Nuclear beta decay induced by intense electromagnetic fields: Basic theory

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A basic formalism is developed for the theory of the effect on nuclear beta decay of an intense, plane-wave electromagnetic field. Interactions of the field with both the nuclear particles and the decay electron are included. The formalism is developed from first principles, including a derivation of transition probabilities between explicitly time-dependent asymptotic states. Interaction of the field with the nucleus is analyzed in terms of separation of the nucleus into an inert core and a fragment. The field interacts with the fragment, consisting of the nucleons which are candidates for beta decay, plus any other nucleons angular-momentum coupled to them in initial or final states. A separation of variables in the dynamical equations for the nucleus into center-of-mass and relative coordinates for the core and fragment shows direct charge coupling even for a fragment consisting entirely of neutrons. The transition formalism involves specific intense-field wave functions both for the nucleus and for the beta particle. Complete results are presented for total transition probability per unit time for intense-field-coupled nuclear beta decay. A much simplified formalism is given for the special case of very high field intensity at very low frequency. The results then bear a formal resemblance to ordinary beta decay theory, but they contain specific field effects in the beta particle spectral function, and in the nuclear interaction matrix elements. This is the first of a series of papers on this subject.

[RADIOACTIVITY Intense-field-induced β decay. β decay forbiddenness removal.]

I. INTRODUCTION

A. The physical problem

It is conventional to presume that photons will interact with atomic nuclei only via first-order interactions, in which the photons possess an energy typical of nuclear transition energies. By contrast, the processes considered here involve specific intense-field electromagnetic interactions, in which nuclear transitions can be affected or effected by very low frequency, but very intense electromagnetic fields. The implication of the phrase "intense-field electromagnetic interactions" is that virtual processes of all orders in the externally applied field are summed over. Thus, whether net absorption or emission of photons is of low order or of high order, the overall interaction implicitly is of arbitrarily high order because of the summed virtual processes.

The nuclear process treated in this paper is beta decay. A major reason for this choice is that forbidden beta decays can show dramatic increases in transition probability when subjected to intense fields. This is easily explained in terms of the photon as a pseudovector particle.[1] Nuclear beta decay is forbidden (i.e., strongly suppressed) if the difference between initial and final nuclear angular momentum exceeds one unit, and/or if nuclear parity changes in the decay. When external photons intercede in the decay, however, an additional angular momentum of up to the net number of photons absorbed or emitted is made available; and an additional parity change or no change is introduced, depending on whether the net photon number is odd or even. Thus, forbidden decays can be modified to allowed decays, albeit at the expense of an electromagnetic field interaction in addition to the weak interaction. With sufficiently high field intensity, the penalty in transition probability paid for this additional interaction is not nearly as great as that paid for forbiddenness in the beta decay. This can lead to major increases in transition probability. This is especially true for high-order forbidden decays, since intense field phenomena have the property that high-order interactions can be competitive with, and even dominate the lowest-order interaction.[2,3]

There are several mechanisms for enhancement of beta decay transition probabilities by intense-field effects, although the removal of forbiddenness is generally the most spectacular. Apart from the forbiddenness-changing mechanism, enhancement can occur in both allowed and forbidden decays through a change in nuclear wave function overlap as a result of the opening up of extra angular momentum channels due to the field. Also, electron spectral integrals experience an increase due to intense-field interactions. This can be viewed as an enlargement of the phase space available to the leptons as a result of electromagnetic field interactions. Yet another mechanism for beta decay alteration is available in energy conditions. A decay that is energetically forbidden can become accessible by energy contributions from the field. This last mechanism is less important than the others when very low applied field frequencies [such as radiofrequency (rf)] are considered. By contrast, the removal of angular momentum and/or parity forbiddenness by field interaction requires only a negligible energy contribution from the field. The spin and parity of each photon is 1^- irrespective of the energy of the photon.

B. Background

There is little history of work on causing changes in the rates of beta radioactivity. The common understanding is that it is an immutable natural process. Somewhat more research has been done on rate alterations in the closely related process of electron capture in nuclei, and on the more distantly connected process of internal conversion. Electron capture and internal conversion have the common feature that they involve interaction between the nucleus and the surrounding atomic electrons. These processes are thus affected by changes in the distribution of atomic electrons brought about by chemical or other external means. Considerable theoretical and experimental work has been done on this subject.[4] Changes in the chemical environment lead typically to changes in the decay rate of a few parts in 10^3, although effects as much as ten times larger have been observed in special cases.[5] Developments in high pressure technology using diamond-anvil presses have led to the observation of increases as large as six parts in 10^3 in the decay constant of an electron capture reaction[6] due to pressure effects.

The cases of electron capture and internal conversion just discussed involved direct participation of atomic electrons in a nuclear process. However, even ordinary β^- and β^+ decays are influenced to some degree by the Coulomb field experienced by the beta particle as it departs from the nucleus. This suggests the possibility that the distribution of atomic electrons can also affect such decays. It has been estimated theoretically that chemical effects might lead to changes of the order of a few parts in 10^4 in beta decay rates.[7]

The work reviewed above is somewhat peripheral to the present subject, which is the modification of beta decay rates by externally applied fields. There are two theoretical treatments of the influence on beta decay of extremely intense constant magnetic fields.[8] These studies conclude that there would be essentially no effects for fields up to about 10^{12} G, but above about 10^{13} G beta decay rates would be increased noticeably. The problem is that the largest field that can be produced in the laboratory at present is about 10^6 G.

