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Graphical analyses of connected-kernel scattering equations

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Simple graphical techniques are employed to obtain a new (simultaneous) derivation of a large class of connected-kernel scattering equations. This class includes the Rosenberg, Bencze-Redish-Sloan, and connected-kernel multiple scattering equations as well as a host of generalizations of these and other equations. The basic result is the application of graphical methods to the derivation of interaction-set equations. This yields a new, simplified form for some members of the class and elucidates the general structural features of the entire class.

[NUCLEAR REACTIONS Graphical techniques and connected-kernel equations. Rosenberg, Bencze-Redish-Sloan, and interaction-set equations.]

I. INTRODUCTION

Derivations of the members of a certain class of connected-kernel scattering equations (which contains, e.g., the Rosenberg,[1] Bencze-Redish-Sloan[2] (BRS), sigma set,[3-5] and connected-kernel multiple scattering[3,6] equations), or the general interaction-set equations[6] which govern[3-5] the class, often rely heavily on algebraic[1-6] or algebraic-combinatoric[6-9] methods at some stage of the development. Although these derivations can be particularly cogent for some purposes, including the exploration of the physical[3] and mathematical[10] content of the structures which arise in the theory, they tend to be both tedious and convoluted. Since the members of this class of equations are too complex for direct computational applications, they serve primarily to display the answers to questions concerning the general characteristics of the many-particle problem and to provide points of departure for realistic approximation schemes. Thus, it is often advantageous[11] to possess simple graphical[11-13] pictures of the essential structure of the equations in order to guide both formal investigations and approximation schemes, e.g., in effective interaction or optical potential theories. Unfortunately, the essential structure of the equations is often obscured by the aforementioned derivations.

Recently, Kowalski[11] emphasized the simple graphical features which underlie the interaction-set equations when only pair-wise (two-body) interparticle forces are present in the many-particle Hamiltonian. The removal of the restriction to pair-wise forces is evidently desirable, both to incorporate the possibility of fundamental many-body forces and, especially, to accommodate the more general types of effective many-particle forces which arise in some truncated nuclear reaction theories.[14,15] The development of Ref. 11, however, appears to be tied to the narrow notion of almost-connected graphs[1,11] which arises naturally in the case of only pair-wise forces.

In this paper we obtain the extension of the ideas of Refs. 1 and 11 to the circumstance wherein there are arbitrary many-particle forces. This result, which is obtained through a straightforward generalization of the concept of almost connectedness, provides a simple graphical picture of the entire class of interaction-set equations. As a by-product of these considerations we obtain a new, structurally transparent form of the interaction-set equations of Ref. 6 which should enhance their usefulness in regard to realistic nuclear reaction theories and models.

II. INTERACTION SET EQUATIONS AND CONNECTIVITY

In this section, we briefly review those aspects of the interaction-set equations which are required for the graphical analysis of Sec. III and for understanding the generality of that analysis.

Our considerations are posed primarily in terms of the transition operator,

$$T = VGG_0^{-1}, \tag{1}$$

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and, in particular, the operators T_p:

$$T=\sum_p T_p=\sum_p [V]_p G G_0{}^{-1} . \tag{2}$$

In Eq. (1), V is the sum of all interparticle interactions in the N-particle system, G_0 is the free Green's function, and G is the full Green's function for the system

$$G^{-1}=G_0{}^{-1}-V . \tag{3}$$

In Eq. (2) the sum runs over the full set of distinct partitions p of the N-particle system, where a partition of a many-particle system is a division of the system into distinct groups or clusters of particles. The notation $[V]_p$ signifies the part of V (if any) which has connectivity p, that is to say, the part of V which connects all the particles internal to each particular cluster of p but connects no particles which are contained in different clusters of p. The decomposition of V in Eq. (2) represents the usual[16,17] (string or C_0) connectivity expansion of V. It is important to note that, since the p span the full set of partitions, the potentials $[V]_p$ span the set of possible interparticle potentials, including both fundamental many-body potentials and the more general effective potentials which arise in some reaction theories.[14,15]

The interaction-set operators T_p satisfy the interaction-set integral equations[6]

$$T_p=\sum_q B_{p,q}+\sum_q (K_I)_{p,q} T_q , \tag{4}$$

where

$$(K_I)_{p,q}=\sum_{m,n} B_{p,m} Q_1 (P_2+\Gamma)_{m,n} G_0 Q_1 \bar{\Delta}_{n,q} Q_0 \tag{5}$$

and

$$B_{p,q}=\{[T_p]_q+\delta_{p,1}[V]_1[G G_0{}^{-1}]_q\}\bar{\delta}_{1,q} . \tag{6}$$

