

## Three-body reactions for a local potential in coordinate space

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The three-body theory of reactions is formulated for local potentials in coordinate space on the basis of the Faddeev equation. The wave function is expressed as a sum of all possible channel wave functions. In solving, the physically important pole terms are taken as the zeroth order terms. The higher order terms are treated as the perturbation. The kernels that appear in the perturbational series are regular at the origin and confined in a finite region of space, thus enabling us to calculate the breakup process numerically. We can expect that this perturbational series is rapidly convergent. The equation is finally reduced to a coupled set of integral equations for the elastic and breakup amplitudes.

NUCLEAR REACTIONS Three-body model. The Faddeev equation for local potential in coordinate space. Coupled channels including breakup. Perturbational calculation. Physically important terms as lowest order terms. Anticipated rapid convergence.

### I. INTRODUCTION

After pioneering work on three body problems,<sup>1</sup> now called the Faddeev theory, a considerable amount of work has been performed to study the properties of few-body systems,<sup>2</sup> and also to extend the method to nuclear reactions.<sup>3</sup> Contrary to conventional theories of nuclear reactions, the Faddeev approach renders an impetus to the treatment of three-body breakup processes on a mathematically rigorous basis.

So far, the Faddeev equation has been solved for the breakup process by three different approaches: (i) by invoking a separable potential in momentum space,<sup>4</sup> (ii) by using the Padé approximant in momentum space,<sup>5</sup> and (iii) by solving the partial differential equation in coordinate space.<sup>6</sup> In contrast to these approaches, it has been the author's desire for a long time to formulate the Faddeev equation in such a manner that the theory resembles the conventional theory of reactions as closely as possible, at the same time having the equation soluble with a minimum of ambiguities. To reiterate: (1) We want to formulate the wave function in the form of an integral equation consisting of a sum of the elastic, inelastic, rearrangement, breakup, and

closed channels. (2) We want to formulate the theory to be used for a local potential in coordinate space. We prefer not to use the hyperspherical coordinates. These considerations will make our theory accessible to a wide class of nuclear physicists. (3) The theory should be formulated so that all numerical calculations are performed within a finite region and the contributions from large distances are treated analytically. This requirement is not necessary if we confine ourselves to the elastic, inelastic, or rearrangement channels, but it is necessary if we treat the breakup process.

The formulation of the three-body theory with the purpose (1)–(3) has never been tried before. Let us explain the difficulties that we encounter in the course of the formulation, and describe how they are removed.

In handling three-body problems, one is often tempted to make a separable expansion of the two-body scattering matrix ( $t$  matrix) with the hope that the expansion approaches the exact one with an increasing number of separable terms.<sup>7</sup> Although this approach is powerful insofar as we deal with the problem approximately at relatively low energies, it is not suitable for treating the problem exactly.<sup>8</sup> Even from the approximate point of view, this ap-

proach is not desirable, because with an increasing number of separable terms the number of nodes in the separable expansion increases, and the computation task becomes rapidly unmanageable even on a powerful computer, while convergence becomes slow. Furthermore, at high energies or for larger angular momenta where the Born approximation becomes effective, the use of a separable expansion with many terms is simply absurd. As an alternative, we have proposed an iterative approach to the three-body problem in which physically important contributions, such as the primary singularity leading to the elastic channel, the effects of a virtual state, or resonances, are treated as the lowest term, while all the remaining higher order terms are dealt with as the perturbation.<sup>9</sup>

To achieve this purpose, the wave function is decomposed into channels as a sum,<sup>10</sup> and the two-body  $t$  matrix that appears in the Faddeev wave function is expressed as a sum of a separable term that describes the effects of the virtual state or resonances and a nonseparable term that may be treated as the perturbation. Such a separation of the  $t$  matrix was proposed by us.<sup>11,12</sup>

At this point it is appropriate to discuss the method of a nonsingular kernel in the two-body problem.<sup>13</sup> This method has attracted physicists concerning the convergence of iteration. In this method, a modified equation is related to the original equation in a simple manner. There are many ways of choosing the modified equation. However, if we confine ourselves to the question of convergence, we may not claim the advantage of the method of a nonsingular kernel so strongly, because there are many other ways of obtaining convergent results.

Here, we should emphasize that one of the advantages of the method of the nonsingular kernel lies in the fact that this method yields the off-shell  $t$  matrix as a sum of one nonseparable and one separable term. From a practical point of view, it is important that the number of separable terms does not increase to improve the numerical result. This is very different from the usual separable expansions.