The work just cited is of interest in an astrophysical context. Another astrophysical treatment of beta decay modification treats photon effects on beta decay in a stellar interior. The mechanism is one in which the photon produces a virtual electron-positron pair, with the positron being absorbed by the nucleus in lieu of beta-particle emission.[9] The process can become of importance at temperatures of the order of 10^8 K.

Specific intense-field effects on free neutron decay have been calculated,[10] following earlier work on field effects on other elementary particles.[11–13]

The earliest work treating intense-field modification of beta decay as it occurs in nuclei was done by this author.[14] These results are, at least in part, reproduced below. Included in this work was a treatment of the field interaction with the nuclear particles as well as with the beta particle. The principal thrust of this investigation was the case of forbidden beta decay. An application of the general results of Ref. 14 to allowed decays has been described briefly.[15] Recently, Becker *et al.*[16] considered laser effects on beta decay, confining themselves to the allowed case, treating only effects on the decay electron, and only for the algebraically simple case of circular polarization of the applied field. In response to criticism[17] that calculated results were overstated, Becker *et al.* at first defended[18] the initial results. They then decided[19,20] that circular polarization indeed gave small effects, but that linear polarization, under some conditions, appeared to be much more promising. Criticisms[17,21] about the observability of laser enhancement of beta decay as proposed by Becker *et al.*[16] will be discussed in a later paper.

C. Plan of the paper

The basic formalism employed is stated in Sec. II. To ascertain the effect of an externally applied electromagnetic field on the internal coordinates of a nucleus, the nucleus is considered to consist of two parts: a "core" and a "fragment." The core is a stable subnucleus of zero total angular momentum, and the fragment contains the nucleon (or nucleons) which is a candidate for beta decay, plus any other nucleons which are angular momentum coupled to it in initial or final states. The equation of motion is then separated into center-of-mass (c.m.) and relative coordinate equations giving, respectively, the dynamical equations for the motion of the center of mass of the entire nucleus and the relative motion of the fragment with respect to the core. It is this latter equation which must be solved.

The theory of induced beta decay involves a coupling of the nuclear fragment both to the external electromagnetic field and to the weak (beta decay) interaction. The coupling constant of the weak interaction is very small. On the other hand, the coupling constant to the electromagnetic field is very much larger, particularly in view of the relatively large intensity of the applied field. Furthermore, the field can be regarded as being on for a time approaching infinity before and after the beta decay occurs. Therefore, the weak interaction is treated as a perturbation which causes a transition of the nucleus-plus-field system from one state to another. Since the combined nuclear-electromagnetic field system is explicitly time dependent, the standard derivation of the perturbation formalism of beta decay (based on stationary nuclear states) is not appropriate. That is, the Fermi Golden Rule is not valid. A derivation is presented which is applicable in the presence of explicit time dependence. The result has the standard form.

The perturbation theory just described requires a knowledge of the state vector for the nuclear fragment in the presence of the field. The interacting nuclear wave function employed is the momentum translation approximation.

The electron emitted in the beta decay does not appear until the decay has occurred, and so its interaction with the field might be thought to be of no consequence. However, the field intensity parameter associated with induced beta decay is so large (and the mass of the electron sufficiently small) that the onset of effective interaction of the electron with the field occurs on a shorter time scale than the Heisenberg uncertainty time of the beta decay interaction. The onset of field-electron interaction is also much faster than the transit time of the newly created beta particle across the nucleus. The electron is therefore represented by a Volkov wave function, which is an exact solution for a free charged particle in the presence of an electromagnetic field.

In Sec. III, a general expression for the transition probability for induced beta decay is developed. Coupling of the electromagnetic field to the beta particle causes the transition probability to split into three parts corresponding to the following: direct interaction of the field with the electron charge, interaction of the field with the spin of the electron, and an interference between the direct and spin terms. For the field intensities of interest here, the direct term and the spin term are of approximately equal importance for the more energetic beta decays, although the direct term dominates for low energy decays.

The final form for the transition probability per unit time, or equivalently, for the half-life for induced beta decay, is written for any order of forbiddenness which is to be overcome by the inducing field, and for any number of nucleons in the fragment.

The formalism developed in Sec. III is quite general. In Sec. IV, a much simplified theory is developed for the special case of very high intensity and very low frequency. The field intensity domain considered is one in which the order of magnitude of the intensity parameter describing field-nucleus interaction is in the neighborhood of unity. The algebraic form of the results for transition probability resemble conventional beta decay theory, but they differ in that the electron spectral integral contains effects of the applied field, and the interaction matrix elements also contain field effects. These matrix elements are generalizations of the usual Fermi and Gamow-Teller matrix elements.

D. Companion papers

The present paper is the first of a series of papers based largely on the physics developed in Ref. 14. The next paper in the series will present several explicit calculational examples showing how the formalism given in this first paper is applied. Each of the examples has a different degree of forbiddenness, ranging up to fourth forbidden. Another paper in the series will be confined to the case of allowed beta decay. There will also be a paper dedicated to some of the novel fundamental features introduced by the problem of low-frequency intense-field induced nuclear beta decay. Because this problem involves nontypical values for physical parameters like field frequency and intensity, several rules of thumb regarded as reliable in nuclear physics, atomic physics, and field theory are shown to be misleading when applied to the present problem. Considerations such as gauge transformations and invariance, Lorentz invariance, low frequency limits, and convergence of perturbation theory will be discussed. There will also be papers on unusual aspects of experiments involving induced beta decay, as well as some early experimental results.

