\left[a+\beta-\frac{2\epsilon}{\sigma_p{}^2(1-\rho^2)}\right]-\omega_0{}^2-\frac{2\epsilon\rho}{\sigma_p\sigma_x(1-\rho^2)}\frac{\omega_0{}^2}{\omega_1{}^2}=0 \tag{18}$$

and yields for $F(\xi)$ the differential equation

$$-\left[a+\beta-\frac{2\epsilon}{\sigma_p{}^2(1-\rho^2)}\right]\xi\frac{\partial F}{\partial\xi}=\epsilon\frac{\partial^2}{\partial\xi^2} . \tag{19}$$

Equation (19) has the formal solution

$$F(x,p;t)=F_0[(a+\beta)/\epsilon-2/(\sigma_p{}^2(1-\rho^2))]^{1/2}\int_{-\infty}^{p-a(t)x}d\xi\exp\{-\tfrac{1}{2}\xi^2((a+\beta)/\epsilon-2/\sigma_p{}^2(1-\rho^2))\} , \tag{20}$$

where $F_0$ is a constant. It is shown below that $a(t)>0$. Then, with $F_0=(2\pi)^{-1/2}$ we obviously have $F\to+1$ for $x\to-\infty$, and $F\to0$ for $x\to+\infty$, which shows that $F$ fulfills the boundary conditions formulated above. These statements hold if

$$a(t)+\beta-2\epsilon/[(1-\rho^2)\sigma_p{}^2]>0 ,$$

which we presently show follows from $a(t)>0$.

To discuss the behavior of $a(t)$ we recall that[14] $\sigma_p{}^2$, $\sigma_x{}^2$, and $\rho$ have the asymptotic $(t\to\infty)$ values $T/\mu$, $T/(\mu\omega_1{}^2)$, and zero, respectively, and that these values are attained at a time scale given by $\beta^{-1}$. For times $t\gg\beta^{-1}$, the coefficients in the differential equation (18) are constants, and $a(t)$ itself is then also constant and a solution of the quadratic equation $(\rho\to0)$:

$$a^2-\beta a-\omega_0{}^2=0 .$$

This is the very same equation as used by Kramers, and by choosing the solution with $a>0$ we ensure

that $F$ has, for $t \gg \beta^{-1}$, the correct asymptotic behavior. Solving the differential equation (18) backward in time, we arrive at the desired solution $a(t)$ which we use henceforth.

The construction just described, together with the fact that $P_0$, $F$, and hence also $P$, tend towards Kramers's solution for $t \gg \beta^{-1}$, automatically ensures that our decay rate will tend to Kramers's value in the same limit. On the other hand, a glance at the form (16) shows that for very small times, $P(x,p;t)$ is confined to a vicinity of $x_1$, and that the current over the barrier is exponentially small. We can thus expect that by calculating the rate of escape over the barrier from the approximate expressions just discussed, we obtain reasonable results.

To discuss the function $a(t)$ we observe that Eq. (18) can be written in the form

$$\frac{da}{dt}+[a+\alpha(t)-\Omega(t)][a+\alpha(t)+\Omega(t)]=0 . \tag{21}$$

Here,

$$\alpha(t)=\beta/2-\epsilon/(\sigma_p^2(1-\rho^2)) ,$$
$$\Omega(t)=[\alpha^2(t)+\omega_0^2+2\epsilon\rho\omega_0^2/(\sigma_p\sigma_x\omega_1^2(1-\rho^2))] . \tag{22}$$

It is useful to discuss the solution (21) first for the case when $\alpha$ and $\Omega$ are independent of time. Then, $a_0=-\alpha+\Omega$ is also independent of time and equals the solution used by Kramers. If a different initial condition is chosen, the general solution with the correct asymptotic behavior has the form

$$a(t)=-\alpha-\Omega\frac{C\exp(-2\Omega t)+1}{C\exp(-2\Omega t)-1} ,$$

with $C$ an arbitrary constant to obey the initial condition. This form shows that $a(t)\to a_0$ over a time of order $(2\Omega)^{-1}$. A similar statement involving the exponential

$$\exp\left\{-2\int\Omega(t')dt'\right\}$$

can be obtained in the case where both $\alpha$ and $\Omega$ do depend on time by assuming that $a(t)$ differs from $-\alpha(t)+\Omega(t)$ by terms which are small of first order. These arguments together with our numerical findings suggest that $a(t)$ tends towards $-\alpha(t)+\Omega(t)$ over times of order $|2\Omega(t)|^{-1}$. Since $\Omega(t)\geq\omega_0$, the time involved is less than $(2\omega_0)^{-1}\simeq 3\times 10^{-22}$ sec. This time is much smaller than the time needed for the onset of the rise of $\lambda_f(t)$ in all cases studied. Hence, $a=-\alpha+\Omega(t)$ is sufficiently accurate for our purposes, and the properties of $a(t)$ claimed above follow.

