Bound states in the pion-nucleus velocity-dependent potential

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The possibility of bound states in a velocity-dependent potential of the Kisslinger type, such as that used to describe the low-energy pion-nucleus interaction, is discussed. It is shown in a specific example that, for a real potential, the number of bound states is finite and their binding energies are real, in contradiction with general results claimed by other authors.

[NUCLEAR REACTIONS Pion-nucleus optical potential; nuclear bound states of pions in nuclei; velocity-dependent potential.]

The interaction of pions with nuclei at low energies is very commonly described by means of a Schrödinger equation with an optical potential of the Kisslinger type[1]

$$2\mu\mathscr{V}(\vec{r})=\vec{\nabla}\cdot\alpha(\vec{r})\vec{\nabla}+\mathscr{Q}(\vec{r}), \tag{1}$$

where $\alpha(\vec{r})$ and $\mathscr{Q}(\vec{r})$ are complex functions closely related to the nuclear density and μ represents the reduced mass of the pion-nucleus system. Four years ago, Ericson and Myhrer[2] (EM) raised the question of the possible existence of pion-nuclear strongly bound states. By using an oversimplified model for $\alpha(\vec{r})$ and $\mathscr{Q}(\vec{r})$, namely constant values for a particle confined in a spherical box, they showed that an infinity of bound states, of indefinitely high binding energy, does exist. By qualitative considerations, they concluded that the same result should be obtained for any real function $\alpha(\vec{r})$, provided $\alpha(\vec{r})>1$ in some region. Moreover, in the case of $\alpha(\vec{r})$ varying with $\vec{r}$ in such a manner that $\alpha(\vec{r})$ can be considered linear in the neighborhood of the critical point $\vec{r}_0$ where $\alpha(\vec{r}_0)=1$, they found that the wave function presents a logarithmic singularity at $\vec{r}_0$ and the binding energies have a nonvanishing imaginary part, even in the limit of a purely real potential. These features of velocity-dependent potentials have been discussed by Ericson[3,4] in connection with several π-nuclear effects.

The possibility of observing those bound states, due to the velocity-dependent part of the π-nucleus optical potential, has been considered by Friedman, Gal, and Mandelzweig.[5] The same authors[6] have analyzed the anomalies of velocity-dependent potentials and suggested procedures to avoid their influence in the π-nucleus system.

Velocity-dependent potentials of the form

$$2\mu\mathscr{V}(\vec{r})=\vec{\nabla}\cdot\alpha(\vec{r})\vec{\nabla} \tag{2}$$

present peculiarities that are far from being understood, in spite of the attention they had deserved in the past[7-9] and recently.[6] In particular, the results reported by EM establish the puzzling problem of a Hermitian (Hamiltonian) operator having nonreal eigenvalues. These eigenvalues (under the denomination of non-Hermitian solutions) have been investigated by Mandelzweig, Gal, and Friedman[6] (MGF) by considering two simple models for $\alpha(\vec{r})$ which make the Schrödinger equation analytically solvable. The first model corresponds to a spherically symmetric step shape

$$\alpha(\vec{r})=\alpha_0\theta(R-r),$$

with $\mathrm{Re}\alpha_0>1$. In this case, the infinity of bound states can be shown explicitly, their binding energies being given by the zeros of a certain combination of Bessel functions, but the imaginary part of these energies vanishes as $\mathrm{Im}\alpha_0$ tends to zero. The case of purely real α_0 does not present anomalies[9] either in the wave function or in the eigenvalues. Non-Hermitian solutions do not occur, but this fact does not invalidate the results of EM because in this model there is no critical point $\vec{r}_0$ where $\alpha(\vec{r}_0)=1$. The second model makes use of a spherically symmetric parabolic shape,

$$\alpha(\vec{r})=1-(r^2-r_0^2)/R^2,$$

for which the solution of the Schrödinger equation can be obtained analytically in terms of Legendre functions. For real $\alpha(\vec{r})$, an infinity of bound states

