Nuclear beta decay induced by intense electromagnetic fields: Forbidden transition examples

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A formalism developed earlier for the effect on nuclear beta decay of an intense plane-wave electromagnetic field is applied to three examples of forbidden beta transitions. The examples represent cases where the nuclear "fragment" contains one, two, and three nucleons; where the nuclear fragment is defined to be that smallest sub-unit of the nucleus containing the nucleon which undergoes beta decay plus any other nucleons directly angular-momentum coupled to it in initial or final states. The single-nucleon-fragment example is ^{113}Cd, which has a fourth-forbidden transition. The two-nucleon-fragment example is ^{90}Sr, which is first-forbidden. The three-nucleon-fragment example is ^{87}Rb, which is third-forbidden. An algebraic closed-form transition probability is found in each case. At low external-field intensity, the transition probability is proportional to z^L, where z is the field intensity parameter and L is the degree of forbiddenness. At intermediate intensities, the transition probability behaves as $z^{L-(1/2)}$. At higher intensities, the transition probability contains the $z^{L-(1/2)}$ factor, a declining exponential factor, and an alternating polynomial in z. This high-intensity transition probability possesses a maximum value, which is found for each of the examples. A general rule, $z=q^2(2L-1)$, where q is the number of particles in the fragment, is found for giving an upper limit on the intensity at which the maximum transition probability occurs. Field-induced beta decay half-lives for all the examples are dramatically reduced from natural half-lives when evaluated at the optimum field intensity. Relative half-life reduction is greater the higher the degree of forbiddenness.

[RADIOACTIVITY ^{87}Rb, ^{90}Sr, ^{113}Cd intense-field-induced β decay minimum half-lives, half-life reduction forbiddenness dependence.]

I. INTRODUCTION

Reference 1 (hereafter designated as I) gave the basic theory of the interaction of intense plane-wave electromagnetic fields with nuclear beta decay. When the applied electromagnetic field is very intense, processes of high order in that field can occur with substantial probability. Thus it is possible for the field to intercede in the beta decay in terms, for example, of modifying the conservation conditions associated with the beta decay. Of particular interest in the present paper is the case of field-induced forbidden beta decay. Each photon of the applied field carries with it one unit of angular momentum, and an intrinsic negative parity. If a field-free beta transition is a forbidden transition of order L, then the net participation of L photons from the applied field can remove that forbiddenness, independently of the energy of each photon. With sufficient field intensity, the penalty in transition probability paid for the extra field interaction can be far less than the penalty paid for forbiddenness. This tends to be the case for high orders of forbiddenness more than for low orders. However, there is a limit to how much transition probability enhancement can be achieved by increasing the field intensity.[2,3]

The theory presented in I exhibits field dependence in terms of two intensity parameters, one arising from interaction of the field with the nucleus, and the other stemming from the field interaction with the beta particle. The field-nucleus intensity parameter is

$$z=(eaR_0)^2\ , \tag{1}$$

where e is the proton charge, a is the amplitude of the electromagnetic plane-wave vector potential in Coulomb gauge, and R_0 is the nuclear radius. "Natural" units, with $\hbar=c=1$, are used. The field-electron intensity parameter is

$$z_f=\tfrac{1}{2}(ea/m)^2 , \quad (2)$$

where m is the electron mass. This is the intensity parameter which always appears in intense-field interactions with a free, charged particle.[4] It is related to the bound-state intensity parameter of Eq. (1) in that (apart from the factor $\frac{1}{2}$), one need only replace the bound-state radius R_0 in Eq. (1) by the electron Compton wavelength in order to obtain Eq. (2). Equations (1) and (2) are expressed in Coulomb gauge, and they appear to be gauge dependent. When expressed in terms of four-dimensional vector potentials, however, they are easily shown[5] to be both gauge invariant and Lorentz invariant. Another way to express the intensity parameter in a gauge-invariant way is to write it in terms of field quantities. For instance, Eq. (1) is

$$z=(e\,|\vec{E}\,|R_0/\omega)^2=(e\,|\vec{B}\,|R_0/\omega)^2 ,$$

when written in terms of the electric field amplitude $|\vec{E}|$ or the amplitude of the magnetic induction $|\vec{B}|$. The inverse squared dependence on field frequency ω in these expressions demonstrates the important fact that the intensity parameter (for a given energy density of the field) increases as the square of the wavelength. Large intensity parameters are easier to achieve at low frequencies. For example, if R_0 is taken to be 5×10^{-13} cm, and a field frequency of 1 MHz is selected, then a value of unity for the intensity parameter is associated with an electric field strength of 8×10^5 V/m and a magnetic induction of 3×10^{-3} T. These are reasonable values attainable in a transmission line with conventional radio-frequency (rf) field sources.

In Sec. II, the appropriate results from I are reproduced, and specialized to the particular conditions needed here. Specifically, the examples to be calculated are forbidden transitions whose field-enhanced transition probabilities are to be expressed in an intensity domain encompassing the field intensity for optimum transition probability. This means that results pertaining to the intense-field, low-frequency case are employed. Furthermore, all the examples given involve a change in isospin, so only the Gamow-Teller terms are retained.

The first example, ^{113}Cd, is treated in Sec. III. This example has a single-particle "fragment." Coupling of the field to the nucleus is determined by a separation of the nucleus into a "core" and a "fragment." The fragment is that smallest portion of the nucleus which contains the nucleon (or nucleons) which is a candidate for beta decay, plus any other nucleons angular-momentum coupled to it in initial or final states, as determined in the single-particle shell model. The core is the remnant nucleus of spin and parity 0^+. ^{113}Cd has a fourth-forbidden beta decay ($L=4$) under no-field conditions.

Section IV is devoted to ^{90}Sr, an example of a nucleus with a two-nucleon fragment. This entails extra complexity in the calculation beyond the single-nucleon fragment. ^{90}Sr is first-forbidden ($L=1$). ^{87}Rb is treated in Sec. V. This has a three-nucleon fragment, with an additional measure of complexity beyond the two-nucleon case. ^{87}Rb has $L=3$.

The final results for transition probability per unit time found in Secs. III—V are analyzed in Sec. VI in the case $z\ll1$, and are analyzed also for general algebraic behavior for arbitrary z. The $z\ll1$ case is not strictly a low-intensity limit, since the results used here require $z_f\gg1$ always. This is because the analytical results from I employ an asymptotic form for the generalized Bessel functions that occur there. Transition probability in the $z\ll1$ case behaves as $z^{L-(1/2)}$. For arbitrary z, the transition probability contains the small-z factor, times an exponential which decays with increasing z, times a polynomial in z. This implies the existence of a maximum in transition probability as a function of z. These maxima are found for the three examples treated. The existence of a maximum transition probability is a feature of intense-field theories.[2,3] It means that beyond that maximum, an *increase* in field intensity parameter causes a *decrease* in transition probability, a rather counter-intuitive result. A simple, general algebraic result is derived which gives an upper limit, as a function of degree of forbiddenness and size of the fragment, on the location of the maximum transition probability. The actual maxima found for the three examples are then translated into the corresponding minimum field-enhanced half-lives in Sec. VII.

II. BASIC THEORETICAL RESULTS

The results stated in Sec. IV of I are to be employed here. These results are derived from the most general form of the theory by considering the case where $z_f\gg1$. Since z_f and z are related by

$$z_f=\tfrac{1}{2}(mR_0)^{-2}z\approx(3\times10^3)z , \quad (3)$$

the value of z need not be large even when $z_f\gg1$.