Now, let us compare various kinds of methods of the nonsingular kernel. The most general expression of the off-shell  $t$  matrix is given by Eq. (51) of Ref. 11. As discussed in Refs. 12 and 14, it is known at this moment that this equation is related to four kinds of nonsingular equations: the Jost-type, the standing-wave-type, the Kowalski-Noyes-type, and the one presented by us,<sup>11,12</sup> each of which is related to a modified convergent equation for a two-body problem. For instance, the Jost solution is one of these. However, the Jost solution is not suitable to use for the two-body sector in the three-body equa-

tion, because it is not regular at the origin and as a result, some matrix elements diverge.<sup>15</sup> The standing wave equation is not suitable, because it can diverge on the real momentum axis. The Kowalski-Noyes-type equation is not suitable, either, because the resulting separable term of the  $t$  matrix has no relation to physical poles. After all, the Kowalski-Noyes-type equation serves only to obtain a convergent result. In contrast to these, our ameliorated equation<sup>11,12</sup> is free from all of these difficulties; it is regular at the origin, and all physically important poles are involved in one separable term of the  $t$  matrix.

We have thus a modified equation [as given by Eq. (30) of Ref. 12] that is convergent at the two-body level. However, it is not evident *ab initio* whether this modified equation embedded in the three-body equation is convergent or not. In this situation, it is highly desirable to have a further method by which we can get a convergent result even if the original Neumann series is divergent. As a preliminary for handling this problem, we have proposed a method of acceleration.<sup>14,16</sup> It has been shown for two-body equations that this method is very efficient in getting a rapid convergence even when the original Neumann series is divergent. Since the type of equations handled in Ref. 14 is the same that appears in the three-body problem (the Fredholm type), we expect that the same acceleration method is also successfully applicable to the series involved in the three-body system such as Eqs. (19) and (41) of this paper.

In the Faddeev equation, the scattering process is described in a manner that the interactions take place successively between one pair of particles after another. Therefore, the elastic scattering is described as a scattering from a nonlocal potential.<sup>17</sup> After some efforts<sup>16-18</sup> we have reached a conclusion that a method of treating the  $n$ - $d$  elastic scattering as a scattering from a nonlocal potential, which is rapidly convergent for "high" energy such as 20 MeV, is usually extremely slowly convergent at "low" energies such as 1 MeV. To overcome this difficulty, a method of nonsingular kernel for a nonlocal potential has been proposed and combined with the method of acceleration.<sup>14</sup> If we use this method, we get a convergent value very rapidly, even at low energies.

The breakup process gives rise to the following difficulty. If we use the usual Green's function and try to make a simple-minded iteration, we are forced to calculate numerically up to infinitely large distances, at each order of multiple scattering series. Evidently, this is simply impossible. In the method of solving the partial differential equation,<sup>6</sup> this difficulty is avoided by putting the boundary condition at the asymptotic region. This method has the fol-

lowing shortcoming, however. With decreasing energy, the asymptotic region escapes to larger and larger distances. Especially if there exists a final state interaction between a pair, as in the case of the singlet nucleon-nucleon interaction, the pair is subject to this interaction no matter how large the distance between the spectator and the c.m. of the pair is. As a result, we have to perform numerical calculations up to infinitely large distances. This difficulty is avoided by smoothly joining the boundary condition of this region with that of the breakup region in the six-dimensional space.<sup>6</sup> However, even with this clever boundary condition, we cannot avoid the numerical difficulty coming from the fact that the asymptotic region is far away from the origin at low energies. Because of these difficulties, we do not attempt to solve the partial differential equation, but we formulate the kernel of the Faddeev equation so that the numerical calculations are performed only in a finite region and the contribution from large distances is calculated analytically.<sup>12</sup>

In concluding the Introduction, we are now in a position to construct a three-body theory with purposes (1)–(3) from our elements.<sup>12,14</sup> In Sec. II we present our three-body theory as concisely as possible. For the purpose of clear presentation, we neglect antisymmetrization<sup>19</sup> and the Coulomb potential<sup>20</sup> without loss of generality. Our theory can, of course, be easily generalized to a more general class of three-body systems, e.g., those having many discrete excited states. In Sec. III conclusions are given.

## II. THEORY

To avoid complications due to reduced masses, etc., and to demonstrate our theory as clearly as possible, we keep the neutron-deuteron system in mind without loss of generality. For the same reason, we neglect spin until Sec. IIF.