II. BASIC FORMALISM

A. Separation of variables

In the cases of interest here, one can consider the initial nucleus to consist of a stable, relatively tightly bound core, plus a fragment of one or several nucleons outside the core. This fragment contains the nucleon which is a candidate for beta decay, plus any other nucleons which couple with it to provide the observed total angular momentum and parity of the nucleus. The core will always be such as to have spin and parity 0^+. For example, consider ^{90}Sr, which has 38 protons, 52 neutrons, and a total spin of zero and positive intrinsic parity ($J^\pi=0^+$). The core nucleus can be considered to be ^{88}Sr, which has 50 neutrons, $J^\pi=0^+$, and is the principal stable isotope of strontium. ^{88}Sr is particularly stable since the neutron number of 50 is a magic number, and the proton number of 38 corresponds to completed $p_{3/2}$ and $f_{5/2}$ shells beyond the magic number of 28. The fragment constituents of two neutrons in ^{90}Sr outside the ^{88}Sr core are both $d_{5/2}$ neutrons, coupled together to give an overall 0^+ state. One of these two neutrons will decay to a $p_{1/2}$ proton, which will couple with the remaining $d_{5/2}$ neutron to form a 2^- state in the daughter ^{90}Y nucleus.

The problem posed here is first to write a nonrelativistic Schrödinger equation for the nucleus in terms of the coordinates of core and fragment; then to introduce coordinates for the center of mass of the entire nucleus, and for the relative coordinate between the core and fragment portions; and finally to see if the Schrödinger equation can be separated into independent c.m. and relative coordinate equations.

Assign the subscript 1 to quantities (mass, charge, position vector) associated with the nuclear fragment, and subscript 2 to core quantities. The Schrödinger equation in $\vec{r}_1,\vec{r}_2$ coordinates, expressed in Coulomb gauge, is

$$i\partial_t\psi(\vec{r}_1,\vec{r}_2)=\left[\frac{1}{2m_1}(-i\vec{\nabla}_1-e_1\vec{A})^2+\frac{1}{2m_2}(-i\vec{\nabla}_2-e_2\vec{A})^2+V(|\vec{r}_1-\vec{r}_2|)\right]\psi(\vec{r}_1,\vec{r}_2)\,, \quad (1)$$

where, for simplicity, a central potential has been assumed to represent the binding between fragment and core. This equation is written in so-called "natural" units ($\hbar=c=1$) in the Gaussian system. The electromagnetic field occurs in the equation through a vector potential $\vec{A}(t)$, presumed to be sinusoidal in time. The vector potential has the same form as a plane wave treated in long-wavelength approximation.[22]

The transformation to the relative coordinates $\vec{r}$ and center-of-mass (c.m.) coordinates $\vec{R}$ is accomplished by

$$\vec{r}=\vec{r}_1-\vec{r}_2\,,$$
$$(m_1+m_2)\vec{R}=m_1\vec{r}_1+m_2\vec{r}_2\,,$$

which has the inverse

$$\vec{r}_1=\vec{R}+\frac{m_2}{m_1+m_2}\vec{r}\,,$$
$$\vec{r}_2=\vec{R}-\frac{m_1}{m_1+m_2}\vec{r}\,.$$

Substitution of these quantities in Eq. (1) gives an equation which can be separated upon introduction of the product wave function $\psi=\psi_r(\vec{r})\psi_R(\vec{R})$ to give[23]

$$i\partial_t\psi_R=\frac{1}{2m_t}(-i\vec{\nabla}_R-e_t\vec{A}(t))^2\psi_R\,, \quad (2)$$

$$i\partial_t\psi_r=\left[\frac{1}{2m_r}(-i\vec{\nabla}_r-\tilde{e}\vec{A}(t))^2+V(r)\right]\psi_r\,, \quad (3)$$

where m_t and e_t are the total mass and total charge

$$m_t=m_1+m_2,\quad e_t\equiv e_1+e_2\,, \quad (4)$$

and m_r and $\tilde{e}$ are the reduced mass and reduced charge

$$m_r\equiv\frac{m_1m_2}{m_1+m_2},\quad \tilde{e}\equiv\frac{e_1m_2-e_2m_1}{m_1+m_2}\,. \quad (5)$$

The Schrödinger equation for c.m. motion, Eq. (2), is exactly what one would expect for a system of mass m_t, charge e_t subjected to the potentials $\vec{A}$. The dynamical equation for the internal nuclear motion, Eq. (3), is less obvious. The mass is the usual reduced mass m_r, but the charge is the less familiar $\tilde{e}$. Were this separation of variables done for a single-electron atomic problem, then $e_1=-e$, $e_2=e$, and so $\tilde{e}=-e$ as usual. For the present nuclear problem, let ζ and ν be the number of protons and of neutrons, respectively, in the nuclear fragment, and let Z and N be the same numbers for the entire nucleus. Then

$$\tilde{e}=e\frac{(\zeta N-\nu Z)}{A}\,, \qquad (6)$$

where $A=Z+N$ is the total nuclear mass number.

