Chew-Low model and the potential description of the πN interaction

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The inverse scattering problem for the Chew-Low model is solved and the solution is used to construct three different forms for the off-shell πN T matrix. The three forms differ in their treatment of the nucleon pole and the crossing cut. One of the forms is shown to be equivalent to a separable potential model with an energy dependent strength. The analysis gives some insight into the question of the range of the πN interaction.

[NUCLEAR REACTIONS Inverse problem for Chew-Low model; relation between field theory and potential descriptions of πN interaction.]

I. INTRODUCTION

Many approaches for determining the pion-nucleon T matrix have been developed over the years. Reference 1 gives a brief outline of the various approaches, as well as extensive references to the literature. A fairly recent, and very thorough, discussion of the pion-nucleon interaction is given in the review article of Thomas and Landau.[2] Relevant articles that have appeared even more recently are given in Refs. 3–11.

The purpose of the present work is not to develop a realistic model of the pion-nucleon interaction, but rather to illuminate the connection between the field theory for this interaction and the potential description. For simplicity the field theory that we shall consider is the well-known Chew-Low[12,13] model. The inverse scattering problem for this model will be solved and the solution will be used to construct three possible off-shell extensions of the on-shell scattering amplitude. The three possibilities differ in the treatment of the nucleon pole and the left hand or crossing cut. Each off-shell T matrix is written in the standard N/D form. The first case is the original Chew-Low[12,13] T matrix in which the D function carries the nucleon pole, the unitarity cut, and the crossing cut. The N function is essentially determined by the cutoff function of the field theory.[12,13] For the second possibility the crossing cut is transferred to the N function. This form is appropriate to the P_{11} channel, since the nucleon pole remains as a zero in the D function.[2] In the third form the nucleon pole and the crossing cut appear in the N function, while the D function carries only the unitarity cut. It will be shown that this off-shell extension of the T matrix can be derived from a separable potential of the type introduced by Londergan *et al.*,[14] in which inelasticity effects are accounted for by an energy dependent potential strength. The inversion formulas for this strength and for the potential form factors are of the same structure as they have obtained.

II. THE INVERSE PROBLEM AND THE T MATRIX

We write the on-shell T matrix in the form

$$T_\alpha(p,p;\omega_p+i\epsilon)=-\frac{p^2v^2(p)}{\pi\omega_p}h_\alpha(\omega_p+i\epsilon)\ , \quad (1)$$

where p is the magnitude of the pion's three momentum and ω_p is its energy given by

$$\omega_p=(p^2+\mu^2)^{1/2}\ , \quad (2)$$

with μ the pion mass. The function $v(p)$ is a cutoff function normalized to one at $p=0$. The index α labels the four P-wave channels distinguished by the total isospin T and total angular momentum J according to

$$\alpha=1,2,3,4;\ \ 2T,2J=11,13,31,33\ . \quad (3)$$

The function $h_\alpha(z)$ is a real, analytic function of the complex variable z and has the representation[13,15]

$$h_\alpha(z)=\frac{\lambda_\alpha}{z}+\frac{1}{\pi}\int_\mu^\infty d\omega_p p^3v^2(p)\left[\frac{1}{\eta_\alpha(p)}\frac{|h_\alpha(\omega_p+i\epsilon)|^2}{\omega_p-z}+\sum_\beta A_{\alpha\beta}\frac{1}{\eta_\beta(p)}\frac{|h_\beta(\omega_p+i\epsilon)|^2}{\omega_p+z}\right]\ , \quad (4)$$

where

$$\lambda_\alpha = \frac{2}{3}\left[\frac{f}{\mu}\right]^2 (-4,-1,-1,2) \text{ for } \alpha=(1,2,3,4) , \tag{5}$$

with f the renormalized coupling constant. The parameter $\eta_\alpha(p)$ is the ratio of the elastic to the total cross section in the channel α. The matrix A is given by

$$A = \frac{1}{9}\begin{bmatrix} 1 & -4 & -4 & 16 \\ -2 & -1 & 8 & 4 \\ -2 & 8 & -1 & 4 \\ 4 & 2 & 2 & 1 \end{bmatrix} , \tag{6}$$

and appears in the crossing relation

$$h_\alpha(z) = \sum_{\beta=1}^{4} A_{\alpha\beta} h_\beta(-z) . \tag{7}$$

We see from (4) that $h_\alpha(z)$ has a simple pole at $z=0$ with residue λ_α, a right hand cut (RHC) beginning at $z=\mu$, and a left hand cut (LHC) beginning at $z=-\mu$.