### A. Decomposition into channels

In the Faddeev theory, the total wave function of the system is expressed as a sum of three components  $\Psi^{(i)}$  ( $i=1,2,3$ )

$$\Psi = \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)}. \quad (1)$$

Here,  $\Psi^{(i)}$  represents the wave function in which the pair of particles  $j$  and  $k$  are subject to the final state interaction  $V_i \equiv V_{jk}$ , while the particle  $i$  stands as the spectator. Although Faddeev introduced this decomposition for mathematical reasons, the physical meaning of Eq. (1) is stated in Ref. 12. We denote by  $t_i$  the  $t$  matrix for the pair  $jk$ , by  $G_0$  the Green's function in the three-body free space, by

$\phi(jk)$  the radial wave function of the target (deuteron), by  $f(i)$  that of the incoming particle, and by  $|\alpha_0\rangle$  the (isospin-, spin-) angular-wave function of the initial state. The Faddeev equation then reads

$$\Psi^{(1)} = \phi(23)f(1)|\alpha_0\rangle + G_0 t_1 (\Psi^{(2)} + \Psi^{(3)}), \quad (2)$$

with two other equations obtained by the cyclic permutations of 1, 2, and 3. For simplicity, we shall use the permutation operator  $Q$  defined by

$$Q\Psi^{(1)} \equiv \Psi^{(2)} + \Psi^{(3)}. \quad (3)$$

The angular momenta of the interacting pair and the spectator will be represented by  $L$  and  $l$ , respectively. We designate by  $x_1$  the distance between particles 2 and 3, and by  $y_1$  the relative distance between particle 1 and the c.m. of the pair 23,

$$\begin{aligned} \vec{x}_1 &= \vec{r}_2 - \vec{r}_3, \\ \vec{y}_1 &= \vec{r}_1 - (\vec{r}_2 + \vec{r}_3)/2. \end{aligned} \quad (4)$$

We use the usual notations for the spherical Bessel and Hankel functions;  $j_l(kr)$  and  $h_l^{(+)}(kr)$  for momentum  $k$  and coordinate  $r$ . We represent the nucleon mass by  $m$ , the energy of the system by  $E$ , and the energy of the bound pair 23 by  $|E_{23}|$ . The initial momentum  $p_0$  of the incident particle is given by

$$p_0 = \left[ \frac{4m}{3\hbar^2} (E + |E_{23}|) \right]^{1/2}. \quad (5)$$

With these notations, the initial state  $\phi(23)f(1)|\alpha_0\rangle$  in Eq. (2) is written more explicitly as

$$\phi_L(x_1)j_l(p_0 y_1)|\alpha_0(L, l)\rangle.$$

Let us introduce a complete orthonormal set of functions  $\{\phi_{1,p}\}$ , where  $\phi_{1,p}$  denotes the product of the normalized plane wave for the spectator

$$u_l(p, y_1) = \left[ \frac{2}{\pi} \right]^{1/2} p j_l(p y_1) \quad (6)$$

and the angular wave function  $|\alpha_1\rangle$ ,

$$\phi_{p\alpha_1} = u_l(p, y_1)|\alpha_1\rangle. \quad (7)$$

The spectator energy  $E_p$  and the energy of the interacting pair  $E_q$  are related by

$$E_q = E - E_p, \quad E_p = \frac{3\hbar^2}{4m} p^2. \quad (8)$$

The Green's function  $G_0$  in Eq. (2) is a function of  $\vec{x}_1$  and  $\vec{y}_1$ . We expand it by  $\{\phi_{1,p}\}$  and express the function  $\Psi^{(1)}$  as a sum of four components,<sup>10</sup>

$$\begin{aligned} \Psi^{(1)} = & \phi(23)f(1) |\alpha_0\rangle + |\phi(23)\rangle |\alpha_0\rangle G_1 \langle \alpha_0 | \langle \phi(23) | V_{23} | Q\Psi^{(1)} \rangle \\ & + \sum_{\alpha_1} \int_0^E dE_p |\phi_{p\alpha_1}\rangle G_{1,q} t_{1,q} \langle \phi_{p\alpha_1} | Q\Psi^{(1)} \rangle + C_I^{(1)} | Q\Psi^{(1)} \rangle . \end{aligned} \quad (9)$$

On the right-hand side, the first (second, third, fourth) term represents the initial (elastic, breakup, closed) channel wave function. Here,  $G_1$ ,  $G_{1,q}$ , and  $t_{1,q}$  are two-body operators defined by using the kinetic energy operator  $K_{23}$  ( $K_1$ ) for the relative motion of the pair 23 (the spectator 1 and the c.m. of the pair 23) as

$$G_1 = \frac{1}{E + |E_{23}| - K_1 + i\epsilon} , \quad (10)$$

$$G_{1,q} = \frac{1}{E_q - K_{23} + i\epsilon} , \quad (11)$$

and

$$t_{1,q} = V_1 + V_1 G_{1,q} t_{1,q} , \quad V_1 = V_{23} . \quad (12)$$

The notation  $C_I^{(1)}$  represents the sum of operators for the closed channel as given by Eqs. (10b) and (10c) of Ref. 10. Since the explicit form of this operator is not necessary in the present article, we do not write it here.