The implication of Eq. (6) is that the fragment behaves as if it has a positive charge when there is a preponderance of protons in the fragment, a negative charge when neutrons predominate, and a near-zero charge when equal numbers of protons and neutrons exist in the nuclear fragment. Note, in particular, that a fragment consisting of two neutrons (an important special case) has an effective charge of approximately $-e$. That is, the coupling of the relative motion of the nuclear fragment to the electromagnetic field has about the magnitude associated with a full proton charge even though the fragment consists entirely of neutral nucleons.

B. S-matrix formalism

In the derivation to be presented below of the beta decay transition probability as induced by an applied electromagnetic field, it is appropriate to view the asymptotic states as states which contain the full influence of the applied field, and the transition-causing perturbation will be the beta decay interaction. This means that the asymptotic states are explicitly time dependent, and not the stationary states normally employed. The Fermi Golden Rule thus cannot be used. A suitable transition formalism is given below, based on an S-matrix approach within a Hamiltonian formalism.

The derivation will be accomplished without consideration of explicit Dirac space properties. They can be inserted at the end. The equation of motion for the unperturbed system is

$$(i\partial_t-H_0)\phi=0$$

with a Green's operator satisfying

$$(i\partial_t-H_0)G(t,t_0)=\delta(t-t_0)1\,.$$

The equation of motion for the complete system including the perturbation V is

$$(i\partial_t-H_0-V)\psi=0\,.$$

In these expressions, H_0, V, G are operators, ϕ, ψ, 1 are vectors, t is a parameter external to the Hilbert space, and H_0 and V may both be time dependent. The desired S-matrix element for transitions caused by V can be defined by

$$S_{fi}=\lim_{t\to\infty}(\phi_f,\psi_i^{(+)})\,,$$

which is the probability amplitude that a complete in-state with initial quantum numbers (indicated by subscript i) will evolve into a particular noninteracting final state (subscript f).

The Green's operators are explicitly

$$G^{(+)}(t,t_0)=-i\theta(t-t_0)\sum_{\alpha}|\alpha,t\rangle\langle\alpha,t_0|\,,$$

$$G^{(-)}(t,t_0)=i\theta(t_0-t)\sum_{\alpha}|\alpha,t\rangle\langle\alpha,t_0|\,,$$

where it is convenient to introduce Dirac bra-ket notation for these expressions with the correspondence to the earlier vector notation that

$$|\alpha,t\rangle\leftrightarrow\phi_\alpha\,,$$

where $\{\alpha\}$ is the set of all quantum numbers. The correspondence between these retarded and advanced operators is

$$G^{(-)}(t,t_0)=G^{(+)\dagger}(t_0,t)\,,$$

and their action on the asymptotic states is

$$G^{(+)}(t,t_0)\phi_\alpha(t_0)=-i\theta(t-t_0)\phi_\alpha(t)\,,$$

$$G^{(-)}(t,t_0)\phi_\alpha(t_0)=i\theta(t_0-t)\phi_\alpha(t)\,.$$

The formal solution for the interacting state ψ is

$$\psi^{(\pm)}(t)=\phi(t)+\int_{-\infty}^{t}dt_1G^{(\pm)}(t,t_1)V(t_1)\psi^{(\pm)}(t_1)\,.$$

By direct substitution, the S-matrix element is

$$S_{fi}=\lim_{t\to\infty}(\phi_f,\phi_i)+\lim_{t\to\infty}\int_{-\infty}^{t}dt_1(\phi_f(t),G^{(+)}(t,t_1)V(t_1)\psi_i^{(+)}(t_1))$$
$$=\delta_{fi}+\lim_{t\to\infty}\int_{-\infty}^{t}dt_1(G^{(-)}(t_1,t)\phi_f(t),V(t_1)\psi_i^{(+)}(t_1))\,.$$

From the property that $G^{(-)}(t_1,t)$ propagates $\phi_f(t)$ to $\phi_f(t_1)$, and the fact that

$$\lim_{t\to\infty}\theta(t-t_1)=1\,,$$

it follows that

$$S_{fi}=\delta_{fi}-i\int_{-\infty}^{\infty}dt_1(\phi_f(t_1),V(t_1)\psi_i^{(+)}(t_1))\,,$$

and also

$$S_{fi}=\delta_{fi}-i\int_{-\infty}^{\infty}dt_1(\phi_f(t_1),V(t_1)\phi_i(t_1))$$
$$-i\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2(\phi_f(t_1),V(t_1)G^{(+)}(t_1,t_2)V(t_2)\psi_i^{(+)}(t_2)) .$$

To lowest order in V, the result is simply

$$S_{fi}\approx\delta_{fi}-i\int_{-\infty}^{\infty}dt(\phi_f,V\phi_i) ,$$

which is a standard result, even though here the states ϕ_f,ϕ_i are themselves explicitly time dependent.