The final result for transition probability per unit time when $z_f\gg1$ is given in Eqs. (125), (133), and (138) of I in the form

$$W_{\text{tot}}=\frac{G^2m^5}{2\pi^3}f_{\text{tot}}\,|M_{\text{ind}}|^2 , \quad (4)$$

where G is the beta-decay coupling constant, f_{tot} is the total spectral integral for field-induced beta decay, and M_{ind} is the field-dependent transition matrix element. The total spectral integral is the sum of three parts

$$f_{\text{tot}}=f_1+f_2+f_3 \tag{5}$$

arising from direct interactions of the beta particle with the field, from spin interactions with the field, and from an interference between these terms, respectively. The necessary expressions are given in Eqs. (127), (128), (134), (135), (139), and (140) of I. The squared transition matrix element normally contains both Fermi and Gamow-Teller terms. However, none of the examples treated here obey the isospin selection rule $\Delta T=0$ (Ref. 6) for Fermi transitions, so only Gamow-Teller terms are retained.

From I, the squared transition matrix element is

$$|M_{\text{ind}}|^2=\frac{\kappa^2}{4\pi(2z_f)^{1/2}}(|\vec{M}_{fi}^{\cos}|^2+|\vec{M}_{fi}^{\sin}|^2)\,, \tag{6}$$

where

$$\vec{M}_{fi}^{\cos}=\frac{1}{(2j_i+1)}\times\sum_{m_i}\sum_{m_f}\left[\psi_f,\cos\left[\frac{z^{1/2}}{q}u_k\cos\theta_k\right]\vec{\sigma}\psi_i\right], \tag{7}$$

$$\vec{M}_{fi}^{\sin}=\frac{1}{(2j_i+1)}\times\sum_{m_i}\sum_{m_f}\left[\psi_f,\sin\left[\frac{z^{1/2}}{q}u_k\cos\theta_k\right]\vec{\sigma}\psi_i\right]. \tag{8}$$

In Eq. (6), κ is the ratio of the strength of the axial vector coupling to the vector coupling in beta decay. It has the experimental value[7]

$$\kappa=1.23\pm0.01\ . \tag{9}$$

In Eqs. (7) and (8), j_i is the total angular momentum of the initial state, m_i and m_f are the polar-axis projections of the initial and final total angular momenta, q is the number of nucleons in the fragment, the index k specifies which of the q particles in the fragment undergoes beta decay, u_k is the dimensionless radial coordinate

$$u_k=r_k/R_0\ , \tag{10}$$

θ_k is the polar-angle coordinate of particle k, and $\vec{\sigma}$ is the vector Pauli spin operator. The factor $(2j_i+1)^{-1}$ times the sum over m_i is an average over orientations of the initial angular momentum, and the sum over m_f is a sum over orientations of the final angular momentum.

In practice, only one of the two terms in Eq. (6) will be nonzero. When ψ_f and ψ_i have the same parity, only $\vec{M}_{fi}^{\cos}$ contributes, and when they have opposite parity, only $\vec{M}_{fi}^{\sin}$ contributes.

One additional result needed below has to do with inner products in spinor space. The product

$$\chi_{1/2}^{\mu_s\dagger}\vec{\sigma}\chi_{1/2}^{m_s} \tag{11}$$

occurs, where the $\chi_{1/2}^{m_s}$ are two-component Pauli spinors. It will be convenient to deal with the components of $\vec{\sigma}$ in terms of raising and lowering operators rather than in terms of rectangular components. That is, the components

$$\begin{aligned}\sigma_+&=\sigma_x+i\sigma_y\ ,\\ \sigma_-&=\sigma_x-i\sigma_y\ ,\\ \sigma_0&=\sigma_z\ ,\end{aligned} \tag{12}$$

will be used. Since the components of $\vec{\sigma}$ determine the components of $\vec{M}_{fi}^{\cos}$ or $\vec{M}_{fi}^{\sin}$, this means that $(M_+)_{fi}$, $(M_-)_{fi}$, and $(M_0)_{fi}$ will be calculated, rather than rectangular components. Transformation of M_+, M_-, and M_0 back to rectangular components is accomplished by

$$\begin{aligned}M_x&=\tfrac{1}{2}(M_++M_-)\ ,\\ M_y&=\frac{1}{2i}(M_+-M_-)\ ,\\ M_z&=M_0\ .\end{aligned} \tag{13}$$

If the three components $+,-,0$ are identified by an index λ taking the values $\lambda=+1,-1,0$, then the inner product in spinor space stated in Eq. (11) is

$$\chi_{1/2}^{\mu_s\dagger}\sigma_\lambda\chi_{1/2}^{m_s}=\begin{cases}2\delta_{\mu_s,1/2}\delta_{m_s,-(1/2)}, & \lambda=1\\ 2\delta_{\mu_s,-(1/2)}\delta_{m_s,1/2}, & \lambda=-1\\ (-)^{1/2-m_s}\delta_{\mu_s,m_s}, & \lambda=0\ .\end{cases} \tag{14}$$

Equation (14) implies the constraint

$$\mu_s-m_s=\lambda\ . \tag{15}$$

III. ^{113}Cd EXAMPLE

The simplest possible single-particle shell model approach is used here to establish ground-state wave functions. By this approach, $^{113}_{48}Cd_{65}$ has a single-nucleon fragment. The core nucleus, $^{112}_{48}Cd_{64}$, is a stable nuclide with spin and parity of 0^+. The odd neutron in ^{113}Cd has a shell-model assignment of $s_{1/2}$, which determines the total nuclear spin and parity to be $(\frac{1}{2})^+$. Upon beta decay, the $s_{1/2}$ neutron becomes a $g_{9/2}$ proton, which contributes the observed $(\frac{9}{2})^+$ spin and parity of the final ${}^{113}_{49}\mathrm{In}_{64}$ nucleus.

The coordinates in which the wave functions are expressed are relative coordinates between the nuclear fragment and the nuclear core. The nuclear state in general is

$$\psi_{jm}=R_{nl}(r)\sum_{m_l m_s}(lsj\,|\,m_l m_s m)Y_l^{m_l}(\theta,\phi)\chi_s^{m_s}\,, \tag{16}$$

where $R_{nl}(r)$ is the radial wave function, the CG (Clebsch-Gordan) coefficient $(lsj\,|\,m_l m_s m)$ shows how the orbital angular momentum vector $\vec{l}$ (components m_l) and spin angular momentum vector $\vec{s}$ (components m_s) couple together to form the total angular momentum vector $\vec{j}$ (components m), $Y_l^{m_l}$ is a spherical harmonic, and $\chi_s^{m_s}$ is a spin wave function defined in a spinor space. For the initial state, Eq. (16) is

$$\psi_i=R_{20}(r)\sum_{m_s}(0\tfrac{1}{2}\tfrac{1}{2}\,|\,0m_s m_i)Y_0^0\chi_{1/2}^{m_s}$$
$$=R_{20}(r)Y_0^0\chi_{1/2}^{m_i}\,, \tag{17}$$

since the CG coefficient is simply $\delta_{m_s m_i}$. For the final state, Eq. (16) gives

$$\psi_f=R_{04}(r)\sum_{\mu_l\mu_s}(4\tfrac{1}{2}\tfrac{9}{2}\,|\,\mu_l\mu_s m_f)Y_4^{\mu_l}\chi_{1/2}^{\mu_s}\,, \tag{18}$$

where the CG coefficient gives the constraint

$$\mu_l=m_f-\mu_s\,.$$

For ^{113}Cd, only $\vec{M}_{fi}^{\cos}$ will contribute. Equation (7) with Eqs. (17) and (18) is

$$\vec{M}_{fi}^{\cos}=R_0{}^3\int u^2du\,R_{04}(u)R_{20}(u)\tfrac{1}{2}\sum_{m_i}\sum_{m_f}\sum_{\mu_s}(4\tfrac{1}{2}\tfrac{9}{2}\,|\,m_f-\mu_s\,\mu_s\,m_f)$$
$$\times\int d\Omega\,Y_4^{m_f-\mu_s^*}\cos(z^{1/2}u\cos\theta)Y_0^0\chi_{1/2}^{\mu_s\dagger}\vec{\sigma}\chi_{1/2}^{m_i}\,, \tag{19}$$

where the radial integral is written in terms of $u=r/R_0$, which is the form taken by Eq. (10) for a single-nucleon fragment. The solid-angle integral reduces to

$$\int d\Omega\,Y_4^{m_f-\mu_s^*}\cos(g\cos\theta)Y_0^0=\tfrac{3}{2}\delta_{m_f,\mu_s}\int_{-1}^{1}dx\,P_4(x)\cos(gx)\,, \tag{20}$$

where $P_l(x)$ is the Legendre polynomial, and the definition

$$g\equiv z^{1/2}u \tag{21}$$

has been introduced. Since

$$P_4(x)=\tfrac{1}{8}(35x^4-30x^2+3)\,,$$

then Eq. (20) can be evaluated explicitly as

$$\int d\Omega\,Y_4^{m_f-\mu_s^*}\cos(g\cos\theta)Y_0^0=48\delta_{m_f,\mu_s}g^4\sum_{k=0}^{\infty}\frac{(-)^k g^{2k}}{(2k+9)!}\frac{(k+4)!}{k!}\,. \tag{22}$$