### B. Breakup channel

Following Ref. 12, let us bring the breakup term in Eq. (9) to a tractable form. In Eq. (32) of Ref. 12, we expressed the kernel  $G_{1,q} t_{1,q}$  as a sum of one nonseparable term  $\hat{\omega}_{1,q} - 1$  and one separable term as

$$\begin{aligned} G_{1,q} t_{1,q} = & (\hat{\omega}_{1,q} - 1) \\ & - |\bar{\phi}_{1,q}\rangle \frac{1}{D_q} \langle j_L(qx_1) | V_1 \hat{\omega}_{1,q} . \end{aligned} \quad (13)$$

Here, the nonseparable term  $\hat{\omega}_{1,q} - 1$  is regular at the origin and

$$\bar{\phi}_{1,q} \xrightarrow{x_1 \rightarrow \infty} qh_L^{(+)}(qx_1) \sim \frac{e^{i(qx_1 - L\pi/2)}}{x_1} . \quad (14)$$

In Ref. 12, we intentionally accommodate only a finite number of physically important poles as the zeros of the function  $D_q$ ,

$$D_q = 0 . \quad (15)$$

Since these poles are the origin of the divergence or very slow convergence we thus secure the convergence of the series involved in the nonseparable term. At the same time, we can avoid the calculations on a large number of physically insignificant

resonances. (For instance, if  $D_q$  were the Jost function, it would involve an infinite number of zeros, almost all of which are physically insignificant. Furthermore, the calculations are almost impossible because most of the zeros lie outside the analytical region on the complex momentum plane.)

Since  $\hat{\omega}_{1,q} - 1$  vanishes at large distances, we treat this term as a part of the closed channel. This term is combined with  $C_I^{(1)}$  of Eq. (9),

$$\begin{aligned} C^{(1)} = & C_I^{(1)} + \sum_{\alpha_1} \int_0^E dE_p |\phi_{p\alpha_1}\rangle \\ & \times (\hat{\omega}_{1,q} - 1) \langle \phi_{p\alpha_1} | . \end{aligned} \quad (16)$$

We can prove by the stationary-phase theorem<sup>21</sup> that the last term on the right-hand side asymptotically vanishes. If we use Eq. (16), Eq. (9) reads

$$\begin{aligned} \Psi^{(1)} = & \phi(23)f(1) |\alpha_0\rangle \\ & + \phi(23) |\alpha_0\rangle G_1 \langle \phi(23) | \langle \alpha_0 | V_{23} | Q\Psi^{(1)} \rangle \\ & + \sum_{\alpha_1} \int_0^E dE_p |\bar{\phi}_{1,q}\rangle |\phi_{p\alpha_1}\rangle \frac{1}{D_q} A_{q,\alpha_1} \\ & + C^{(1)} | Q\Psi^{(1)} \rangle , \end{aligned} \quad (17)$$

where  $A_{q,\alpha_1}$  is given by

$$A_{q,\alpha_1} = - \langle j_L(qx_1) | \langle \phi_{p\alpha_1} | V_1 \hat{\omega}_{1,q} | Q\Psi^{(1)} \rangle . \quad (18)$$

In Eq. (17), the breakup term is expressed as a sum and an integration over the separable terms. The breakup amplitude  $(1/D_q)A_{q,\alpha_1}$  manifests the effect of the pole in the final state interaction<sup>22</sup> through the factor  $1/D_q$ .

### C. Closed channel

We calculate the closed channel as the perturbation. We define the operator  $(1 - C^{(1)}Q)^{-1}$  by the Neumann series

$$\frac{1}{1 - C^{(1)}Q} = 1 + C^{(1)}Q + C^{(1)}QC^{(1)}Q + \dots . \quad (19)$$

We expect that this perturbation series is convergent, because all effects of poles, the primary singularity causing the elastic scattering and the poles

that affect the breakup channel, are excluded from the closed channel. We do not know of course whether the Neumann series (19) is actually convergent or not. Fortunately, this anxiety is relieved if

we adopt the method of acceleration which in many cases changes the divergent series into a convergent one.<sup>14,16</sup>

We write Eq. (17) as

$$\Psi^{(1)} = \frac{1}{1-C^{(1)}Q} \phi(23)F(1) |\alpha_0\rangle + \sum_{\alpha_1} \int_0^E dE_p \frac{1}{1-C^{(1)}Q} |\bar{\phi}_{1,q}\rangle |\phi_{p\alpha_1}\rangle \frac{1}{D_q} A_{q,\alpha_1}, \quad (20)$$

where the function  $F(1)$  stands for the wave function of the elastic channel,

$$F(1) = f(1) + G_1 \langle \phi(23) | \langle \alpha_0 | V_{23} | Q\Psi^{(1)} \rangle. \quad (21)$$

Since  $C^{(1)}$  is real and vanishes at large distances, we may say that in Eq. (20) the elastic and breakup waves are "dressed."