To apply this result to beta decay, the following correspondence will be made:

$$\phi_f\to\Psi_f\Psi^{(e)},\quad \phi_i\to\Psi_i\Psi^{(\nu)} ,$$
$$V\to\frac{G}{2^{1/2}}\gamma^{\mu}(1-\kappa\gamma^5) ,$$

where Ψ_f,Ψ_i are nuclear states in the presence of the applied field, $\Psi^{(e)}$ and $\Psi^{(\nu)}$ are electron and neutrino states, also in the applied field; the products like $\Psi_f\Psi^{(e)}$ are meant to imply direct products of the respective spinor spaces; and the expression for V acts in both spinor spaces, but with a value of $\kappa=1$ in the lepton space. The empirical value for κ in the nuclear space is $\kappa=1.23$. The factor $2^{(-1/2)}$ attached to the Fermi coupling constant is of historical origin in beta decay theory. Since δ_{fi} is irrelevant in a real transition, the final form for the S-matrix element is

$$S_{fi}=-i\frac{G}{2^{1/2}}\int d^4x[\overline{\Psi}_f\gamma_\mu(1-\kappa\gamma^5)\Psi_i]\times[\overline{\Psi}^{(e)}\gamma^{\mu}(1-\gamma^5)\Psi^{(\nu)}] . \quad (7)$$

This has precisely the appearance of the standard result, except that here it must be remembered that the nuclear and leptonic states are states containing the full effects of the applied electromagnetic field.

C. Interacting nuclear states

The calculational procedure developed above for induced beta emission is to substitute wave functions including the effects of the applied electromagnetic field. The formalism is otherwise the standard beta decay calculation. The nuclear wave function to be used must represent the effects of the applied field to an order of interaction which is at least as large as the order of forbiddenness of the natural beta decay. It must also be valid in the presence of electromagnetic fields of such intensity that the convergence of conventional perturbation theory is suspect. A technique ideally suited to the present problem is the momentum translation approximation.[24]

The momentum translation expression for the nuclear wave function in interaction with the electromagnetic field is

$$\Psi(\vec{r},t)=\exp(ie\vec{A}\cdot\vec{r})\Phi(\vec{r},t) , \quad (8)$$

where $\Phi(\vec{r},t)$ is the nuclear wave function with no electromagnetic field. Validity conditions for the approximation in Eq. (8) are[24]

$$eaR_0\omega/E\ll 1 , \quad (9)$$

$$\omega R_0\ll 1 , \quad (10)$$

where a is the amplitude of $\vec{A}$, R_0 is the nuclear radius, ω is the energy of a photon of the applied field, and E is the total nuclear transition energy. For optimal transition probability, eaR_0 should be of order unity. Also, ω/E will be many orders of magnitude less than unity. Equation (9) is thus easily satisfied. Equation (10) states essentially that the ratio of the nuclear radius to the wavelength of the applied field is very small, which is amply satisfied for all fields of possible interest. One further condition for applicability of the momentum translation approximation is that no intermediate nuclear states are accessible through interaction with a small number of applied-field photons. This is certainly not possible here. Hence, Eq. (8) is an excellent approximation to employ.

With the standard product solution for the noninteracting wave function

$$\Phi(\vec{r},t)=\psi(\vec{r})e^{-iEt} ,$$

the initial nuclear wave function in the presence of the field is, from Eq. (8),

$$\Psi_i(\vec{r},t)=e^{i\tilde{e}_i\vec{A}\cdot\vec{r}}\psi_i(\vec{r})e^{-iE_it} , \quad (11)$$

and the final nuclear wave function to be used is

$$\Psi_f(\vec{r},t)=e^{i\tilde{e}_f\vec{A}\cdot\vec{r}}\psi_f(\vec{r})e^{-iE_ft} . \quad (12)$$

The reduced charges $\tilde{e}_i$ and $\tilde{e}_f$ are the appropriate forms of Eq. (5) or (6), and $\psi_i(\vec{r})$, $\psi_f(\vec{r})$ are stationary state nuclear wave functions with no field present.

D. Interacting lepton states

The leptons emitted in β^- decay are an electron and an antineutrino.[25] The antineutrino is uncharged, and possesses no coupling to the electromagnetic field. The antineutrino is therefore described by an ordinary free-particle wave function. The emitted antineutrino is treated as a neutrino in the initial state with reversed four-momentum, i.e.,

$$\Psi^{(\nu)}=\frac{1}{(2E_{(\nu)}V)^{1/2}}u^{(\nu)}(k_{(\nu)},s_{(\nu)})e^{ik_{(\nu)}\cdot x} . \quad (13)$$

In Eq. (13), $k_{(\nu)}$ is the four-momentum with time part $E_{(\nu)}$, $u^{(\nu)}$ is a spinor, $s_{(\nu)}$ is the spin parameter, and V is the normalization volume. The scalar product indicated in the exponential is a four-vector product

$$k_{(\nu)}\cdot x=k^{\mu}_{(\nu)}x_{\mu}$$
$$=E_{(\nu)}t-\vec{k}_{(\nu)}\cdot\vec{r} .$$

The electron emitted in beta decay is a charged particle whose coupling to the electromagnetic field is very significant when the field intensity is high. In ordinary beta decay theory, the electron is treated as a free particle, although Coulomb corrections are sometimes introduced. In the present situation, the free particle electron solution is replaced by the Volkov solution,[26] which is an exact wave function for a free, charged particle in the presence of a plane wave electromagnetic field. The equation whose solution is required is the Dirac equation[27]

$$(i\partial\!\!\!/+eA\!\!\!/-m)\Psi^{(e)}=0 , \quad (14)$$

which has the solution (the Volkov solution)

$$\Psi^{(e)}=\left[\frac{m}{E_eV}\right]^{1/2}\exp\left[-ip_e\cdot x+\frac{i}{2p_e\cdot k}\int_{(k\cdot x)_0}^{(k\cdot x)}d(k\cdot x)'(2ep_e\cdot A+e^2A^2)\right]\left[1-\frac{e}{2p_e\cdot k}k\!\!\!/A\!\!\!/\right]u^{(e)}(p_e,s_e) , \quad (15)$$