Equation (19) is now

$$\vec{M}_{fi}^{\cos}=24z^2\sum_{m_i}\sum_{\mu_s}(4\tfrac{1}{2}\tfrac{9}{2}\,|\,0\mu_s\mu_s)R_0{}^3\int du\,u^6R_{04}(u)R_{20}(u)$$
$$\times\sum_{k=0}^{\infty}\frac{(-)^k z^k u^{2k}}{(2k+9)!}\frac{(k+4)!}{k!}\chi_{1/2}^{\mu_s\dagger}\vec{\sigma}\chi_{1/2}^{m_i}\,. \tag{23}$$

Evaluation of Eq. (23) will be done in terms of the components $(M_\lambda^{\cos})_{fi}$, where $\lambda=+1,-1,0$. The only two CG coefficients required for Eq. (23) are

$$(4\tfrac{1}{2}\tfrac{9}{2}\,|\,0\tfrac{1}{2}\tfrac{1}{2})=(4\tfrac{1}{2}\tfrac{9}{2}\,|\,0-\tfrac{1}{2}-\tfrac{1}{2})=\frac{\sqrt{5}}{3}\,. \tag{24}$$

The components of $\vec{M}_{fi}^{\cos}$ are, from Eqs. (14), (23), and (24):

$$(M_+^{\cos})_{fi}=(M_-^{\cos})_{fi}=16\sqrt{5}R_0{}^3z^2\sum_{k=0}^{\infty}\frac{(-)^k(k+4)!z^k}{(2k+9)!k!}\int_0^\infty du\, u^{2k+6}R_{04}(u)R_{20}(u)\,,$$
$$(M_0^{\cos})_{fi}=0\,. \tag{25}$$

The radial integral will now be evaluated. This requires the choice of a model potential for the nuclear force. However, the value obtained for the radial integral is not sensitive to the model selected, and so, as is commonly done, harmonic oscillator wave functions will be used because of their analytical tractability. These radial wave functions are real, and are given by

$$R_{nl}(u)=\left[\frac{2(n)!}{R_0{}^3\Gamma(n+l+\frac{3}{2})}\right]^{1/2}u^le^{-(1/2)u^2}L_n^{l+(1/2)}(u^2)\,, \tag{26}$$

where $L_n^{l+(1/2)}(u^2)$ is a Laguerre polynomial.[8] The particular Laguerre polynomials required for Eq. (25) are

$$L_0^{9/2}(u^2)=1,\;\; L_2^{1/2}(u^2)=\tfrac{1}{8}(4u^4-20u^2+15)\,.$$

The radial integral is

$$\int_0^\infty du\,u^{2k+6}R_{04}R_{20}=\frac{4}{45}\left[\frac{2}{7\pi}\right]^{1/2}\frac{1}{R_0{}^3}\int_0^\infty du\,e^{-u^2}u^{2k+10}(4u^4-20u^2+15)$$
$$=\left[\frac{2}{7}\right]^{1/2}\frac{1}{R_0{}^3}\frac{(2k+9)!!(k+4)(k+3)}{2^{k+2}45}\,,$$

so the matrix element components are

$$(M_+^{\cos})_{fi}=(M_-^{\cos})_{fi}=\left[\frac{5}{14}\right]^{1/2}\frac{z^2}{90}\sum_{k=0}^{\infty}\left[-\frac{z}{4}\right]^k\frac{(k+4)(k+3)}{k!}\,. \tag{27}$$

The sum over k in Eq. (27) can be accomplished in terms of exponentials to give the closed-form result

$$(M_+^{\cos})_{fi}=(M_-^{\cos})_{fi}=\frac{1}{3}\left[\frac{2}{5\cdot 7}\right]^{1/2}e^{-z/4}z^2\left[1-\frac{z}{6}+\frac{z^2}{192}\right]. \tag{28}$$

From Eq. (13), rectangular components of $\vec{M}_{fi}^{\cos}$ are

$$(M_x^{\cos})_{fi}=(M_+^{\cos})_{fi}\,,$$
$$(M_y^{\cos})_{fi}=0\,,$$
$$(M_z^{\cos})_{fi}=0\,.$$

The squared induced matrix element is, from Eqs. (6) and (28),

$$|M_{\rm ind}|^2=\frac{\kappa^2}{4\pi(2z_f)^{1/2}}\frac{2}{3^2\cdot 5\cdot 7}e^{-z/2}\times z^4\left[1-\frac{z}{8}\right]^2\left[1-\frac{z}{24}\right]^2. \tag{29}$$

This, when inserted into Eq. (4), gives the induced transition probability per unit time.

IV. ^{90}Sr EXAMPLE

^{90}Sr has a two-nucleon fragment ($q=2$) with $L=1$. The core nucleus of $^{90}_{38}\mathrm{Sr}_{52}$ is $^{88}_{38}\mathrm{Sr}_{50}$. The two $d_{5/2}$ neutrons in ^{90}Sr beyond the magic number of 50 constitute the fragment. They must be considered as a pair because initially they are angular-momentum coupled to 0^+, and it is impossible to say which of the two will decay. After decay, the remaining $d_{5/2}$ neutron will couple to the newly formed $p_{1/2}$ proton to give the 2^- state of the $^{90}_{39}\mathrm{Y}_{51}$ daughter nucleus.

A properly antisymmetrized initial state wave function is

$$\psi_i=\frac{1}{2^{1/2}}\sum_{m_1,m_2}(\tfrac{5}{2}\tfrac{5}{2}0\,|\,m_1m_20)[\psi_{5/2,m_1}(1)\psi_{5/2,m_2}(2)-\psi_{5/2,m_1}(2)\psi_{5/2,m_2}(1)]\,, \tag{30}$$

where each of the $\psi_{5/2,m}$ states is of the type given in Eq. (16). Quite arbitrarily, it will be supposed that the neutron labeled (1) will be the one to undergo beta decay, so the final state will have a proton labeled (1) as well as the "spectator" neutron carrying the label (2). The antisymmetrized wave function, Eq. (30), ensures that the contributions of both neutrons to the beta decay are considered. A basic property of the CG coefficients gives

$$(\tfrac{5}{2}\tfrac{5}{2}0\,|\,m_1m_20)=-(\tfrac{5}{2}\tfrac{5}{2}0\,|\,m_2m_10)\;.$$

Then if the summation indices m_1,m_2 are interchanged in the second (exchange) term in Eq. (30), this gives an *additive* contribution identical to the first (direct) term. This is physically obvious because the two neutrons are in every way identical. Thus there is no need to treat the exchange term separately. It will simply be dropped, and the direct term doubled to compensate for it. Thus, Eq. (30) becomes

$$\psi_i=2^{1/2}\sum_{m_1,m_2}(\tfrac{5}{2}\tfrac{5}{2}0\,|\,m_1m_20)\psi_{5/2,m_1}(1)\psi_{5/2,m_2}(2)$$
$$=\frac{1}{3^{1/2}}\sum_{m_1}(-)^{(1/2)-m_1}\psi_{5/2,m_1}(1)\psi_{5/2,-m_1}(2)\;, \tag{31}$$

when the CG coefficient value of

$$(\tfrac{5}{2}\tfrac{5}{2}0\,|\,m_1m_20)=(-)^{(1/2)-m_1}(2\cdot3)^{-(1/2)}\delta_{m_1,-m_2}$$

is used. The final state is simply

$$\psi_f=\sum_{\mu_1,\mu_2}(\tfrac{1}{2}\tfrac{5}{2}2\,|\,\mu_1\mu_2m_f)\psi_{1/2,\mu_1}(1)\psi_{5/2,\mu_2}(2)\;. \tag{32}$$