The operators T_p and the Eqs. (4) are referred to as interaction-set operators and equations, respectively, since they are indexed by the same set of partitions which characterize the interparticle interactions. It is shown in Ref. 6 that the kernel of Eq. (4) becomes connected after a single interaction. In fact, Eqs. (4)—(6) represent the generalization of the Rosenberg[1] equations to accommodate arbitrary many-particle forces.[6] In Eq. (5), P_n (Q_n) projects on (off) the set of partitions consisting of exactly n clusters,[18] while Γ and $\bar{\Delta}$ are matrices in partition space which are defined by[6]

$$\Gamma=Q_2\Delta^t C Q_1 , \tag{7}$$

$$(\bar{\Delta})_{a,b}=1-(\Delta)_{a,b} , \tag{8}$$

$$C_a=-(\Delta^{-1})_{1,a} , \tag{9}$$

with

$$\Delta_{a,b}=1 \quad b\subseteq a$$
$$=0 \quad b\not\subset a . \tag{10}$$

In the above, $\subset$ is the usual "contained in" notation, Δ^t is the transpose of Δ, $[V]_1$ is the fully-connected part of V which corresponds to the unique single cluster partition (labeled 1), and $\bar{\delta}_{i,j}=1-\delta_{i,j}$. Evidently, the structure of the interaction-set equations is quite obscure in the above. In Sec. III we find that a graphical analysis can be employed to rewrite them in a much more transparent way.

In Sec. III our analysis is confined to the interaction-set operators T_p and Eqs. (4)—(6). The interaction-set is more general than indicated above,[6] however, and, since the methods and results of Sec. III can be taken over immediately to the entire interaction set, we outline the generalization of the above before proceeding to Sec. III.

The interaction-set idea is extended to include transition operators of the generic form (α arbitrary),

$$T^\alpha=V^\alpha G G_\alpha{}^{-1} , \tag{11}$$

by simply adopting a different notion of connectivity.[6] In Eq. (11), V^α is the interaction external to partition α and G_α is the α-channel Green's function, so that

$$G^{-1}=G_\alpha{}^{-1}-V^\alpha , \tag{12a}$$

$$V=V^\alpha+V_\alpha , \tag{12b}$$

and

$$G_\alpha{}^{-1}=G_0{}^{-1}-V_\alpha , \tag{12c}$$

where V_α is the interaction internal[6] to partition α. Let $[A]^\alpha_b$ denote the b-connected part of the operator A in the C_α connectivity classification, where C_α connectivity is defined in analogy to C_0 connectivity except that in C_α one classifies the connectivity of an operator according to the interactions *external to* α *only*.[3,6] That is, the interactions internal to α are treated, for purposes of classification only, as if they were disconnected operators (in the usual sense). For example, both V_α and G_α are completely disconnected in C_α. It is then easy to see[3,6] that the decomposition [cf., Eq. (2)]

$$T^\alpha=\sum_p T^\alpha_p=\sum_p [V^\alpha]^\alpha_p G G_\alpha{}^{-1} \tag{13}$$

leads to interaction-set equations for the T^α_p that are

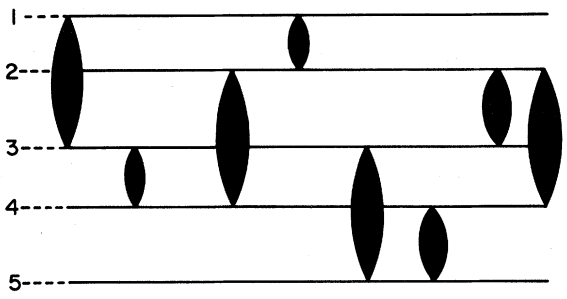

FIG. 1. An example of the graphical pictures (for $N=5$) which are employed throughout Sec. III. The solid horizontal lines represent particles (labeled 1–5 in the figure) and the solid blobs represent the interparticle interactions. The number of lines, n, intersecting the blob represent the n-body nature of the potential. A free Green's function G_0 is understood to act between blobs. This particular graph represents a particular term (for $N=5$) of T_p, with $p=(123)(4)(5)$, which is fully connected. This graph also represents Eq. (17) with $\lambda=(1234)(5)$ and $q=(1)(2)(345)$ since it becomes connected (from the left) at the fifth interaction.

completely analogous to Eqs. (4)–(6) (which correspond simply to the special case where α is the unique N-cluster partition). All that is required are the replacements[6] [in Eqs. (4)–(6), e.g.]

$$T_p \rightarrow T_p^{\alpha} , \tag{14a}$$

$$G_0 \rightarrow G_{\alpha} , \tag{14b}$$

and

$$[V]_p \rightarrow [V^{\alpha}]_p^{\alpha} = [V]_p^{\alpha} - V_{\alpha}\delta_{p,0} . \tag{14c}$$

The fact that the kernel of the resulting integral equation becomes C_0 connected after a single interaction follows from the obvious fact that any operator which is C_{α} fully connected is necessarily C_0 fully connected.