valid when $A^{\mu}(x)$ represents any arbitrary packet of plane wave components with a common $\vec{k}$ direction of propagation. In Eqs. (14) and (15), the slash notation is defined to mean $A\!\!\!/\equiv\gamma^{\mu}A_{\mu}$; p_e is a constant four-vector with time part E_e which satisfies the mass-shell condition $p_e{}^2=m^2$; k^{μ} is the propagation four-vector for the electromagnetic field which is lightlike, or $k^2=0$; A^{μ} is the four-vector potential of the transverse electromagnetic field, with the transversality condition giving $k\cdot A=0$, and with A^{μ} a function only of the phase $k\cdot x$; and where $u^{(e)}$ is a spinor satisfying the condition $(p\!\!\!/_e-m)u^{(e)}=0$, and is a function both of p_e^{μ} and of the spin parameter s_e.

The circumstances which the Volkov solution are to describe are that the electron suddenly appears at some time (say $t=0$) in an electromagnetic field which has been on for a long time prior to the creation of the electron. It is thus appropriate to consider the field to be monochromatic. The transient response of the electron is contained in the Volkov solution, arising as a result of the choice $(k\cdot x)_0=0$ for the lower limit of the integral in Eq. (15). The electromagnetic field is specified as

$$A^{\mu}=a\epsilon^{\mu}\cos(k\cdot x+\rho) , \quad (16)$$

where ρ is a phase shift reflecting the fact that the beta decay cannot be expected to occur in phase with the field. The polarization four-vector ϵ^{μ} in Eq. (16) has the scalar invariant $\epsilon^2=-1$. Equation (16) will be converted to its long-wavelength approximation form later. The end result of using Eq. (16) in Eq. (15), with $(k\cdot x)_0=0$, gives

$$\Psi^{(e)}=\left[\frac{m}{E_eV}\right]^{1/2}\exp\{-i[p_e\cdot x+\eta k\cdot x+\zeta\sin(k\cdot x+\rho)+\tfrac{1}{2}\eta\sin2(k\cdot x+\rho)-\zeta\sin\rho-\tfrac{1}{2}\eta\sin2\rho]\}$$

$$\times\left[1-\frac{e}{2p_e\cdot k}k\!\!\!/A\!\!\!/\right]u^{(e)}(p_e,s_e) , \quad (17)$$

with the definitions

$$\zeta\equiv-\frac{eap_e\cdot\epsilon}{p_e\cdot k},\quad \eta\equiv\frac{e^aa^2}{4p_e\cdot k} . \quad (18)$$

(The minus sign is introduced in the definition of ζ to account for the fact that a gauge with $\epsilon^0=0$ will be used, in which case $p_e\cdot\epsilon=-\vec{p}_e\cdot\vec{\epsilon}$.)

III. TRANSITION PROBABILITY

A. Squared S matrix

The starting point is the S-matrix element, Eq. (7), which gives

$$|S_{fi}|^2=\frac{G^2}{2}\int d^4x\int d^4x'[\overline{\Psi}_f(x)\gamma_\mu(1-\kappa\gamma^5)\Psi_i(x)][\overline{\Psi}^{(e)}(x)\gamma^\mu(1-\gamma^5)\Psi^{(\nu)}]$$
$$\times[\overline{\Psi}^{(\nu)}(x')(1+\gamma^5)\gamma^\nu\Psi^{(e)}(x')][\overline{\Psi}_i(x')(1+\kappa\gamma^5)\gamma_\nu\Psi_f(x')] .$$

The initial and final nuclear states are given in Eqs. (11) and (12), the antineutrino state comes from Eq. (13), and Eq. (17) specifies the electron state. With these substitutions, the squared S-matrix element is

$$|S_{fi}|^2=\frac{G^2m}{4E_eE_{(\nu)}V^2}\int d^4x\int d^4x'\exp\{i[(E_f-E_i)(t-t')-(\tilde{e}_f-\tilde{e}_i)(\vec{A}\cdot\vec{r}-\vec{A}'\cdot\vec{r}')+(p_e+k_{(\nu)})\cdot(x-x')$$
$$+\eta k\cdot(x-x')+\zeta\sin(k\cdot x+\rho)-\zeta\sin(k\cdot x'+\rho)$$
$$+\tfrac{1}{2}\eta\sin2(k\cdot x+\rho)-\tfrac{1}{2}\eta\sin2(k\cdot x'+\rho)]\}$$
$$\times[\overline{\psi}_f(\vec{r})\gamma_\mu(1-\kappa\gamma^5)\psi_i(\vec{r})]\left[\overline{u}^{(e)}\left[1-\frac{e}{2p_e\cdot k}A\not k\right]\gamma^\mu(1-\gamma^5)u^{(\nu)}\right]$$
$$\times\left[\overline{u}^{(\nu)}(1+\gamma^5)\gamma^\nu\left[1-\frac{e}{2p_e\cdot k}\not k A'\right]u^{(e)}\right][\overline{\psi}_i(\vec{r}')(1+\kappa\gamma^5)\gamma_\nu\psi_f(\vec{r}')] , \tag{19}$$

where the notation A' refers to the fact that the argument bears a prime; i.e., $A^{\mu'}\equiv A^\mu(x')$, $\vec{A}'\equiv\vec{A}(x')$. There is no selection of final spin states, so a sum over the spins of electron and antineutrino will be carried out. This is accomplished with the theorems

$$\sum_{s_{(\nu)}}u^{(\nu)}\overline{u}^{(\nu)}=\not k_{(\nu)} ,$$
$$\sum_{s_e}u^{(e)}\overline{u}^{(e)}=\frac{1}{2m}(\not p_e+m) . \tag{20}$$