Only $\vec{\mathrm{M}}_{fi}^{\sin}$ will contribute in this case, since initial and final parities are opposite. From Eqs. (8), (31), (32), and the knowledge that $j_i=m_i=0$, the transition matrix element is

$$\vec{\mathrm{M}}_{fi}^{\sin}=\frac{1}{3^{1/2}}\sum_{m_f}\sum_{\mu_1,\mu_2}\sum_{m_1}(\tfrac{1}{2}\tfrac{5}{2}2\,|\,\mu_1\mu_2m_f)(-)^{(1/2)-m_1}$$
$$\times\left[\psi_{1/2,\mu_1}(1),\sin\left[\frac{z^{1/2}}{2}u_1\cos\theta_1\right]\vec{\sigma}\psi_{5/2,m_1}(1)\right](\psi_{5/2,\mu_2}(2),\psi_{5/2,-m_1}(2))\;. \tag{33}$$

Orthonormality as used with nucleon (2) gives

$$(\psi_{5/2,\mu_2}(2),\psi_{5/2,-m_1}(2))=\delta_{\mu_2,-m_1}\;, \tag{34}$$

and the CG coefficient in Eq. (33) gives

$$\mu_2=m_f-\mu_1\;. \tag{35}$$

Equations (34) and (35) together yield

$$m_f=\mu_1-m_1\;. \tag{36}$$

The inner product for nucleon (1) is

$$(\psi_{1/2,\mu_1},\sin(g_1\cos\theta_1)\vec{\sigma}\psi_{5/2,m_1})=R_0{}^3\int du_1u_1{}^2R_{11}(u_1)R_{12}(u_1)$$
$$\times\sum_{\mu_l,\mu_s}\sum_{m_l,m_s}(1\tfrac{1}{2}\tfrac{1}{2}\,|\,\mu_l\mu_s\mu_1)(2\tfrac{1}{2}\tfrac{5}{2}\,|\,m_lm_sm_1)$$
$$\times\int d\Omega_1Y_1^{\mu_l^*}(1)\sin(g_1\cos\theta_1)Y_2^{m_l}(1)\chi_{1/2}^{\mu_s\dagger}\vec{\sigma}\chi_{1/2}^{m_s}\;, \tag{37}$$

in which Eq. (16) has been used for the wave functions, and where the definition

$$g_1\equiv\tfrac{1}{2}z^{1/2}u_1 \tag{38}$$

has been introduced. The CG coefficients in Eq. (37) yield

$$\mu_l=\mu_1-\mu_s \; , \tag{39}$$

$$m_1=\mu_1-\mu_s+m_s \; . \tag{40}$$

The solid angle integral in Eq. (37) can be written in the form

$$\int d\Omega_1 Y_1^{\mu_l *}\sin(g_1\cos\theta_1)Y_2^{m_l}=\tfrac{1}{2}(3\cdot 5)^{1/2}C_{\mu_l}\delta_{\mu_l,m_l} \; , \tag{41}$$

where

$$C_{\mu_l}\equiv\left[\frac{(1-\mu_l)!}{(1+\mu_l)!}\frac{(2-\mu_l)!}{(2+\mu_l)!}\right]^{1/2}\int_{-1}^{1}dx\, P_1^{\mu_l}(x)P_2^{\mu_l}(x)\sin g_1 x \; , \tag{42}$$

and the $P_n^{\mu_l}(x)$ in Eq. (42) are associated Legendre functions.[9] The C_{μ_l} have the property

$$C_{\mu_l}=C_{-\mu_l} \; . \tag{43}$$

The result of the index constraints from Eqs. (34)–(36), (39), (40), and the integral in Eq. (41), is that Eq. (33) becomes

$$\begin{aligned}\vec{\mathrm{M}}_{fi}^{\sin}=\frac{5^{1/2}}{2}R_0^3\int du_1 u_1^2 R_{11}(u_1)R_{12}(u_1)\\ \times\sum_{\mu_1}\sum_{\mu_s}\sum_{m_s}(-)^{(1/2)-\mu_1+\mu_s-m_s}\\ \times(\tfrac{1}{2}\tfrac{5}{2}2\,|\,\mu_1-\mu_1+\mu_s-m_s\,\mu_s-m_s)(1\tfrac{1}{2}\tfrac{1}{2}\,|\,\mu_1-\mu_s\,\mu_s\,\mu_1)\\ \times(2\tfrac{1}{2}\tfrac{5}{2}\,|\,\mu_1-\mu_s\,m_s\,\mu_1-\mu_s+m_s)C_{(\mu_1-\mu_s)}\chi_{1/2}^{\mu_s\dagger}\vec{\sigma}\chi_{1/2}^{m_s} \; .\end{aligned} \tag{44}$$

From the $\lambda=+1$ component of Eq. (44), Eq. (14) specifies the values of μ_s and m_s, so only one sum remains. The expression is

$$\begin{aligned}(M_+^{\sin})_{fi}=\tfrac{1}{3}\left[\tfrac{5}{2}\right]^{1/2}R_0^3\int du_1 u_1^2 R_{11}(u_1)R_{12}(u_1)\\ \times\sum_{\mu_1}(2\tfrac{1}{2}\tfrac{5}{2}\,|\,1-\mu_1\;-\mu_1+1)(\tfrac{3}{2}-\mu_1)^{1/2}(\tfrac{7}{2}-\mu_1)^{1/2}C_{\mu_1-(1/2)} \; .\end{aligned} \tag{45}$$

With the aid of Eq. (43), the expansion of the μ_1 sum in Eq. (45) gives

$$(M_+^{\sin})_{fi}=\tfrac{1}{3}R_0^3\int du_1 u_1^2 R_{11}(u_1)R_{12}(u_1)(3^{1/2}C_0+4C_1) \; . \tag{46}$$

When the sine function in the integrand in Eq. (42) for the C_{μ_l} is expanded, the integral is elementary, yielding

$$\begin{aligned}C_0&=4\sum_{k=0}^{\infty}\frac{(-)^k g_1^{2k+1}(k+1)}{(2k+1)!(2k+3)(2k+5)} \; ,\\ C_1&=2\cdot 3^{1/2}\sum_{k=0}^{\infty}\frac{(-)^k g_1^{2k+1}}{(2k+1)!(2k+3)(2k+5)} \; ,\\ 3^{1/2}C_0+4C_1&=4\cdot 3^{1/2}\sum_{k=0}^{\infty}\frac{(-)^k g_1^{2k+1}(k+3)}{(2k+1)!(2k+3)(2k+5)} \; .\end{aligned} \tag{47}$$

Equation (47), substituted in Eq. (46), leads to

$$(M_+^{\sin})_{fi}=\frac{2}{3^{1/2}}z^{1/2}\sum_{k=0}^{\infty}\frac{(-z)^k(k+3)}{2^{2k}(2k+1)!(2k+3)(2k+5)}R_0^3\int du_1 u_1^2 R_{11}(u_1)R_{12}(u_1)u_1^{2k+1} \; , \tag{48}$$

when the definition (38) is used.