We evaluate $\lambda_f(t)$ as the ratio of the current $J(t)$ at the saddle point,

$$J(t)=\int_{-\infty}^{+\infty}dp\,pP(0,p;t)$$

to the normalization

$$N(t)=\int_{-\infty}^{+\infty}dp\int_{-\infty}^{0}dx\,P(x,p;t) .$$

We evaluate $J(t)$ by using a harmonic approximation to $U(x)$ near $x=0$, and $N(t)$ by using a harmonic approximation to $U(x)$ near $x=x_1$. This yields in a straightforward fashion

$$\lambda_f(t)=[(2\pi)^{1/2}(1-\rho^2)^{1/2}\sigma_x\sigma_p]^{-1}\sigma_p^2(1-\rho^2)\times\left[\frac{(\beta+a)\sigma_p^2(1-\rho^2)-2\epsilon}{(\beta+a)\sigma_p^2(1-\rho^2)-\epsilon}\right]\times\exp\{-E_f/((1-\rho^2)\sigma_x^2\mu\omega_1^2)\} . \tag{23}$$

It is easy to check that for $t\gg\beta^{-1}$, $\lambda_f(t)\to\lambda_k$ as given by Eq. (7). The last factor on the rhs of Eq. (23) is exponentially small at $t=0$, where $(1-\rho^2)\sigma_x^2\mu\omega_1^2$ is typically 0.3 MeV or so; as $\sigma_x^2$ increases with time, this factor approaches the asymptotic value $\exp(-E_f/T)$ typical of the Kramers expression. The other factors in Eq. (23) approach their asymptotic values oscillatingly or monotonically depending on whether the motion in the first minimum is underdamped ($\beta<2\omega_1$) or overdamped ($\beta>2\omega_1$).

Values of $\lambda_f(t)$ calculated from Eq. (23) are shown as dashed-dotted lines in Figs. 4 and 5. We first observe that there is a close agreement between these and the dotted curves for both temperatures and $\beta<1.0$. While the overshooting seen for $\beta=1.0$ and $T=1.0$ MeV is naturally not reproduced, we observe that the oscillatory structure or lack of the same is reproduced at least qualitatively in all cases. We now understand the origin of these oscillations as being due to the underdamping of the motion near $x=x_1$. Upon closer inspection, the agreement for $\beta<1.0$ is surprisingly good: The behavior of $\lambda_f(t)$ is governed by the exponential, the last factor on the rhs of Eq. (23). It contains $\sigma_x^2$ in the denominator, which is approximated by the analytical expression that applies to diffusion in a harmonic oscillator.[14] Figure 2 shows that this approximation *underestimates* the correct value of $\sigma_x^2$. Hence, we should expect Eq. (23) to yield a curve which rises *more slowly* than the numerically calculated one. That this is not the case must be attributed to a slowing down of the real diffusion process as it reaches the saddle point, to the correction embodied in the factors multiplying the exponential in Eq. (23), or to both. The situation is even more striking for $\beta=5.0$, where our formula leads to an *underes-*

*timation* of $\tau$, rather than to the converse situation. We nevertheless conclude that Eq. (23) is capable of reproducing $\lambda_f(t)$ with sufficient accuracy for a range of $\beta$ values of main physical interest, $0.1<\beta<2.0$ or so, and 0.5 MeV $<T<E_f$.

Equation (23) also yields the following semiquantitative estimate of the time $\tau$ between initiation of the diffusion process and the attainment of quasistationarity. The governing factor on the rhs is the exponential. We simplify the exponent by setting $\rho=0$, and by omitting the oscillatory functions in $\sigma_x{}^2$. This yields for the exponent the approximate expression

$$-E_f/[T(1-\exp(-\beta t))] .$$

The rate $\lambda_f(t)$ will be approximately equal to 90% of the equilibrium value if at time $t=\tau$ the above-mentioned exponent equals $-E_f/T-0.1$. This yields

$$\tau\simeq\beta^{-1}\ln(10E_f/T) . \tag{24}$$

Equation (24) accounts qualitatively for the (weak) temperature dependence (for equal values of $\beta$ and $E_f$) displayed in Figs. 4 and 5. For $E_f/T=4$ we have $\ln(10E_f/T)=3.7$, which agrees with our earlier[5] estimate $\tau\simeq\pi\beta^{-1}$.