#### D. Elastic channel

Next, we discuss the wave function  $F(1)$  given by Eq. (21). Substituting Eq. (20) into Eq. (21), thereby using the fact that the two components  $\Psi^{(2)}$  and  $\Psi^{(3)}$  of  $Q\Psi^{(1)}$  give the same contribution, we find

$$F(1) = f(1) + G_1 2 \langle \phi(23) | \langle \alpha_0 | V_{23} \frac{1}{1-C^{(2)}Q} | \alpha_0 \rangle | \phi(31) \rangle F(2) \\ + G_1 2 \sum_{\alpha_2} \int_0^E dE_p \langle \phi(23) | \langle \alpha_0 | V_{23} \frac{1}{1-C^{(2)}Q} |\bar{\phi}_{2,q}\rangle |\phi_{p\alpha_2}\rangle \frac{1}{D_q} A_{q,\alpha_2}. \quad (22)$$

In this equation, the matrix element

$$\langle \phi(23) | \langle \alpha_0 | V_{23} \frac{1}{1-C^{(2)}Q} | \alpha_0 \rangle | \phi(31) \rangle$$

is a nonlocal potential, which we denote by  $U^I(1,2)$ ,

$$U^I(1,2) \equiv \langle \phi(23) | \langle \alpha_0 | V_{23} \frac{1}{1-C^{(2)}Q} | \alpha_0 \rangle | \phi(31) \rangle. \quad (23)$$

The explicit form of this nonlocal potential has been given in one of our previous papers.<sup>17</sup> (There, the effect of the closed channel  $C^{(2)}Q$  is not taken into account.)

A function representing the coupling of the elastic and the breakup channels is as follows:

$$U_{p,\alpha_2}^{II}(1) = \langle \phi(23) | \langle \alpha_0 | V_{23} \frac{1}{1-C^{(2)}Q} |\bar{\phi}_{2,q}\rangle |\phi_{p\alpha_2}\rangle. \quad (24)$$

Using Eqs. (23) and (24), we express Eq. (22) simply as

$$F(1) = f(1) + G_1 2 U^I(1,2) F(2) + G_1 2 \sum_{\alpha_2} \int_0^E dE_p U_{p,\alpha_2}^{II}(1) \frac{1}{D_q} A_{q,\alpha_2}. \quad (25)$$

The function (24) needs a closer look. When  $x_2$  becomes very large, the closed channel operator  $C^{(2)}$  vanishes. By Eq. (14), the function  $(1-C^{(2)}Q)^{-1} |\bar{\phi}_{2,q}\rangle$  behaves as the spherical Hankel function  $qh_{L_2}^{(+)}(qx_2)$  in this limit. Since  $x_2$  is a function of  $\vec{r}_1$ , as seen by Eq. (4), and  $y_1$  is also a function of  $\vec{r}_1$ ,  $U_{p,\alpha_2}^{II}(1)$  is an oscillatory function of  $y_1$  up to an infinitely large distance. Therefore, the overlap of this function with the Green's function  $G_1$  in Eq. (25) could cause some difficulty. However, we need not worry about this situation. When  $y_1$  becomes large, the last term of Eq. (25) behaves as

$$G_1 2 \sum_{\alpha_2} \int_0^E dE_p U_{p,\alpha_2}^{II}(1) \frac{1}{D_q} A_{q,\alpha_2} \rightarrow -p_0 h_l^{(+)}(p_0 y_1) \sum_{\alpha_2} \int_0^E dE_p \xi_l(p_0, p) \frac{1}{D_q} A_{q,\alpha_2}, \quad (26)$$

where the amplitude  $\xi_l(p_0, p)$  is given by

$$\begin{aligned}\xi_I(p_0, p) &= \langle \phi(23) | \langle j_I(p_0 y_1) | \langle \alpha_0 | V_{23} \frac{1}{1-C^{(2)}Q} | \bar{\phi}_{2,q} \rangle | \phi_{p\alpha_2} \rangle \\ &= \langle \phi(23) | \langle j_I(p_0 y_1) | \langle \alpha_0 | V_{23} \frac{1}{1-C^{(2)}Q} | \bar{\phi}_{2,q} \rangle | \alpha_2 \rangle | j_{I_2}(p y_2) \rangle \left[ \frac{2}{\pi} \right]^{1/2} p.\end{aligned}\quad (27)$$