The definition is now introduced that

$$\mathscr{M}_{\mu\nu}\mathscr{L}^{\mu\nu}=\frac{m}{4E_eE_{(\nu)}}\sum_{s_{(\nu)}}\sum_{s_e}[\overline{\psi}_f(\vec{r})\gamma_\mu(1-\kappa\gamma^5)\psi_i(\vec{r})]\left[\overline{u}^{(e)}\left[1-\frac{e}{2p_e\cdot k}A\not k\right]\gamma^\mu(1-\gamma^5)u^{(\nu)}\right]$$
$$\times\left[\overline{u}^{(\nu)}(1+\gamma^5)\gamma^\nu\left[1-\frac{e}{2p_e\cdot k}\not k A'\right]u^{(e)}\right][\overline{\psi}_i(\vec{r}')(1+\kappa\gamma^5)\gamma_\nu\psi_f(\vec{r}')] , \tag{21}$$

where the nuclear part is

$$\mathscr{M}_{\mu\nu}=[\overline{\psi}_f(r)\gamma_\mu(1-\kappa\gamma^5)\psi_i(\vec{r})][\overline{\psi}_i(\vec{r}')(1+\kappa\gamma^5)\gamma_\nu\psi_f(\vec{r}')] , \tag{22}$$

and the leptonic part is

$$\mathscr{L}^{\mu\nu}=\tfrac{1}{4}\mathrm{Tr}\left[\left[1-\frac{e}{2p_e\cdot k}A\not k\right]\gamma^\mu(1-\gamma^5)\frac{\not k_{(\nu)}}{E_{(\nu)}}(1+\gamma^5)\gamma^\nu\left[1-\frac{e}{2p_e\cdot k}\not k A\right]\left[\frac{\not p_e+m}{2E_e}\right]\right] . \tag{23}$$

The spin sum results (20) have been used in Eq. (23). In the combination $\not p_e+m$, the m term will not contribute, since the trace of the product of an odd number of Dirac matrices always vanishes. The expression $\mathscr{L}^{\mu\nu}$ will be treated in three parts defined by

$$\mathscr{L}_1^{\mu\nu}=\frac{1}{8E_eE_{(\nu)}}\text{Tr}[\gamma^\mu(1-\gamma^5)\not k_{(\nu)}(1+\gamma^5)\gamma^\nu\not p_e] , \quad (24)$$

$$\mathscr{L}_2^{\mu\nu}=\frac{1}{8E_eE_{(\nu)}}\left[\frac{e}{2p_e\cdot k}\right]^2\text{Tr}[\not A\not k\gamma^\mu(1-\gamma^5)\not k_{(\nu)}(1+\gamma^5)\gamma^\nu\not k\not A'p_e] , \quad (25)$$

$$\mathscr{L}_3^{\mu\nu}=\frac{1}{8E_eE_{(\nu)}}\left[-\frac{e}{2p_e\cdot k}\right]\text{Tr}[\gamma^\mu(1-\gamma^5)\not k_{(\nu)}(1+\gamma^5)\gamma^\nu\not k\not A'p_e+\not A\not k\gamma^\mu(1-\gamma^5)\not k_{(\nu)}(1+\gamma^5)\gamma^\nu\not p_e] , \quad (26)$$

with

$$\mathscr{L}^{\mu\nu}=\mathscr{L}_1^{\mu\nu}+\mathscr{L}_2^{\mu\nu}+\mathscr{L}_3^{\mu\nu} . \quad (27)$$

The quantity $\mathscr{M}_{\mu\nu}$ is a squared nuclear transition current, and $\mathscr{L}^{\mu\nu}$ is a squared leptonic transition current. The separation of $\mathscr{L}^{\mu\nu}$ into three terms, as in Eqs. (24)–(27), has physical meaning. The Volkov solution, as it appears in Eq. (15) or (17), contains the factor

$$(1-e\not k\not A/2p_e\cdot k) .$$

$\mathscr{L}_1^{\mu\nu}$ arises from the square of the first term in this factor, $\mathscr{L}_2^{\mu\nu}$ comes from the square of the second term in the factor, and the cross terms give $\mathscr{L}_3^{\mu\nu}$. The term

$$-e\not k\not A/2p_e\cdot k$$

represents a spin interaction. One way to see this is to note that

$$\not k\not A=-i\sigma^{\mu\nu}k_\mu A_\nu ,$$

where

$$\sigma^{\mu\nu}\equiv(i/2)(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)$$

is the four-dimensional spin operator. The Volkov solution for a scalar (spin zero) particle has exactly the same exponential function as Eq. (15), but it lacks the $\not k\not A$ term. Thus, the portion of the squared S matrix associated with $\mathscr{L}_1^{\mu\nu}$ will be called the "direct" part, $\mathscr{L}_2^{\mu\nu}$ gives rise to the "spin" part, and $\mathscr{L}_3^{\mu\nu}$ is the "interference" term. Transition probabilities coming from the direct and spin terms must always be positive, but the interference term can be of either sign, corresponding to constructive or destructive interference between the direct and spin terms.