To evaluate the radial integral in Eq. (48), the radial wave functions in Eq. (26) are employed, with the Laguerre polynomials

$$L_1^{3/2}(u^2)=\tfrac{5}{2}-u^2,\quad L_1^{5/2}(u^2)=\tfrac{7}{2}-u^2\ .$$

The final result for the radial integral is

$$\int_0^{\infty} du_1 u_1^{2k+3} R_{11}(u_1) R_{12}(u_1)=\frac{1}{R_0{}^3}\,\frac{(2k+5)!!(2k^2+4k+7)}{3\cdot 5(2\cdot 7)^{1/2}2^k}\ . \tag{49}$$

This integral, when substituted in Eq. (48), gives

$$(M_+^{\sin})_{fi}=\frac{1}{3\cdot 5}\left[\frac{2}{3\cdot 7}\right]^{1/2} z^{1/2}\sum_{k=0}^{\infty}\frac{(-z)^k(k+3)(2k^2+4k+7)}{2^{4k}k!}\ . \tag{50}$$

The sum in Eq. (50) can be accomplished in closed form to give

$$(M_+^{\sin})_{fi}=\frac{1}{5}\left[\frac{2\cdot 7}{3}\right]^{1/2} e^{-z/16} z^{1/2}\left[1-\frac{31}{2^4\cdot 3\cdot 7}z+\frac{1}{2^4\cdot 3\cdot 7}z^2-\frac{1}{2^{11}\cdot 3\cdot 7}z^3\right]\ . \tag{51}$$

For the $\lambda=-1$ component, it is easily shown that

$$(M_-^{\sin})_{fi}=-(M_+^{\sin})_{fi}\ . \tag{52}$$

Equations (33) and (14) lead to the $\lambda=0$ component of the matrix element

$$\begin{aligned}(M_0^{\sin})_{fi}=\frac{5^{1/2}}{2}R_0{}^3\int du_1 u_1{}^2 R_{11}(u_1)R_{12}(u_1)\\ \times\sum_{\mu_1}\sum_{m_s}(-)^{1-\mu_1-m_s}(\tfrac{1}{2}\tfrac{5}{2}2\,|\,\mu_1\,-\mu_1 0)\\ \times(1\tfrac{1}{2}\tfrac{1}{2}\,|\,\mu_1-m_s\, m_s\,\mu_1)(2\tfrac{1}{2}\tfrac{5}{2}\,|\,\mu_1-m_s\, m_s\,\mu_1)C_{(\mu_1-m_s)}\ .\end{aligned} \tag{53}$$

Both sums in Eq. (53) are to be expanded, and the value of the CG coefficients inserted. The property (43) then leads to the form

$$(M_0^{\sin})_{fi}=-\tfrac{1}{3}\left[\tfrac{3}{2}\right]^{1/2}R_0{}^3\int du_1 u_1{}^2 R_{11}(u_1)R_{12}(u_1)(3^{1/2}C_0+2C_1)\ . \tag{54}$$

With the help of Eq. (47), one has

$$3^{1/2}C_0+2C_1=4\cdot 3^{1/2}\sum_{k=0}^{\infty}\frac{(-)^k g_1^{2k+1}(k+2)}{(2k+1)!(2k+3)(2k+5)}\ . \tag{55}$$

Equations (54) and (55), when combined, give rise to a radial integral exactly the same as the one evaluated in Eq. (49). The $\lambda=0$ component is then

$$(M_0^{\sin})_{fi}=-\frac{1}{3\cdot 5\cdot 7^{1/2}}z^{1/2}\sum_{k=0}^{\infty}\frac{(-z)^k(k+2)(2k^2+4k+7)}{2^{4k}k!}$$

in series form, or

$$(M_0^{\sin})_{fi}=-\frac{2\cdot 7^{1/2}}{3\cdot 5}e^{-z/16}z^{1/2}\left[1-\frac{5^2}{2^5\cdot 7}z+\frac{1}{2^8}z^2-\frac{1}{2^{12}\cdot 7}z^3\right] \tag{56}$$

in closed form.

From Eqs. (6), (13), (51), (52), and (56), the squared induced matrix element is

$$|M_{\rm ind}|^2=\frac{\kappa^2}{4\pi(2z_f)^{1/2}}\frac{2\cdot 7}{3^2\cdot 5}e^{-z/8}z$$

$$\times[1-\tfrac{1}{5}z+(1.6786\times 10^{-2})z^2-(7.3408\times 10^{-4})z^3$$

$$+(1.7106\times 10^{-5})z^4-(1.9203\times 10^{-7})z^5+(8.1095\times 10^{-10})z^6] . \quad (57)$$

The coefficients rendered decimally in Eq. (57) are all available as rational numbers, but their rational expressions are too complicated to be useful.

V. ^{87}Rb EXAMPLE

^{87}Rb has a three-nucleon fragment ($q=3$) with $L=3$. The odd proton in $^{87}_{37}\text{Rb}_{50}$ must be part of the fragment because initially this $p_{3/2}$ particle accounts for the entire ^{87}Rb spin and parity of $(\frac{3}{2})^-$. The beta decay itself involves a neutron, not the odd proton, and since the beta decay neutron is initially paired with another to give 0^+, then both of these neutrons must also be assigned to the fragment. In the final state, the $g_{9/2}$ neutron which beta decays to a $p_{3/2}$ proton will couple to 0^+ with the initial odd proton, while the remaining $g_{9/2}$ neutron accounts for the $(\frac{9}{2})^+$ spin and parity of the $^{87}_{38}\text{Sr}_{49}$ daughter nucleus.

The initial state of the ^{87}Rb fragment has two $g_{9/2}$ neutrons coupled together to give zero angular momentum, and the odd $p_{3/2}$ proton in ^{87}Rb is coupled to the paired neutrons to give an overall angular momentum of $\frac{3}{2}$. The neutron part of the initial state is

$$\psi_{i,\rm neut}(1,2)=\frac{1}{\sqrt{2}}\sum_{m_1m_2}(\tfrac{9}{2}\tfrac{9}{2}0\,|\,m_1m_20)[\psi_{9/2,m_1}(1)\psi_{9/2,m_2}(2)-\psi_{9/2,m_1}(2)\psi_{9/2,m_2}(1)] ,$$

with the further coupling to the proton given by

$$\psi_i=\sum_{m_3}(0\tfrac{3}{2}\tfrac{3}{2}\,|\,0m_3m_i)\psi_{i,\rm neut}(1,2)\psi_{3/2,m_3}(3) .$$

As in the ^{90}Sr example, there are direct and exchange terms because of the antisymmetrization of the identical particles. As in the ^{90}Sr case, the contributions of the two neutrons are identical and additive. With that knowledge, and with

$$(\tfrac{9}{2}\tfrac{9}{2}0\,|\,m_1m_20)=\frac{(-)^{(1/2)+m_2}}{(2\cdot 5)^{1/2}}\delta_{m_1,-m_2} ,$$

$$(0\tfrac{3}{2}\tfrac{3}{2}\,|\,0m_3m_i)=\delta_{m_3,m_i} ,$$

the initial state is

$$\psi_i=\frac{1}{\sqrt{5}}\sum_{m_2}(-)^{(1/2)+m_2}\psi_{9/2,-m_2}(1)\psi_{9/2,m_2}(2)\psi_{9/2,m_i}(3) . \quad (58)$$

It is convenient in a three-nucleon fragment problem to assign the index 2 to the coordinates of the beta decay particle. Other possibilities are, of course, accounted for by antisymmetrization of identical particles. In the final state, the identical particles are $p_{3/2}$ protons coupled together to zero angular momentum, or

$$\psi_{f,\rm prot}(2,3)=\frac{1}{\sqrt{2}}\sum_{\mu_2,\mu_3}(\tfrac{3}{2}\tfrac{3}{2}0\,|\,\mu_2\mu_30)[\psi_{3/2,\mu_2}(2)\psi_{3/2,\mu_3}(3)-\psi_{3/2,\mu_2}(3)\psi_{3/2,\mu_3}(2)] .$$

This pair then couples to the remnant $g_{9/2}$ neutron to give the $\frac{9}{2}$ final state,

$$\psi_f=\sum_{\mu_1}(\tfrac{9}{2}0\tfrac{9}{2}\,|\,\mu_10m_f)\psi_{9/2,\mu_1}(1)\psi_{f,\rm prot}(2,3) .$$