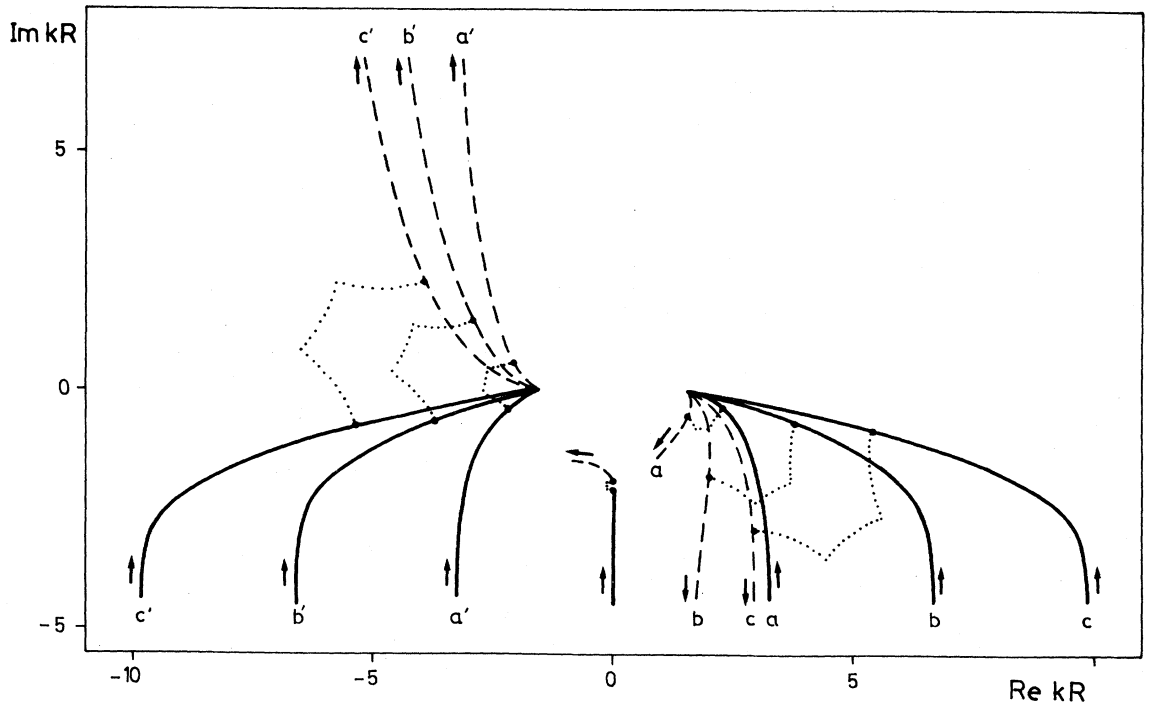

FIG. 1. S-wave poles of the S matrix for a velocity-dependent potential of finite-range parabolic shape. The continuous lines indicate the trajectories of the poles as the intensity A of the potential (see the text) goes from 0 to 1. The dashed lines give the limit position, as $\mathrm{Im}A \to 0^+$, of the trajectories followed by the poles for complex A, with $\mathrm{Re}A$ increasing from 1. The dotted lines indicate the motion of the poles when A takes the values shown (with a dotted line also) in Fig. 2. The arrows indicate the motion of the poles as $\mathrm{Re}A$ increases.

with real binding energies appears, but a non-Hermitian solution does not appear, in contrast with the predictions of EM. Furthermore, if one considers complex $\alpha(\vec{r})$ by allowing r_0 to become complex or by introducing a complex global factor $A_R + iA_I$, $A_R > 1$, all bound states suddenly disappear.

In order to disentangle this confused situation, we consider in this paper a third model for $\alpha(\vec{r})$, namely a finite-range spherically symmetric parabolic shape,

$$\alpha(\vec{r}) = A(1 - r^2/R^2)\theta(R - r). \tag{3}$$

This model presents, with respect to the first model of MGF, the advantage of showing at

$$r_0 = R(1 - A^{-1})^{1/2}$$

the criticality condition $\alpha(r_0) = 1$ discussed by EM. Also, the finite range of our potential makes it much more realistic than the second model of MGF.