III. GRAPHICAL ANALYSIS

From the definition Eq. (2) of the T_p it is evident that T_p consists of the sum of all parts of T which end on the left with a p-connected interaction. The contributions to T_p (and to T) are easily visualized in terms of graphical pictures of the terms of T_p which arise when an expansion of G, using the resolvent relation[19] [cf., Eq. (2)]

$$G = G_0 + G_0 V G , \tag{15}$$

is inserted into the definition $T_p = [V]_p G G_0^{-1}$. An example of the types of graphs which we have in mind throughout this section is displayed in Fig. 1.

We have the C_0 decomposition,

$$T_p = [T_p]_{\text{dis}} + [T_p]_1 , \tag{16}$$

where dis refers to the sum of all of the disconnected parts of T_p and 1 refers to the fully-connected part. In order to simplify the discussion, it is convenient at this point to assume that $[V]_1=0$, but to leave the disconnected parts of V unrestricted. This restriction is removed later.

Consider now the fully-connected part of Eq. (16), and let us use X, Y, and Z to denote arbitrary terms or graphs of T. If $[X_p]_1$ denotes a fully-connected graph contained in T_p then it possesses the unique decomposition

$$[X_p]_1 = [Y_p]_{\lambda} G_0 Z_q , \tag{17}$$

where $\lambda \neq 1$ and $\lambda \cup q = 1$. The union of two partitions, $\lambda \cup q$, is defined[17] to be the partition with the *largest* number of clusters which contains *both* λ and q. Thus the condition $\lambda \cup q = 1$ ensures that $[X_p]_1$ is fully connected. Equation (17) and its uniqueness obtain, because if we follow the graph of $[X_p]_1$ from the left we eventually encounter (for the first time) an interaction which makes it fully connected. The part of the graph preceding this interaction determines $[Y_p]_{\lambda}$ uniquely, while the remainder of the graph determines Z_q uniquely (see Fig. 1).

If we now sum Eq. (17) over all fully-connected graphs (which end with $[V]_p$) we obtain

$$[T_p]_1 = \sum_{\lambda \neq 1} \sum_{q} [T_p]_{\lambda} G_0 \delta_{\lambda \cup q, 1} T_q . \tag{18}$$

Upon inserting Eq. (18) into Eq. (16) we find that

$$T_p = [T_p]_{\text{dis}} + \sum_{\lambda \neq 1} \sum_{q} [T_p]_{\lambda} G_0 \delta_{\lambda \cup q, 1} T_q , \tag{19}$$

or, in matrix form,

$$\vec{\mathrm{T}} = [\vec{\mathrm{T}}]_{\text{dis}} + \vec{\mathrm{W}} Q_1 G_0 \vec{\delta}\, \vec{\mathrm{T}} , \tag{20}$$

where $\vec{\mathrm{T}}$ is the column matrix with row labels p, and the matrices $\vec{\mathrm{W}}$ and $\vec{\delta}$ are defined by

$$(\vec{\mathrm{W}})_{p,\lambda} = [T_p]_{\lambda} , \tag{21}$$

$$(\vec{\delta})_{p,\lambda} = \delta_{p \cup \lambda, 1} , \tag{22}$$

and we recall that Q_1 projects off the unique single-cluster partition. The kernel of Eq. (19) or (20) becomes connected after a single iteration since

$$(\vec{\mathrm{K}}^2)_{a,b} = \sum_{c,d,e \neq 1} [T_a]_d G_0 [T_c]_e G_0 \delta_{d \cup c, 1} \delta_{b \cup e, 1} , \tag{23}$$

so that each term on the right-hand side of Eq. (23) is at least $(d \cup c)$ connected which, in view of the

first δ, is fully connected.

The structure of Eq. (19) or Eq. (20) is completely transparent. Namely, the structure of the kernel guarantees that each $[T_p]_\lambda$ couples to only those T_q *with respect to which* it is almost connected. Equation (19) or Eq. (20) represents the obvious generalization of the Rosenberg equations and the notion of almost-connected graphs.