Again, the direct and exchange terms give equal and additive contributions, and the CG coefficients can be obtained directly as

$$(\tfrac{3}{2}\tfrac{3}{2}0\,|\,\mu_2\mu_3 0)=\frac{(-)^{(3/2)-\mu_2}}{2}\delta_{\mu_2,-\mu_3}\,,$$

$$(\tfrac{9}{2}0\tfrac{9}{2}\,|\,\mu_1 0 m_f)=\delta_{\mu_1,m_f}\,.$$

The final state is then

$$\psi_f=\frac{1}{\sqrt{2}}\sum_{\mu_2}(-)^{(3/2)-\mu_2}\psi_{9/2,m_f}(1)\psi_{3/2,\mu_2}(2)\psi_{3/2,-\mu_2}(3)\,. \tag{59}$$

Initial and final parities in ^{87}Rb decay are opposite, so only $\vec{\mathrm{M}}_{fi}^{\sin}$ will contribute. From Eqs. (8), (58), (59), and with $j_i=\frac{3}{2}$, the transition matrix element is

$$\vec{\mathrm{M}}_{fi}^{\sin}=\frac{1}{4(2\cdot5)^{1/2}}\sum_{m_i}\sum_{m_f}\sum_{\mu_2}\sum_{m_2}(-)^{(3/2)-\mu_2}(-)^{(1/2)+m_2}(\psi_{9/2,m_f}(1),\psi_{9/2,-m_2}(1))$$
$$\times\left[\psi_{3/2,\mu_2}(2),\sin\left[\frac{z^{1/2}}{3}u_2\cos\theta_2\right]\vec{\sigma}\psi_{9/2,m_2}(2)\right](\psi_{3/2,-\mu_2}(3)\psi_{3/2,m_i}(3))\,. \tag{60}$$

Orthonormality applied to nucleons (1) and (3) gives

$$(\psi_{9/2,m_f}(1),\psi_{9/2,-m_2}(1))=\delta_{m_f,-m_2}\,, \tag{61}$$

$$(\psi_{3/2,-\mu_2}(3),\psi_{3/2,m_i}(3))=\delta_{-\mu_2,m_i}\,. \tag{62}$$

The inner product for nucleon (2) is

$$(\psi_{3/2,\mu_2},\sin(g_2\cos\theta_2)\vec{\sigma}\psi_{9/2,m_2})=R_0{}^3\int du_2 u_2{}^2 R_{11}(u_2)R_{04}(u_2)$$
$$\times\sum_{\mu_l,\mu_s}\sum_{m_l,m_s}(1\tfrac{1}{2}\tfrac{3}{2}\,|\,\mu_l\mu_s\mu_2)(4\tfrac{1}{2}\tfrac{9}{2}\,|\,m_l m_s m_2)$$
$$\times\int d\Omega_2 Y_1^{\mu_l *}\sin(g_2\cos\theta_2)Y_4^{m_l}\chi_{1/2}^{\mu_s\dagger}\vec{\sigma}\chi_{1/2}^{m_s}\,, \tag{63}$$

in which Eq. (16) has been used for the wave functions, and where the definition

$$g_2\equiv\tfrac{1}{3}z^{1/2}u_2 \tag{64}$$

has been introduced. The CG coefficients in Eq. (63) yield

$$\mu_2=\mu_l+\mu_s\,, \tag{65}$$

$$m_2=m_l+m_s\,. \tag{66}$$

The solid angle integral in Eq. (63) is

$$\int d\Omega_2 Y_1^{\mu_l *}\sin(g_2\cos\theta_2)Y_4^{m_l}=\frac{3^{3/2}}{2}C'_{\mu_l}\delta_{\mu_l,m_l}\,, \tag{67}$$

where, in analogy with Eq. (42), the definition is introduced that

$$C'_{\mu_l}\equiv\left[\frac{(1-\mu_l)!}{(1+\mu_l)!}\frac{(4-m_l)!}{(4+m_l)!}\right]^{1/2}\int_{-1}^{1}dx\,P_1^{\mu_l}(x)P_4^{\mu_l}(x)\sin g_2 x\,. \tag{68}$$

The C'_{μ_l} have the same evenness property expressed in Eq. (43). The result of Eqs. (64)–(67) is that Eq. (63) becomes

$$\vec{M}_{fi}^{\sin}=\frac{3^{3/2}}{(2^7\cdot 5)^{1/2}}\sum_{\mu_l,\mu_s}\sum_{m_s}(-)^{(3/2)-\mu_s}(-)^{(1/2)+m_s}$$

$$\times R_0{}^3\int du_2 u_2{}^2 R_{11}(u_2)R_{04}(u_2)$$

$$\times(1\tfrac{1}{2}\tfrac{3}{2}\,|\,\mu_l\,\mu_s\,\mu_l+\mu_s)(4\tfrac{1}{2}\tfrac{9}{2}\,|\,\mu_l\,m_s\,\mu_l+m_s)C'_{\mu_l}\chi_{1/2}^{\mu_s\dagger}\vec{\sigma}\chi_{1/2}^{m_s}\,. \tag{69}$$

When the $\lambda=+1$ component of Eq. (69) is stated with the help of Eq. (14), the two CG coefficients can be evaluated immediately, and the outcome is

$$(M_+^{\sin})_{fi}=-\frac{1}{(2^5\cdot 5)^{1/2}}R_0{}^3\int du_2 u_2{}^2R_{11}(u_2)R_{04}(u_2)\sum_{\mu_l}(2+\mu_l)^{1/2}(5-\mu_l)^{1/2}C'_{\mu_l}\,. \tag{70}$$

The sum over μ_l is

$$\sum_{\mu_l}(2+\mu_l)^{1/2}(5-\mu_l)^{1/2}C'_{\mu_l}=(2\cdot 5)^{1/2}C'_0+(2+2^{1/2})3^{1/2}C'_1\,,$$

when the property in Eq. (43) is employed. The C'_{μ_l} can be evaluated straightforwardly when the sine function is expanded, to give the results

$$\begin{aligned}
C'_0&=2^6\sum_{k=0}^{\infty}\frac{(-)^k g_2^{2k+1}k(k+1)^2(k+2)(k+3)}{(2k+7)!}\,,\\
C'_1&=2^6\left[\frac{5}{2}\right]^{1/2}\sum_{k=0}^{\infty}\frac{(-)^k g_2^{2k+1}k(k+1)(k+2)(k+3)}{(2k+7)!}\,,\\
(2\cdot 5)^{1/2}C'_0+(2+2^{1/2})3^{1/2}C'_1&=-2^6\cdot 5^{1/2}\sum_{k=0}^{\infty}\frac{(-)^k g_2^{2k+3}(k+4)!(2^{1/2}k+2^{3/2}+3^{1/2}+6^{1/2})}{(2k+9)!k!}\,.
\end{aligned} \tag{71}$$

In the evaluation of C'_1, a value for the associated Legendre function $P_4{}^1(x)$ is required, which is not to be found in most brief tabulations of that function. The required function is

$$P_4{}^1(x)=-\tfrac{5}{2}(1-x^2)^{1/2}x(7x^2-3)\,.$$

Equation (71) puts Eq. (70) into the form

$$(M_+^{\sin})_{fi}=\frac{2^{7/2}}{3^3}z^{3/2}\sum_{k=0}^{\infty}\frac{(-z)^k(k+4)!}{3^{2k}(2k+9)!k!}(2^{1/2}k+2^{3/2}+3^{1/2}+6^{1/2})R_0{}^3\int du_2u_2{}^2R_{11}(u_2)R_{04}(u_2)u_2^{2k+3}\,, \tag{72}$$

when the definition (64) is used.