We introduce in the usual way the denominator function $g_\alpha(z)$ defined by

$$g_\alpha(z) = \frac{\lambda_\alpha}{z h_\alpha(z)} . \tag{8}$$

Assuming $h_\alpha(z)$ has no zeros, we see that $g_\alpha(z)$ is a real, analytic function of z whose only singularities are a RHC and a LHC beginning at $z=\mu$ and $z=-\mu$, respectively. Furthermore,

$$g_\alpha(0) = 1 , \tag{9}$$

and $g_\alpha(\infty)$ is finite. From (4) and (8), it follows that

$$\operatorname{Im} g_\alpha(\omega_p + i\epsilon) = -\frac{\lambda_\alpha}{\omega_p}\frac{p^3 v^2(p)}{\eta_\alpha(p)}, \quad \omega_p \ge \mu . \tag{10}$$

We can write

$$g_\alpha(\omega \pm i\epsilon) = |g_\alpha(\omega + i\epsilon)| e^{\mp i\delta_\alpha(\omega)}, \quad \omega \ge \mu ,$$
$$g_\alpha(-\omega \mp i\epsilon) = |g_\alpha(-\omega - i\epsilon)| e^{\mp i\Delta_\alpha(\omega)}, \quad \omega \ge \mu , \tag{11}$$

where the fact that the phases have opposite signs above and below the cuts follows from the real, analytic nature of $g_\alpha(z)$. From (1), (8), (10), and (11), it follows that

$$T_\alpha(p,p;\omega_p + i\epsilon) = -\frac{\eta_\alpha(p)}{\pi p \omega_p} e^{i\delta_\alpha(\omega_p)} \sin\delta_\alpha(\omega_p) . \tag{12}$$

This form for the on-shell T matrix has been used previously in the inverse scattering problem[14,15] and is convenient for parametrizing data.[16] Below the inelastic threshold η_α is equal to one and δ_α becomes the usual phase shift. According to (11), (8), (7), (1), and (12), we can determine the other phase Δ_α from

$$e^{2i\Delta_\alpha(\omega_p)} = \frac{\sum_\beta A_{\alpha\beta}\eta_\beta(p) e^{i\delta_\beta(\omega_p)} \sin\delta_\beta(\omega_p)}{\sum_\gamma A_{\alpha\gamma}\eta_\gamma(p) e^{-i\delta_\gamma(\omega_p)} \sin\delta_\gamma(\omega_p)} . \tag{13}$$

By writing a dispersion relation for $\ln[g_\alpha(z)/g_\alpha(\infty)]$ and using (9) to eliminate $g_\alpha(\infty)$, it is straightforward to show that

$$g_\alpha(z) = \exp\left\{-\frac{z}{\pi}\int_\mu^\infty \frac{d\omega}{\omega}\left[\frac{\delta_\alpha(\omega)}{\omega - z} - \frac{\Delta_\alpha(\omega)}{\omega + z}\right]\right\} . \tag{14}$$

The cutoff function $v(p)$ can be obtained from (10). According to (11) and (5), we must have $\delta_\alpha < 0$ for $\alpha=1$, 2, 3, and $\delta_4 > 0$, in order that $v(p)$ be real. This form for the solution of the inverse problem is similar to that obtained previously[10] for a separable potential model of the π-N interaction. It differs from the solution obtained by Ernst and Johnson[15] in that they have to solve nonlinear equations in order to account for the crossing relation given by (7).

We now consider three possible off-shell extensions of the T matrix. The original analysis of Chew and Low[12,13] suggests the form

$$T_\alpha^{(1)}(p,q;z) = -\frac{1}{\pi}\frac{p v(p)}{\omega_p^{1/2}}\frac{\lambda_\alpha}{z g_\alpha(z)}\frac{q v(q)}{\omega_q^{1/2}} . \tag{15}$$

Here the nucleon pole, the unitarity cut (RHC), and the crossing cut (LHC) all appear in the variable z. This separation of the variables can lead to a violation of unitarity in the treatment of pion scattering from a system of more than one nucleon.[2,17]

Using (14), we see that another possibility is

$$T_\alpha^{(2)}(p,q;z) = -\frac{1}{\pi}\frac{p w_\alpha(p)}{\omega_p^{1/2}}\frac{\lambda_\alpha}{z G_\alpha(z)}\frac{q w_\alpha(q)}{\omega_q^{1/2}} , \tag{16}$$

where

$$G_\alpha(z) = \exp\left[-\frac{z}{\pi}\int_\mu^\infty \frac{d\omega}{\omega}\frac{\delta_\alpha(\omega)}{\omega - z}\right] \tag{17}$$

and

$$w_\alpha(p) = v(p)\exp\left[-\frac{\omega_p}{2\pi}\int_\mu^\infty \frac{d\omega}{\omega}\frac{\Delta_\alpha(\omega)}{\omega + \omega_p}\right] . \tag{18}$$

In this form the nucleon pole and the unitarity cut are in the variable z, while the crossing cut has been transferred to the variables p and q. This type of off-shell extension was found[18] to arise naturally in the derivation of three-particle equations from a crossing-symmetric extension of the Lee model. In three particle models of the π-N system this off-shell form is appropriate for the P_{11} channel.[19,20]