Our derivation of expression (23) for $\lambda_f(t)$ lacks essential ingredients in the case of overdamped motion, $\beta>2\omega_1$. We have not paid great attention to this case not only because it is perhaps unphysical for the nuclear situation, but also because a complete analytical treatment has been given by van Kampen.[15]

Comparing the analytical results with our numerical values for the case of a small fission barrier (Fig. 6), we see that for $T=1.0$ MeV $<E_f$, the agreement is again close, while for $T>E_f$ the overshooting once again is not reproduced.

We now proceed to use Eq. (23) to evaluate the fission probability (the probability of first-chance fission) $P_f$ as given by Eq. (9). This can be done by straightforward numerical integration. In order to be able to compare with the Bohr-Wheeler formula, we multiplied all our rates with the energy-dependent factor $(T/\omega_1)H(T)$, where $H$ differs by unity slightly and only for small excitation energies to correctly reproduce the energy dependence of the integral $\int dE\,\rho(E)$ appearing in the Bohr-Wheeler expression. The result is shown in Fig. 7 for the quantity $\log\alpha$ with $\alpha=P_f/(1-P_f)$ which equals $\Gamma_f/\Gamma_n$ in the absence of transients. The curve labeled *TT* corresponds to the Bohr-Wheeler formula without the inclusion of any transients; the solid curves show the effect of $\beta$ again without transients; i.e., the deviation from the transition-state results due to friction, while the dashed curves include

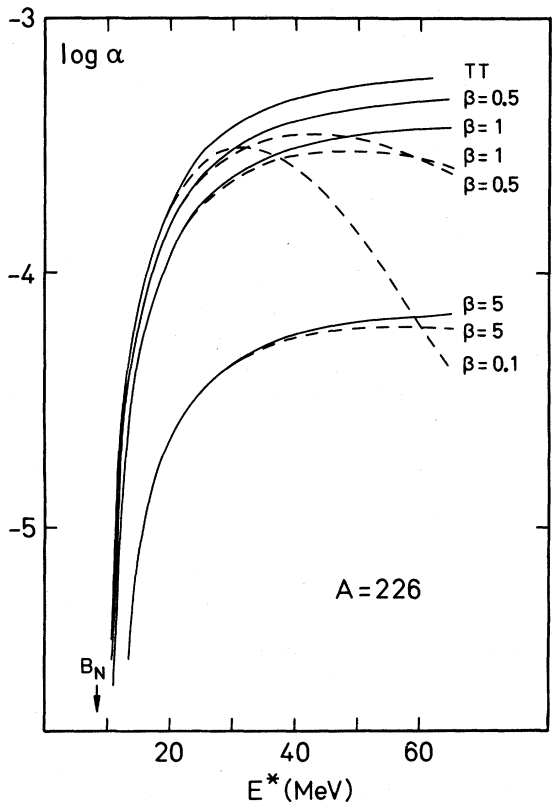


FIG. 7. The quantity $\alpha$ defined in the text is plotted versus excitation energy $E^*$ for various strengths of the nuclear friction constant, and for two situations: (i) constant probability flow over the barrier for all times—full curves; (ii) inclusion of transient behavior by use of Eq. (21)—dashed curves. For the dashed curves, we have used Eq. (21) and the following parameter values: $A=226$, $\mu=A/4$, $c=7.6$ fm, $b=5.0$ fm, $g=0.06749$ fm$^{-2}$ sec$^{-2}$, which yields $|x_1|=3.49$ fm, $E_f=8.5$ MeV, and $x_2=5.44$ fm and a potential depth $U(x_2)=-27.8$ MeV. The level density parameter was taken as $A/10$, and the friction constant $\beta$ as indicated in the figure.

the transient behavior and show that at the energies where $\hbar/\Gamma_n$ becomes comparable to $\tau$, a very significant reduction of the probability of first-chance fission ensues. Our results are, by the way, in good agreement with those of Ref. 5. They suggest that a further study of the effect of transients upon the fission rate might help us to understand the experimental observations.[11]

## V. SUMMARY AND CONCLUSIONS

Using a Fokker-Planck equation for the fission variable and its canonically conjugate momentum, and a potential which is sufficiently realistic to simulate the actual fission barrier, we have studied the time evolution of the probability density both

numerically and analytically, emphasizing the study of transients, i.e., of those processes which occur before the quasistationary probability flow over the barrier is attained. We have calculated the "time-dependent fission width" $\hbar\lambda_f(t)$ in terms of the current passing the saddle point. For large times $t$, $\hbar\lambda_f(t)$ approaches the value predicted by Kramers, unless $\beta$, the friction coefficient, is unrealistically small, or if the temperature is comparable with the fission barrier, or larger.