The radial integral in Eq. (72) is evaluated by using the radial wave functions stated in Eq. (26). The required Laguerre polynomials are

$$L_1^{3/2}(u_2)=\tfrac{5}{2}-u^2,\quad L_0^{9/2}(u^2)=1\,.$$

The radial integral is

$$\int_0^{\infty}du_2u_2^{2k+5}R_{11}(u_2)R_{04}(u_2)=\frac{1}{R_0{}^3}\frac{(2k+9)!!(k+3)}{3^2\cdot 5\cdot 7^{1/2}2^{k+1}}\,. \tag{73}$$

Equations (72) and (73) yield the result

$$(M_+^{\sin})_{fi}=\frac{1}{2^{3/2}\cdot 3^5\cdot 5\cdot 7^{1/2}}z^{3/2}\sum_{k=0}^{\infty}\frac{(-z)^k(k+3)(2^{1/2}k+2^{3/2}+3^{1/2}+6^{1/2})}{2^{2k}3^{2k}k!}\,. \tag{74}$$

The series in Eq. (74) can be summed to give the closed form

$$(M_+^{\rm sin})_{fi}=\frac{(4+2\cdot 3^{1/2}+6^{1/2})}{2^2\cdot 3^4\cdot 5\cdot 7^{1/2}}e^{-z/36}z^{3/2}\left[1-\left[\frac{12+2\cdot 3^{1/2}+6^{1/2}}{4+2\cdot 3^{1/2}+6^{1/2}}\right]\frac{1}{2^2\cdot 3^3}z+\frac{1}{(4+2\cdot 3^{1/2}+6^{1/2})}\frac{1}{2^3\cdot 3^5}z^2\right] . \tag{75}$$

The $\lambda=-1$ component can be shown to be

$$(M_-^{\rm sin})_{fi}=(M_+^{\rm sin})_{fi} . \tag{76}$$

The combination of Eqs. (14) and (69) gives the $\lambda=0$ component

$$(M_0^{\rm sin})_{fi}=\left[\frac{3^3}{2^7\cdot 5}\right]^{1/2}\sum_{\mu_l}\sum_{m_s}(-)^{(1/2)-m_s}R_0{}^3\int du_2 u_2{}^2 R_{11}(u_2)R_{04}(u_2)$$
$$\times(1\tfrac{1}{2}\tfrac{3}{2}\,|\,\mu_l\, m_s\, \mu_l+m_s)(4\tfrac{1}{2}\tfrac{9}{2}\,|\,\mu_l\, m_s\, \mu_l+m_s)C'_{\mu_l} .$$

When the two terms in the sum over m_s are written explicitly, the CG coefficients can be evaluated to put the matrix element in the form

$$(M_0^{\rm sin})_{fi}=\frac{1}{(2^7\cdot 5)^{1/2}}R_0{}^3\int du_2 u_2{}^2 R_{11}(u_2)R_{04}(u_2)$$
$$\times\sum_{\mu_l}[(2+\mu_l)^{1/2}(5+\mu_l)^{1/2}-(2-\mu_l)^{1/2}(5-\mu_l)^{1/2}]C'_{\mu_l} . \tag{77}$$

In the μ_l sum in Eq. (77), the square bracket is antisymmetric under a change of sign in μ_l, whereas C'_{μ_l} is symmetric. This means that

$$(M_0^{\rm sin})_{fi}=0 . \tag{78}$$

Equations (6), (13), (75), (76), and (78) give the squared matrix element as

$$|M_{\rm ind}|^2=\frac{\kappa^2}{4\pi(2z_f)^{1/2}}\frac{(4+2\cdot 3^{1/2}+6^{1/2})^2}{2^4\cdot 3^8\cdot 5^2\cdot 7}e^{-z/18}z^3$$
$$\times\left[1-\left[\frac{12+2\cdot 3^{1/2}+6^{1/2}}{4+2\cdot 3^{1/2}+6^{1/2}}\right]\frac{1}{2\cdot 3^3}z+\frac{(35+2^{3/2}+12\cdot 3^{1/2}+6^{3/2})}{(4+2\cdot 3^{1/2}+6^{1/2})^2}\frac{1}{2^3 3^5}z^2\right.$$
$$\left.-\frac{(12+2\cdot 3^{1/2}+6^{1/2})}{(4+2\cdot 3^{1/2}+6^{1/2})^2}\frac{1}{2^4\cdot 3^8}z^3+\frac{1}{(4+2\cdot 3^{1/2}+6^{1/2})^2}\frac{1}{2^6\cdot 3^{10}}z^4\right] . \tag{79}$$

VI. LIMITS AND EXTREMA OF THE TRANSITION PROBABILITY

A. Low intensity behavior

An investigation of the $z \ll 1$ case using the general results presented in I gives the simple result that the transition probability is proportional to z^L. The $z \ll 1$ case can also be explored in simple fashion from the results presented here, but it must be kept in mind that $z \ll 1$ does not mean $z \rightarrow 0$. The reason is that the results given in this paper make use of an analytical approximation developed in I in which it was required that $z_f \gg 1$. In view of Eq. (3), this does not eliminate the possibility of examining $z \ll 1$, but it does set a lower limit on z. An examination of the validity conditions for the use in I of an asymptotic expansion of the generalized Bessel function leads to the general restriction that $\frac{1}{2}z_f \gg 1$. From Eq. (3), this converts to

$$z \gg 6\times 10^{-4} . \tag{80}$$

With this restriction, $z \ll 1$ can be regarded as covering a range between about $z \approx 10^{-3}$ and $z \approx 10^{-1}$.

Equation (4) shows that the intensity dependence of the total transition probability per unit time is given by the squared transition matrix element, $|M_{\text{ind}}|^2$. The $z \ll 1$ limit of Eq. (29) for ^{113}Cd, Eq. (57) for ^{90}Sr, and Eq. (79) for ^{87}Rb all are of the form

$$W_{\text{tot}} \propto z^L / z_f^{1/2}$$

or

$$W_{\text{tot}} \propto z^{L-(1/2)} \tag{81}$$

when the connection between z and z_f given in Eq. (3) is accounted for. In other words, the true $z \to 0$ limit

$$W_{\text{tot}} \propto z^L , \tag{82}$$

predicted by the complete results in I, is modified in the $z \ll 1$, $z_f \gg 1$ domain from Eq. (82) to Eq. (81). Equation (81) should really be termed an intermediate-intensity behavior rather than a true low-intensity behavior.

B. Extrema in the transition probability

1. ^{113}Cd

The transition probability per unit time expressed with the help of Eq. (29) is a positive-definite quantity which approaches zero as $z \to 0$ because of the $z^{7/2}$ factor, and approaches zero as $z \to \infty$ because of the $e^{-z/2}$ factor. That means that there must be at least one maximum for physical (positive) values of z. The existence of a maximum transition probability in the presence of intense fields has been noted before.[2,3] In fact, some transitions can exhibit more than one maximum.[2]

The z dependence of W_{tot} for ^{113}Cd is given by

$$f(z) = e^{-z/2} z^{7/2} (1 - z/8)^2 (1 - z/24)^2 \tag{83}$$

from Eq. (29). Setting $f'(z) = 0$ requires the solution of

$$z^{5/2}(z-8)(z-24)(z^3 - 47z + 544z - 1344) = 0 \ . \tag{84}$$

The cubic factor in Eq. (84) has roots at $z = 3.3939$, 12.8942, and 30.7119. These are all maxima, with the absolute maximum at

$$z = 3.3939 \ . \tag{85}$$

The minima of Eq. (84) are at $z = 0$, 8 and 24. The solution at $z = 0$, while it is physically reasonable, is not truly meaningful here since it falls outside the inequality stated in Eq. (80). All the other roots of Eq. (84) are physical. However, only the maximum at $z = 3.3939$ is of any significance. Equation (83) has both a zero and a minimum at $z = 8$ and 24. The other maxima which occur at larger values of z have very small amplitudes as compared to the absolute maximum at $z = 3.3939$, and they are thus of little physical consequence.