In evaluating this amplitude, the part that we cannot treat numerically, since it extends to infinity, is the asymptotic part of  $qh_{L_2}^{(+)}(qx_2)$  which is the asymptotic limit of  $\bar{\phi}_{2,q}$ . All other parts are confined in a finite region and can be calculated numerically. The term in question is

$$\bar{\xi}_I(p_0, p) \equiv qp \left[ \frac{2}{\pi} \right]^{1/2} \langle \phi(23) | \langle j_I(p_0 y_1) | \langle \alpha_0 | V_{23} | \alpha_2 \rangle | j_{I_2}(p y_2) \rangle | h_{L_2}^{(+)}(qx_2) \rangle. \quad (28)$$

To bring Eq. (28) to an amenable form, we use the formula given in the previous articles,<sup>23,24</sup>

$$\begin{aligned}\langle j_I(p_0 y_1) | \langle \alpha_0 | \alpha_2 \rangle | j_{I_2}(p y_2) \rangle | h_{L_2}^{(+)}(qx_2) \rangle \\ = \frac{1}{p_0 p} \left[ \frac{2}{\pi} \right] \int_{-1}^1 du j_L(\lambda x_1) \Lambda_{p_0 p}(\alpha_0, \alpha_2; u) \int_0^\infty x_2^2 dx_2 j_{L_2}(\lambda_1 x_2) h_{L_2}^{(+)}(qx_2).\end{aligned}\quad (29)$$

Here  $\lambda$ ,  $\lambda_1$ , and  $u$  are defined by

$$\begin{aligned}\vec{\lambda} &= \vec{p}_0/2 + \vec{p}, \\ \vec{\lambda}_1 &= -(\vec{p}_0 + \vec{p}/2), \\ u &= \cos\theta_{\vec{p}_0 \vec{p}}.\end{aligned}\quad (30)$$

The explicit form of the function  $\Lambda_{p_0 p}(\alpha_0, \alpha_2; u)$  is given in Ref. 23, but it is not necessary for the present discussions. We make use of the formula given by Fuda<sup>25</sup> for the last factor of Eq. (29).

$$\int_0^\infty x^2 dx j_L(\lambda x) h_L^{(+)}(qx) = \frac{q^{-1}(\lambda/q)^L}{\lambda^2 - (q^2 + i\epsilon)}. \quad (31)$$

We have discussed in Ref. 12 that if we use Eq. (29) together with Eq. (31) for the right-hand side of Eq. (28), we can calculate it numerically.

Now, we come back to Eq. (25). The Green's function  $G_1$  given by Eq. (10) for the incident neutron is expressed as a sum of a separable term and the standing wave Green's function  $PG_1$  as

$$G_1 = -ip_0 |f(1)\rangle \langle f(1)| + PG_1, \quad (32)$$

where we take

$$f(1) = j_I(p_0 y_1). \quad (33)$$

If we designate by  $\bar{F}(1)$  the elastically scattered standing wave for the nonlocal potential  $2U^I(1,2)$ , it satisfies the equation

$$\bar{F}(1) = f(1) + PG_1 2U^I(1,2) \bar{F}(2). \quad (34)$$

In Ref. 14, it is shown that the iterative calculation of this equation converges very quickly for all ener-

gies if we use a method of the nonsingular kernel together with the method of acceleration. Further, we define a function by

$$\bar{F}_{p,\alpha_2}(1) = G_1 U_{p,\alpha_2}^{II}(1) + PG_1 2U^I(1,2) \bar{F}_{p,\alpha_2}(2), \quad (35)$$

which is calculated in the same way as Eq. (34).

In terms of Eqs. (34) and (35), Eq. (25) is expressed as

$$\begin{aligned}F(1) &= \bar{F}(1) [1 - ip_0 \langle f(1) | 2U^I(1,2) | F(2) \rangle] \\ &+ \sum_{\alpha_2} \int_0^E dE_p \bar{F}_{p,\alpha_2}(1) \frac{1}{D_q} A_{q,\alpha_2}.\end{aligned}\quad (36)$$

The function  $F(2)$  that is involved in the (as yet unknown) solution to the integral equation (25) is now contained in the elastic amplitude

$$\langle f(1) | U^I(1,2) | F_2 \rangle$$

in Eq. (36). In Sec. II E, we shall obtain a set of coupled integral equations for this amplitude and the breakup amplitude  $A_{q,\alpha_2}$ . If we solve this coupled set of equations, we obtain the wave function  $F(1)$  from Eq. (36) and the Faddeev component  $\Psi^{(1)}$  from Eq. (20).