For the time $\tau$ between initiation of the diffusion process and the attainment of the quasistationary situation, we have found that for $\beta<\beta_0$, $\tau$ decreases with increasing $\beta$ [see Eq. (24)], while for $\beta>\beta_0$, $\tau$ increases with increasing $\beta$. Here, $\beta_0$ is given by $2\omega_1$, with $\omega_1$ the frequency of the oscillator potential which osculates the true potential in the first minimum. The condition $\beta<\beta_0$ ($\beta>\beta_0$) is equivalent to the underdamping (overdamping) of the motion in the first well; the dependence of $\tau$ on $\beta$ can qualitatively be understood along these terms.

For times $t<\tau$, the function $\lambda_f(t)$ is an oscillatory (monotonic) function of $t$ depending on whether $\beta<\beta_0$ ($\beta>\beta_0$). This behavior again follows from the underdamping (overdamping) of the motion in the first minimum. We have given an analytical expression for $\lambda_f(t)$ which describes the numerical results with sufficient accuracy in a wide range of temperatures and $\beta$ values to be practically useful. Although our numerical results were obtained for a restricted choice of potential parameters, we believe that the discussion of these results as well as the analytical approximation for $\lambda_f(t)$ shows that their main features are generic and of general validity. We have also found that for $\beta<\beta_0$, our results coincide with those of an earlier model.[5] This domain of validity for the model of Ref. 5—the underdamped motion—was qualitatively expected.

Folding the probability for neutron emission with the time-dependent fission probability leads to the expression for the probability for first-chance fission, in our model a function of excitation energy and friction constant. We found that $P_f$ is modified—compared to the value calculated on the basis of the Bohr-Wheeler formula—in two ways: (i) $P_f$ suffers an overall reduction due to the nonzero value of $\beta$ when transients are not included; (ii) transients change the energy dependence of $P_f$ at excitation energies of several tens of MeV's (depending on $\beta$); $P_f$ tends to be significantly smaller than predicted without the inclusion of transients in this domain.

A significant reduction of the probability of first-chance fission should clearly also lead to an increase of the ratio of neutron over fission decays; a quantitative analysis of this ratio must utilize a cascade program together with our formula for $\lambda_f(t)$; this we have not yet attempted to carry out. It is nonetheless encouraging that our results tend in the direction of accounting for the experimentally observed discrepancies. Another problem derives from the situation encountered in some experiments inducing fission with projectiles heavier than $^{12}C$ or so: The large available angular momentum reduces the depth of the first minimum in such a way that $E_f\simeq T\simeq 2\hbar\omega_1$. In this case, fission may altogether be governed by the transients, since $\tau$ is not substantially smaller than $\hbar/\Gamma_f$. Examples for this type of behavior were also studied.

The estimates for $\lambda_f(t)$ and $\tau$ obtained in this paper relate to the attainment of quasistationarity at the *saddle point.* We have not yet investigated the analogous problem at the *scission point,* where it is physically more meaningful. At this latter point, the approximation leading to Eq. (23) does not apply. One could hope, however, that the form of $\lambda_f(t)$ should apply equally at the scission point, with an additional time delay reflecting the path of the system from saddle to scission. This delay could perhaps be estimated as in Ref. 16. Work on this problem is under way.

Our work contains simplifications. We have neglected the temperature dependence of $\beta$ and of the fission barrier; we have paid no attention to the possibility of a double-humped fission barrier; we have considered only a single degree of freedom instead of a multidimensional potential landscape which could account for shape deformation effects; we have paid no attention to the possibility that the level-density parameter (and, hence, the nuclear temperature) might change with $x$ as a consequence of shell effects, etc. In spite of these simplifications, we believe we have identified in a unique way the influence of transients on the fission process. We hope that further work on this problem will lead to an improved understanding of the fission process, and of the role of nuclear friction.

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