2. ^{90}Sr

Finding the extrema of Eq. (57) for ^{90}Sr requires finding the roots of a power of z times a seventh degree polynomial (septic). The zero root from the power-of-z factor can be ignored since it does not satisfy Eq. (80). The septic is

$$z^7 - (2.8880 \times 10^2) z^6 + (3.1514 \times 10^4) z^5 - (1.6646 \times 10^6) z^4 + (4.6046 \times 10^7) z^3 - (6.6062 \times 10^8) z^2 + (4.1926 \times 10^9) z - (4.9325 \times 10^9) = 0 \ . \tag{86}$$

Equation (86) has two complex roots and five real roots indicating maxima in Eq. (57) at $z = 1.4931$, 40.868, and 104.87, and minima at $z = 22.512$ and 81.385. Of these extrema, only the maximum at

$$z = 1.4931 \tag{87}$$

has any physical importance. Relative to this maximum in the transition probability for ^{90}Sr, the other extrema have amplitudes much too small to be of consequence.

3. ^{87}Rb

A solution for the extrema in the transition probability for ^{87}Rb as it follows from Eq. (79) requires a solution for the roots of the quintic

$$z^5 - (7.6189 \times 10^2) z^4 + (2.0636 \times 10^5) z^3 - (2.3972 \times 10^7) z^2 + (1.1544 \times 10^9) z - 1.6713 \times 10^{10} = 0 \ . \tag{88}$$

Equation (88) has five real roots predicting maxima in the transition probability at $z=24.666$, 118.78, and 296.00, and minima at $z=79.243$ and 243.20. Of these, only the first maximum at

$$z=24.666 \tag{89}$$

is of any importance.

C. General algebraic behavior

It is clear that there are strong algebraic similarities in Eqs. (29), (57), and (79) for the squared induced transition matrix elements for the three examples treated. It is also evident that there are major differences in the locations of the maxima of the three different transition probabilities, as can be seen from Eqs. (85), (87), and (89). The goal posed here is to ascertain a general form for the squared transition matrix element as a function of forbiddenness, L, and number of particles in the fragment, q; and then to see if some estimate can be made for the location of the maximum.

The algebraic form of the transition probability is

$$W \propto z^{L-1/2} e^{-z/\alpha} \times (\text{alternating polynomial in } z) , \tag{90}$$

where the "alternating polynomial in z" in Eq. (90) starts with the zeroth-degree term and proceeds upward in powers of z. The problem is to ascertain the appropriate value of α in Eq. (90). The value of α is found from the coefficient attached to powers of z in the sum over the index k as it occurs in Eq. (27), (50), or (74). The factor that goes with z follows from the combination g^{2k} (or g_1^{2k} or g_2^{2k}). From Eqs. (21), (38), and (64), g is of the form $z^{1/2}u/q$. The factor $(z^{1/2}/q)^{2k}$ leads directly to z/q^2. The u^{2k} factor, after integration over the u variable, yields a further factor of $(1/2^{2k})$. Altogether, z appears in the k sum in the form $(z/4q^2)^k$, which accounts for the final exponential which arises from summing the k series as $\exp(-z/4q^2)$. This factor in the matrix element, when squared, gives $\exp(-z/2q^2)$ in the transition probability. Equation (90) is thus more explicitly stated as

$$W \propto z^{L-(1/2)} e^{-z/2q^2} \times (\text{alternating polynomial in } z) . \tag{91}$$

Were the polynomial in z in Eq. (91) replaced by a constant, W would have a maximum at

$$z_{\max} = q^2(2L-1) . \tag{92}$$

For values of z given by Eq. (92), the constant term in the alternating polynomial in z is, in fact, the dominant term in the polynomial. Because of the alternating quality of the polynomial, the first extremum will actually occur at a smaller value than predicted by Eq. (92). For larger z values, higher-degree terms in the polynomial become increasingly important, and so other extrema at larger z values can occur.

The general conclusion is that Eq. (91) is the general form for the transition probability. This form exhibits a first maximum at a value less than that given in Eq. (92), possibly followed by other extrema. For example, a comparison of Eqs. (85) and (92) for ^{113}Cd (3.39 vs 7), Eqs. (87) and (92) for ^{90}Sr (1.49 vs 4), and Eqs. (89) and (92) for ^{87}Rb (24.67 vs 45), show the general validity as well as the limitations of the conclusions on the first maximum.

VII. HALF-LIVES

The total induced transition probability per unit time is given in Eq. (4). It is related to the half-life for induced beta decay by

$$t_{\text{ind}} = \ln 2 / W_{\text{tot}} . \tag{93}$$

The analog of Eq. (4) for allowed beta decay is

$$W_0 = \frac{G^2 m^5}{2\pi^3} f_0 |M_0|^2 . \tag{94}$$

Equation (94) gives the half-life expression

$$t = \frac{6.19\times 10^3}{f_0 |M_0|^2} = \frac{10^{3.79}}{f_0 |M_0|^2} \tag{95}$$

when the experimental values[10] for the universal weak interaction coupling constant and the Cabibbo angle are used to evaluate G. If an expression like Eq. (95) were used to evaluate induced half-lives, an unrealistically short lifetime would be predicted. The reason is that the simple shell-model wave functions used here would, were no angular momentum and/or parity forbiddenness present, give wave function overlaps more characteristic of superallowed than of ordinary allowed transitions. An empirical remedy for this problem is to use for induced transitions in even-A nuclides

$$t_{\text{ind}} = \frac{10^{5.35}}{f_{\text{tot}} |M_{\text{ind}}|^2} , \tag{96}$$

and for odd-A nuclides

$$t_{\text{ind}} = \frac{10^{5.18}}{f_{\text{tot}} |M_{\text{ind}}|^2} , \tag{97}$$

where the exponents in Eqs. (96) and (97) come from average $\log ft$ values[11] for allowed transitions in even-A and odd-A nuclides.

Evaluation of f_{tot} as required for Eqs. (96) and

TABLE I. Summary of results for the nuclides calculated as examples. The f_n values are spectral integrals. The squared induced transition matrix element and the induced half-life are calculated for the field intensity which maximizes the transition probability.

Nuclide	q	L	ϵ_0	f_1	f_2	f_3	f_{tot}	$\lvert M_{ind}\rvert^2$	t_{ind} (years)
^{113}Cd	1	4	1.581	9.90×10^{-2}	3.09×10^{-2}	-6.4×10^{-3}	0.124	3.08×10^{-5}	1.3×10^{3}
^{90}Sr	2	1	2.068	1.399	0.700	-0.144	1.95	3.50×10^{-4}	10.4
^{87}Rb	3	3	1.535	6.97×10^{-2}	2.03×10^{-2}	-4.2×10^{-3}	8.58×10^{-2}	4.77×10^{-7}	1.2×10^{5}

(97) can be accomplished if the value of $\epsilon_0=E_0/m$ is known, where $E_0=m+T_0$, and T_0 is the maximum kinetic energy (or Q value) available to the electron in beta decay. These ϵ_0 values are listed in Table I for the three examples cited here. This table also lists f_1, f_2, and f_3 values as calculated from Eqs. (127), (128), (134), (135), (139), and (140) of I. Also listed is f_{tot}, the sum of the separate spectral integrals, as shown in Eq. (5). The squared induced transition matrix elements given in Table I are found from Eqs. (29), (57), and (79) when evaluated at their respective maxima, stated in Eqs. (85), (87), and (89). The value of κ is given in Eq. (9), and z_f is related to z as in Eq. (3), with $mR_0\approx\frac{1}{80}$.

The shortest induced half-life shown in Table I for ^{90}Sr, a first-forbidden beta decay, is somewhat faster than the natural decay channel. When combined with the 28.6 year[12] natural decay, the resulting half-life would be 7.6 years. By contrast, the third-forbidden ^{87}Rb natural half-life of 4.80×10^{10} years[13] stands in striking comparison with the shortest induced half-life of 1.2×10^5 years. Even more spectacular is the case of fourth-forbidden ^{113}Cd, where a natural half-life of 9.3×10^{15} years[14] and a shortest induced half-life of 1.3×10^3 years are to be compared. In general, the higher the degree of forbiddenness, the more striking is the possible relative reduction in half-life due to decay induced by external fields.

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