The Schrödinger equation with the potential of Eq. (2) with its shape given by Eq. (3) can be analytically solved in the S-wave case. For a particle of energy E and wave number $k = (2\mu E)^{1/2}$, the radial wave function inside the potential well can be written

$$\psi(r) = F(a,b;c;z),$$
$$z = Ar^2/(A-1)R^2, \quad r < R, \tag{4}$$

where F represents the hypergeometric function[10] of parameters a,b,c, such that $a + b = \frac{3}{2}$, $ab = (kR)^2/4A$, $c = \frac{3}{2}$, and outside the potential well

$$\psi(r) \propto r^{-1}\exp(ikr), \quad r > R. \tag{5}$$

The eigenvalues E of the Hamiltonian are obtained by matching the inside and the outside wave functions, Eqs. (4) and (5), according to the continuity condition[7] of

$$[1 - \alpha(r)](d\psi/dr)/\psi(r).$$

At the edge, $r = R$, of the potential well that condition reads

$$\left[2z\frac{dF(a,b;c;z)/dz}{F(a,b;c;z)}\right]_{z=A/(A-1)} + 1 - ikR = 0. \tag{6}$$

As is well known, the relation also gives the poles, in the complex k plane, of the $l = 0$ component of the S

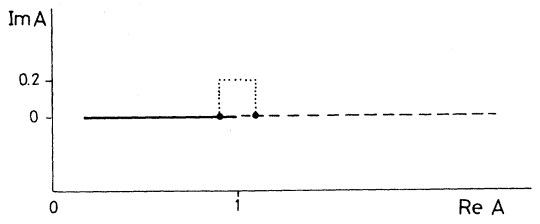

FIG. 2. Values of the intensity of the velocity-dependent potential whose S matrix poles have been shown in Fig. 1.

matrix for the potential under consideration.

The hypergeometric function and its derivative are one-valued analytic functions in the z plane cut along the real axis from 1 to ∞. Therefore, the solution of Eq. (6) does not present any difficulty in the case of complex A. Our interest, however, lies in analyzing the anomalies of real velocity-dependent potentials reported by EM. For this reason we concentrate on real values of A.

Negative values of A are of no interest, as they correspond to repulsive potentials which do not represent the peculiarities we are investigating. For $0<A<1$, Eq. (6) has an infinity of complex solutions for k, some of which have been represented in Fig. 1. As far as they occur in the lower half-plane, all of them correspond to unnormalizable states. They are located symmetrically with respect to the imaginary axis, one of the solutions being precisely on the imaginary axis. For very small A they lie in the infinity of the lower half-plane. As A increases towards 1, they approach the real axis. Obviously, solutions near and below the real axis can be interpreted as resonances. So, our potential produces an infinity of S-wave resonances, the narrower the nearer A is to 1. As A reaches the value 1, all solutions in the right and left half-planes concentrate at the points $kR=\frac{3}{2}$ and $kR=-\frac{3}{2}$, respectively. The solution on the imaginary axis is at $kR=-2i$

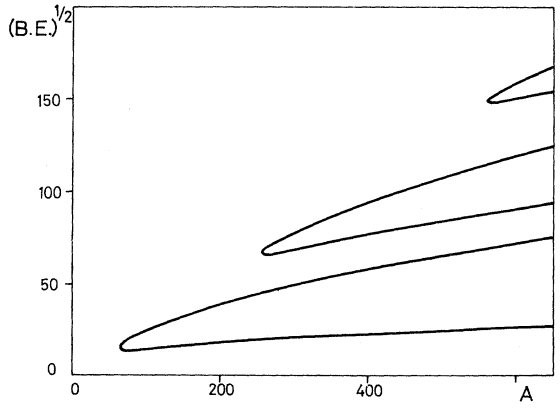

FIG. 3. Binding energy of bound states in the velocity-dependent potential of finite-range parabolic shape as a function of the intensity A of the potential. On the vertical axis the square root of the binding energy, in units of $(2\mu)^{-1/2}R^{-1}$ (μ is the mass of the particle and R is the range of the potential), has been represented.