### E. Integral equations for amplitudes

To simplify the description, we omit the particle indices in relevant quantities. For instance, the amplitude

$$\langle f(1) | U^I(1,2) | F_2 \rangle$$

is expressed simply as  $\langle f | U^I | F \rangle$ . This does not cause any confusion. We write Eqs. (20) and (36) simply as

$$\Psi = \frac{1}{1-CQ} |\phi F \alpha_0\rangle + \sum_{\alpha p} \frac{1}{1-CQ} |\bar{\phi}_q \phi_{p\alpha}\rangle \frac{1}{D_q} A_{q\alpha} \quad (37)$$

and

$$F = \bar{F}(1-2ip_0 \langle f | U^I | F \rangle) + \sum_{\alpha p} 2\bar{F}_{p\alpha} \frac{1}{D_q} A_{q\alpha}, \quad (38)$$

$$A_{q\alpha} = -\langle j_q \phi_{p\alpha} | V \hat{\omega}_q Q \frac{1}{1-CQ} |\phi F \alpha_0\rangle - \sum_{\alpha' p'} \langle j_q \phi_{p\alpha} | V \hat{\omega}_q Q \frac{1}{1-CQ} |\bar{\phi}_{q'} \phi_{p'\alpha'}\rangle \frac{1}{D_{q'}} A_{q'\alpha'}. \quad (40)$$

Further, if we use Eq. (38) in Eq. (40), we get

$$A_{q\alpha} = -\langle j_q \phi_{p\alpha} | V \hat{\omega}_q Q \frac{1}{1-CQ} |\phi \bar{F} \alpha_0\rangle (1-2ip_0 \langle f | U^I | F \rangle) - \sum_{\alpha' p'} \langle j_q \phi_{p\alpha} | V \hat{\omega}_q Q \frac{1}{1-CQ} [2|\phi \bar{F}_{p'\alpha'} \alpha_0\rangle + |\bar{\phi}_{q'} \phi_{p'\alpha'}\rangle] \frac{1}{D_{q'}} A_{q'\alpha'}. \quad (41)$$

On the other hand, if we multiply by  $\langle f | U^I$  on both sides of Eq. (38) from the left, we obtain

$$\langle f | U^I | F \rangle = \langle f | U^I | \bar{F} \rangle (1-2ip_0 \langle f | U^I | F \rangle) + 2 \sum_{\alpha p} \langle f | U^I | \bar{F}_{p\alpha} \rangle \frac{1}{D_q} A_{q\alpha}. \quad (42)$$

Thus our problem has been reduced to solving the set of integral equations (41) and (42). As we have described, all quantities in these equations are feasible for numerical calculations. The region of integration spreads only over a finite region of space. In the course of the calculations, we do not need any contour integral in the complex momentum space, as required in the usual three-body calculations in momentum space.

After solving Eqs. (41) and (42), the breakup amplitude  $(1/D_q)A_{q\alpha}$  is obtained and we calculate the elastic amplitude of the total system by the following expression that is derived from Eq. (25):

$$\Psi_T^{(1)} = \phi(23)f(1)|\alpha_0\rangle + |\phi(23)\rangle|\alpha_0\rangle G_1 \langle \alpha_0 | \langle \phi(23) | V_{23} | Q(\Psi_T^{(1)} + \Psi_S^{(1)}) \rangle + \sum_{\alpha_{T,1}} \int_0^E dE_p |\phi_{p\alpha_1}^T\rangle G_{1,q} t_{1,q}^T \langle \phi_{p\alpha_1}^T | Q(\Psi_T^{(1)} + \Psi_S^{(1)}) \rangle + C_{T,I}^{(1)} | Q(\Psi_T^{(1)} + \Psi_S^{(1)}) \rangle. \quad (44)$$

The equation for the singlet state does not involve the initial wave and the elastic wave. Thus the singlet Faddeev component reads

$$\Psi_S^{(1)} = \sum_{\alpha_{S,1}} \int_0^E dE_p |\phi_{p\alpha_1}^S\rangle G_{1,q} t_{1,q}^S \langle \phi_{p\alpha_1}^S | Q(\Psi_S^{(1)} + \Psi_T^{(1)}) \rangle + C_{S,I}^{(1)} | Q(\Psi_S^{(1)} + \Psi_T^{(1)}) \rangle. \quad (45)$$

respectively. Similarly, the amplitude  $A_{q,\alpha_1}$  defined by Eq. (18) is written as

$$A_{q\alpha} = -\langle j_q \phi_{p\alpha} | V \hat{\omega}_q | Q \Psi \rangle. \quad (39)$$

From Eqs. (37)–(39), we derive a set of integral equations relating the amplitudes  $\langle f | U^I | F \rangle$  and  $A_{q\alpha}$ .

If we use Eq. (37) on the right-hand side of Eq. (39), we get

$$2\langle f | U^I | F \rangle + 2 \sum_{\alpha p} \langle f | U_{p\alpha}^{II} \rangle \frac{1}{D_q} A_{q\alpha}. \quad (43)$$

Having calculated the amplitudes, we get the wave function for the elastic scattering by Eq. (38) and the Faddeev component  $\Psi^{(1)}$  by Eq. (37).