From (10), (14), and (17) it follows that

$$\text{Im}G_\alpha(\omega_p+i\epsilon)=-\frac{\lambda_\alpha}{\omega_p}\frac{p^3 w_\alpha{}^2(p)}{\eta_\alpha(p)},\ \ \omega_p\geq\mu\ . \tag{19}$$

According to this relation and (17), the form factors $w_\alpha(p)$ can be determined directly from a knowledge of δ_α and η_α. There is no need to deal with the crossing relation (7). Since $G_\alpha(z)$ has only a RHC and $G_\alpha(0)=1$, we can write

$$G_\alpha(z)=1-\frac{z}{\pi}\lambda_\alpha\int_\mu^\infty\frac{d\omega_p}{\omega_p{}^2}\frac{p^3 w_\alpha{}^2(p)}{\eta_\alpha(p)}\frac{1}{\omega_p-z}\ . \tag{20}$$

Thus $T_\alpha^{(2)}$ looks like a Chew-Low T matrix[12,13] with neglect of the crossing cut in the denominator function. Here crossing manifests itself by making the form factors $w_\alpha(p)$ channel dependent.

A third possibility for the off-shell T matrix is

$$T_\alpha^{(3)}(p,q;z)=-u_\alpha(p)\frac{\lambda_\alpha}{\pi G_\alpha(z)}u_\alpha(q)\ , \tag{21}$$

where

$$u_\alpha(p)=pw_\alpha(p)/\omega_p\ . \tag{22}$$

Here the nucleon pole has been transferred to the variables p and q. Only the unitarity cut resides in the variable z. From (20) and (22) we find

$$G_\alpha(\infty)=1+\frac{\lambda_\alpha}{\pi}\int_\mu^\infty d\omega_p\frac{pu_\alpha{}^2(p)}{\eta_\alpha(p)}>0\ , \tag{23}$$

where the fact that this is positive follows from representation (17). By solving for the one in (23) and putting it into (20), it is easy to show that

$$T_\alpha^{(3)}(p,q;z)=u_\alpha(p)\frac{\zeta_\alpha}{D_\alpha(z)}u_\alpha(q)\ , \tag{24}$$

where

$$D_\alpha(z)=G_\alpha(z)/G_\alpha(\infty)=1-\zeta_\alpha\int_0^\infty\frac{dp\,p^2}{\eta_\alpha(p)}\frac{u_\alpha{}^2(p)}{z-\omega_p}\ , \tag{25}$$

and

$$\zeta_\alpha=-\frac{\lambda_\alpha}{\pi G_\alpha(\infty)}\ . \tag{26}$$

This T matrix can be obtained as the solution of the Lippmann-Schwinger equation

$$T_\alpha^{(3)}(p,q;z)=V_\alpha(p,q;z)+\int_0^\infty V_\alpha(p,k;z)\frac{k^2dk}{z-\omega_k}T_\alpha^{(3)}(k,q;z)\ , \tag{27}$$

where the energy-dependent potential is given by

$$V_\alpha(p,q;z)=u_\alpha(p)\zeta_\alpha\gamma_\alpha(z)u_\alpha(q)\ , \tag{28}$$

with

$$\frac{1}{\gamma_\alpha(z)}=1+\zeta_\alpha\int_{2\mu}^\infty\frac{d\omega_p\omega_p p u_\alpha{}^2(p)}{\omega_p-z}\left[\frac{1}{\eta_\alpha(p)}-1\right]\ . \tag{29}$$

From (25) and (17) it follows that

$$\text{Im}D_\alpha(\omega_p+i\epsilon)=\pi p\omega_p\frac{\zeta_\alpha u_\alpha{}^2(p)}{\eta_\alpha(p)} \tag{30}$$

and

$$D_\alpha(z)=\exp\left[-\frac{1}{\pi}\int_\mu^\infty d\omega\frac{\delta_\alpha(\omega)}{\omega-z}\right]\ . \tag{31}$$

We see that this off-shell extension of the T matrix is essentially the same as the one arising from the separable potential model of Londergan *et al.*[14] Thus the present work provides a partial justification for the application of their approach to the π-N system.

In conclusion we note that the three form factors considered, $v(p)$, $w_\alpha(p)$, and $u_\alpha(p)$, have quite different singularity structures. In addition to whatever singularities $v(p)$ might have, we see from (18) that $w_\alpha(p)$ also has a LHC in the variable ω_p beginning at $\omega_p=-\mu$. Besides this cut, the potential model form factor given by (22) also has a pole at $\omega_p=0$. For a reasonable cutoff function[15] $v(p)$, $v(p)$ should lead to the shortest range in configuration space, while $u_\alpha(p)$ should produce the longest range. The question of the range of the πN interaction, and the related analytic structure of the T matrix, has been discussed by several authors.[2,9,15,17,21] It is hoped that the analysis given here will clarify the issue.

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