Values of A larger than 1 require a careful treatment. The first term in Eq. (6) is no longer uniquely defined, as the value of z lies on the cut of the hypergeometric function. This corresponds to the fact that we are trying to match the internal and external wave functions at a point (the edge of the potential) on the cut of the internal wave function introduced by its logarithmic branch point at r_0. So, a suitable prescription for determining the internal wave function is needed.

One possibility is to modify A by adding to it a small positive imaginary part, as shown in Fig. 2, then evaluate the first term of Eq. (6), and finally adopt its limit value as $\mathrm{Im}A\to 0$. This corresponds to selecting for the internal wave function its value just above the cut. By doing that one obtains the solutions shown in Fig. 1. Those in the upper half-plane could be interpreted as decaying bound states, and that infinity of bound states of complex binding energies even in the limit $\mathrm{Im}A\to 0$ seems to confirm the predictions of EM. However, this manner of circumventing the problem is quite artificial. Had we added to A a small negative imaginary part, we would have obtained, by the same procedure, the complex conjugate values for the binding energies. Instead of *decaying* bound states, we would have *growing* ones.

There exists nevertheless another more natural possibility of solving the problem. It consists in taking for the wave function the mean of its values just above and below the cut. This is the usual manner of defining special functions like, for instance, Legendre functions on their cuts. It has been adopted by MGF in the treatment of their second model, in order to ensure the reality of the wave function, and by King and Rohrlich[11] in the solution of the Schrödinger equation that arises in the relativistic two-body problem with momentum-dependent interactions. If the wave function is defined in this natural way, no bound states appear in our model unless the intensity parameter A becomes larger than a threshold value $A_2\simeq 62.4$. At this value, Eq. (6) has a double solution for pure imaginary k. As A increases, the two solutions separate and move along the imaginary axis as shown in Fig. 3. So two bound states occur, their binding energies being obviously real. A new double solution of Eq. (6) occurs if A increases further and reaches the value $A_4\simeq 255.7$.

For A above this value, four bound states are found. If A becomes larger than $A_6 \simeq 559$, six bound states occur. And so on.

These results, namely a finite number of bound states of purely real binding energies for real $\alpha(\vec{r})$, manifestly contradict the predictions of EM. Moreover, it turns out from our precedent analysis that a drastic quantitative change in the spectrum of the Hamiltonian occurs when the real $\alpha(\vec{r})$ acquires an infinitesimal imaginary part.

It is not difficult to understand why the results of EM, valid for a nonvanishing (even infinitesimal) imaginary part of $\alpha(\vec{r})$, are not correct in the limit of real $\alpha(\vec{r})$. They obtain a nonvanishing absorption (complex expectation values of the Hamiltonian) even in the limit of a purely real potential. This conclusion is reached starting from their Eq. (5). This equation is obtained from their Eq. (1) by means of a partial integration which is not justified in the limit $\mathrm{Im}\alpha \to 0$, because the wave function presents a branch point in the integration path. In other words, the differential operator $\vec{\nabla}$ cannot be transferred, in the calculation of the expectation value

$$\langle \psi \mid \vec{\nabla} \cdot (1-\alpha(r))\vec{\nabla} \mid \psi \rangle,$$

to the bra $\langle \psi \mid$, as it does not belong to the domain of $\vec{\nabla}$.

In the solution of the Schrödinger equation with a conventional potential, the question of the domain of definition of the Hamiltonian does not play a relevant role and is usually ignored. For velocity-dependent potentials, however, it becomes crucial, as can be seen in our precedent example. As is well known,[12] the Schrödinger Hamiltonian with a real conventional potential is a symmetric operator in its natural domain of definition, $D(H)=C_0^\infty(R^3)$ (infinitely differentiable functions with compact support), but it is not self-adjoint because the domain of definition of its adjoint is larger. It can admit, nevertheless, only one or a family of self-adjoint extensions (according to the common value of its two deficiency indices) and one of them is to be selected by imposing the boundary conditions. In the case of a velocity-dependent potential as given in Eq. (3), the Hamiltonian is also symmetric in $C_0^\infty(R^3)$ for real A and admits self-adjoint extensions. However, solving the Schrödinger equation with complex A and then making $\mathrm{Im}A \to 0$ gives the spectrum of an extension of H but, as it is obvious from the fact that the eigenvalues are not real, not of a self-adjoint one. This explains the great differences between the spectra obtained by making $\mathrm{Im}A \to 0^+$ or $\mathrm{Im}A \to 0^-$ or taking for the wave function on the cut the mean of its values above and below it. This last procedure guarantees the reality of the eigenvalues and allows us to ensure that we are dealing with a self-adjoint extension of the Hamiltonian.