#### F. Treatment of virtual state

So far, we have neglected spin and isospin. The inclusion of these variables is, of course, achieved by extending the definition of the function  $|\alpha\rangle$  to include spin and isospin. However, it is more complex than that. We have to correctly take into account the effect of the singlet virtual state.

Equation (9) discussed so far should be regarded as the equation for the interacting pair in the triplet state. We do not need any modification of Eq. (9) for this component except that the singlet state is now coupled to it. With obvious notations, Eq. (9) now reads

In the triplet state, the effect of the two-body bound state has been extracted out in the form of the elastic term and no effect of the bound state remains in the closed channel, as demonstrated in Ref. 10. On the other hand, in the form of Eq. (45), the contribution from the singlet two-body virtual state is involved in the closed channel denoted by  $C_{S,I}^{(1)}$ . To make the zeroth order term physical and at the same time to get a good convergence in iterations of the closed channel, we extract out the effect of the virtual state in the form of the Sturm-Liouville function. This kind of treatment has been adopted by us<sup>23,26</sup> in calculations of  $^3\text{H}$  and  $^3\text{He}$ . Thus, as a special application of Eq. (51) of Ref. 11, we write for a suitably chosen energy  $E' (> E)$

$$C_{S,I}^{(1)} = \sum_{\alpha_{S,1}} \int_E^{E'} dE_p |\phi_{p\alpha_1}^S\rangle \frac{\lambda_q}{1-\lambda_q} |\hat{\psi}_{1,q}^S\rangle \\ \times \langle \hat{\psi}_{1,q}^S | \langle \phi_{p\alpha_1}^S | + \bar{C}_{S,I}^{(1)}, \quad (46)$$

where  $\hat{\psi}_{1,q}^S$  denotes the normalized Sturm-Liouville function without any node for the singlet interaction with energy  $-|E_q|$ , and  $\lambda_q$  represents the corresponding eigenvalue. Since we have presented the Sturm-Liouville function as well as the operator  $\bar{C}_{S,I}^{(1)}$  in detail in Refs. 23 and 26, we do not repeat it here. As in Eq. (16), the nonseparable term in  $G_{1,q}^S t_{1,q}^S$  of Eq. (45) is combined with  $\bar{C}_{S,I}^{(1)}$  and treated as the perturbation for the singlet states.

Finally, as an extension of Eqs. (41) and (42), we have a set of coupled integral equations for three kind of amplitudes,  $\langle f | U^I | F \rangle$ ,  $A_{q\alpha}^T$ , and  $A_{q\alpha}^S$ . Since the extension of Eqs. (41) and (42) to this set of coupled equations is obvious, they are not written here explicitly.

### III. CONCLUSION

We formulate the three-body theory of reactions with local potentials in coordinate space. In our theory, the calculations are performed according to the following steps. (1) First, we calculate the two-body equations (34) and (35). (2) We put the results in the matrix elements of Eqs. (41) and (42) and calculate the perturbation series  $(1-CQ)^{-1}$  by the method of acceleration. (3) Then, we calculate the matrix elements in Eqs. (41) and (42). (4) Finally,

we solve the set of coupled integral equations (41) and (42) to obtain the amplitudes. (5) To obtain the wave function, we use these results for Eqs. (37) and (38).

As explained in the Introduction and throughout Sec. II, emphasis in our formulation is put on the following points:

(1) The formulation is performed in a manner suitable for comparison between the effects of the elastic, breakup, and closed channels.

(2) In the zeroth order approximation, only contributions from physically important poles are included.

(3) The effect of the pole in the final state interaction is included explicitly by the factor  $1/D_q$  in Eq. (17).

(4) All physically important resonances are included as *one* separable term regardless of their number.<sup>12</sup>

(5) The effects other than those due to poles are treated as the perturbation  $[(1-CQ)^{-1}$  in Eq. (41)].

(6) Improvements of the numerical results are achieved by performing the perturbation iteration up to any desired higher order term. We have a method of doing it in a very efficient way.<sup>14</sup>

(7) All operators involved in perturbation iterations are confined in a finite region of space. The functions that appear in the course of iteration are regular at the origin.<sup>12</sup> The contribution from the asymptotic region of the breakup process is contained only in Eq. (24). This part is calculated analytically. These properties of our theory make numerical calculations feasible. Also, we expect that these properties will make the rigorous mathematical treatment of the three-body scattering theory much easier than the conventional ways.

(8) In the whole course of calculations, we need not take any contour in the complex momentum plane. All of our calculations are performed on the real momentum axis.

(9) The problem is finally reduced to a set of coupled linear integral equations for amplitudes [Eqs. (41) and (42)].

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