If we compare Eq. (19) or Eq. (20) with Eqs. (4)—(6) for the case at hand (recall that $[V]_1=0$, so far), then we see that the driving terms are the same. It is not at all obvious, however, that the complicated kernel of Eq. (4) is equivalent to the simple kernel of Eq. (19), although it must be. To see that the two are equivalent, consider the part of the kernel of Eq. (4):

$$M \equiv Q_1(P_2+\Gamma)Q_1\bar{\Delta}Q_0 \ . \tag{24}$$

If we make use of the identity[6,17]

$$\Delta^t C Q_1 = P_2+\Gamma \ , \tag{25}$$

we see that

$$M = Q_1\Delta^t C Q_1 \bar{\Delta} Q_0 \tag{26}$$

or

$$M_{\lambda,q} = \sum_a \Delta_{a,\lambda} C_a \bar{\delta}_{a,1}\bar{\Delta}_{a,q}\bar{\delta}_{q,0} \ . \tag{27}$$

Upon using Eq. (8) and the obvious identity

$$\Delta_{a,b}\Delta_{a,c} = \Delta_{a,b\cup c} \ ,$$

we find

$$M_{\lambda,q} = \sum_a C_a \bar{\delta}_{a,1}(\Delta_{a,\lambda}-\Delta_{a,\lambda\cup q})\bar{\delta}_{q,0} \ , \tag{28}$$

so that application of the identity[6,17]

$$\sum_a C_a \bar{\delta}_{a,1}\Delta_{a,b} = \bar{\delta}_{b,1} \tag{29}$$

yields

$$M_{\lambda,q} = \bar{\delta}_{\lambda,1}\delta_{\lambda\cup q,1} \ . \tag{30}$$

Use of Eq. (30) in Eqs. (4) and (5) yields Eq. (19), thus establishing the equivalence. The foregoing highlights the advantage of the graphical method, namely, complicated combinatoric or algebraic methods are obviated by the use of simple structural pictures. We remark that the corresponding results of Refs. 3 and 11 follow immediately from Eq. (19) upon restricting V to include only pair-wise forces.

Let us now extend our considerations to the case where $[V]_1\neq 0$. It is evident from their construction that no change is required in Eqs. (19) for $p\neq 1$, other than noting that in the sum over q the $q=1$ term now contributes. Thus, we need only provide an additional equation for

$$T_1 = [T_1]_1 = [V]_1 G G_0^{-1} \ . \tag{31}$$

In view of Eq. (31), which expresses the fact that T_1 has no disconnected part, we write

$$T_1 = [V]_1[GG_0^{-1}]_{\rm dis} + [V]_1[GG_0^{-1}]_1 \ , \tag{32}$$

in contrast to Eq. (16). Iteration of the Green's function G using the resolvent relation Eq. (15) leads to a graphical picture for (GG_0^{-1}) which is analogous to that which we have employed for the T_p. In fact, the same logical line that was used to obtain Eq. (18) can be used to show that

$$[GG_0^{-1}]_1 = \sum_{\lambda\neq 1}\sum_q [GG_0^{-1}]_\lambda G_0 \delta_{\lambda\cup q,1} T_q \ . \tag{33}$$

Applying Eq. (33) in Eq. (32) yields

$$T_1 = [V]_1[GG_0^{-1}]_{\rm dis} + \sum_{\lambda\neq 1}\sum_q [V]_1[GG_0^{-1}]_\lambda G_0\delta_{\lambda\cup q,1}T_q \ , \tag{34}$$

which is the additional equation required.

Combining Eq. (34) with Eq. (19) yields the final result of our graphical analysis:

$$T_p = B_p + \sum_q K_{p,q}T_q \ , \tag{35}$$

where

$$B_p = [T_p]_{\rm dis} + \delta_{p,1}[V]_1[GG_0^{-1}]_{\rm dis} \tag{36}$$

and

$$K_{p,q} = \sum_{\lambda\neq 1}\{[T_p]_\lambda + \delta_{p,1}[V]_1[GG_0^{-1}]_\lambda\} \times G_0\delta_{\lambda\cup q,1}T_q \ . \tag{37}$$

Equations (35)—(37) are easily reconciled with Eqs. (4)—(6) through the use of Eq. (30).

Finally, we emphasize that the methods and results of this section can be extended to the general interaction set equations discussed in Sec. II. This is easily verified by making use of Eqs. (14). We remark also that the transformation from the interaction-set equations to the corresponding channel-labeled equations, e.g., the BRS equations, is easily accomplished through the use of the sigma-set methods of Refs. 3—5. Thus the conceptual clarity of the foregoing graphical analysis applies to the channel-labeled equations as well as to their interaction-set counterparts.

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