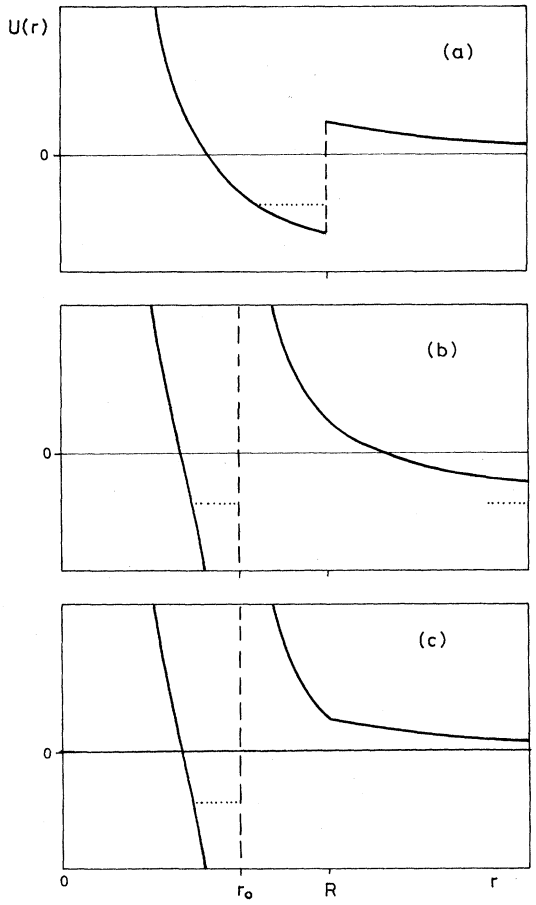

FIG. 4. Energy-dependent potentials "equivalent" to the velocity-dependent ones mentioned in the text of shapes:

(a) $\alpha(r)=1.8\theta(R-r)$,

(b) $\alpha(r)=1.8(1-r^2/R^2)$,

(c) $\alpha(r)=1.8(1-r^2/R^2)\theta(R-r)$.

The value of the energy to which they correspond is indicated by a horizontal dotted line.

In order to understand why the spectra of the Hamiltonians considered by MGF in their two examples are so different from that obtained in the model we are considering, it is interesting to perform a transformation that was proven[7] to be useful to relate a velocity-dependent potential to an "equivalent" energy-dependent one, in the S-wave case. The transformation consists in replacing the radial wave function $\psi(r)$ by another related function

$$\chi(r)=[1-\alpha(r)]d\psi(r)/dr,$$

which has essentially the same asymptotic behavior as $\psi(r)$ and gives, therefore, the same information as $\psi(r)$ about bound states and phase shifts. That function obeys the Schrödinger-type equation

$$d^2\chi(r)/dr^2+(2/r)d\chi(r)/dr+[k^2-U(r)]\chi(r)=0,$$

with the energy-dependent potential

$$U(r)=2/r^2-k^2\alpha(r)/[1-\alpha(r)].$$

We have represented in Fig. 4 the three potentials $U(r)$ equivalent, respectively, to the velocity-dependent ones considered by MGF and by us. In the first case it can be seen that, if the binding energy increases, the intensity of the (attractive) potential increases even more, it being possible in this manner to have an infinity of bound states. In the second case, increasing the binding energy results in a more attractive potential for $0<r<r_0$, more repulsive in the range $r_0<r<R$ and more attractive for $r>R$. The repulsive part is overcompensated by the attractive one and an infinity of bound states results also in this case. For the third case, however, the equivalent potential does not change with the binding energy for $r>R$, where it remains repulsive. Bound states of indefinitely high energy cannot be found for a given intensity A of the velocity-dependent potential.

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