B. Squared transition current

1. Direct term

The $\mathscr{L}_1^{\mu\nu}$ expression in Eq. (24) can be simplified by using

$$(1-\gamma^5)\not k_{(\nu)}(1+\gamma^5)=2(1-\gamma^5)\not k_{(\nu)} , \quad (28)$$

which comes from facts that γ^5 anticommutes with all the γ^μ, and that

$$(1-\gamma^5)^2=2(1-\gamma^5) .$$

The trace expression to be evaluated is straightforward and gives the result

$$\text{Tr}[\gamma^\mu(1-\gamma^5)\not k_{(\nu)}\gamma^\nu\not p_e]=4(k_{(\nu)}^\mu p_e^\nu-g^{\mu\nu}k_{(\nu)}\cdot p_e+k_{(\nu)}^\nu p_e^\mu+i\epsilon^{\mu\rho\nu\lambda}k_{(\nu)\rho}p_{e\lambda}) ,$$

where the part containing the completely antisymmetric tensor $\epsilon^{\mu\rho\nu\lambda}$ comes from the γ^5 term. The squared leptonic current will be written as the sum of two parts,

$$\mathscr{L}_1^{\mu\nu}=\mathscr{L}_{1a}^{\mu\nu}+\mathscr{L}_{1b}^{\mu\nu} ,$$

where

$$\mathscr{L}_{1a}^{\mu\nu}=\frac{1}{E_eE_{(\nu)}}(k_{(\nu)}^\mu p_e^\nu-g^{\mu\nu}k_{(\nu)}\cdot p_e+k_{(\nu)}^\nu p_e^\mu) , \quad (29)$$

$$\mathscr{L}_{1b}^{\mu\nu}=\frac{i}{E_eE_{(\nu)}}\epsilon^{\mu\rho\nu\lambda}k_{(\nu)\rho}p_{e\lambda} . \quad (30)$$

The lepton factors $\mathscr{L}_1^{\mu\nu}$ are to be combined with the nuclear part defined in Eq. (22). The nuclear part can be used in the nonrelativistic limit, which gives

$$\bar\psi_f\gamma^0\psi_i=\psi_f^\dagger\psi_i ,$$
$$\bar\psi_f\vec\gamma\psi_i\approx 0 ,$$
$$\bar\psi_f\gamma^0\gamma^5\psi_i\approx 0 , \quad (31)$$
$$\bar\psi_f\vec\gamma\gamma^5\psi_i=\psi_f^\dagger\vec\sigma\psi_i .$$

This nonrelativistic nuclear approximation leads to

$$\mathcal{M}^{\mu\nu}\mathcal{L}_{1\mu\nu}\approx[\psi_f^\dagger(\vec{r})\psi_i(\vec{r})][\psi_i^\dagger(\vec{r}\,')\psi_f(\vec{r}\,')]\mathcal{L}_{100}+\kappa^2[\psi_f^\dagger(\vec{r})\sigma^j\psi_i(\vec{r})][\psi_i^\dagger(\vec{r}\,')\sigma^l\psi_f(\vec{r}\,')]\mathcal{L}_{1jl}$$
$$-\kappa[\psi_f^\dagger(\vec{r})\psi_i(\vec{r})][\psi_i^\dagger(\vec{r}\,')\sigma^j\psi_f(\vec{r}\,')]\mathcal{L}_{10j}-\kappa[\psi_f^\dagger(\vec{r})\sigma^j\psi_i(\vec{r})][\psi_i^\dagger(\vec{r}\,')\psi_f(\vec{r}\,')]\mathcal{L}_{1j0}\,.$$

The nuclei are unpolarized, and an average over nuclear spins causes the terms linear in κ to vanish, and leads to

$$\sigma^j\sigma^l\rightarrow\tfrac{1}{3}\sigma^j\sigma^j\,.$$

Averaging over nuclear spins thus gives

$$\mathcal{M}^{\mu\nu}\mathcal{L}_{1\mu\nu}\approx[\psi_f^\dagger(\vec{r})\psi_i(\vec{r})][\psi_i^\dagger(\vec{r}\,')\psi_f(\vec{r}\,')]\mathcal{L}_{100}+\tfrac{1}{3}\kappa^2[\psi_f^\dagger(\vec{r})\sigma^j\psi_i(\vec{r})][\psi_i^\dagger(\vec{r}\,')\sigma^j\psi_f(\vec{r}\,')]\mathcal{L}_{1jj}\,. \tag{32}$$

Recall that $\mathcal{L}_{1\mu\nu}$ as given in Eqs. (29) and (30) contains $\mathcal{L}_{1b}^{\mu\nu}$, involving a factor $\epsilon^{\mu\rho\nu\lambda}$. Because of this antisymmetric factor, $\mathcal{L}_{1b}^{\mu\nu}$ makes no contribution to either $\mathcal{L}_{100}$ or $\mathcal{L}_{1jj}$, and so these terms are easily evaluated as

$$\mathcal{L}_{100}=\frac{1}{E_eE_{(\nu)}}(2E_{(\nu)}E_e-k_{(\nu)}\cdot p_e)=\left[1+\frac{\vec{p}_e\cdot\vec{k}_{(\nu)}}{E_eE_{(\nu)}}\right],$$

$$\mathcal{L}_{1jj}=\frac{1}{E_eE_{(\nu)}}(2\vec{k}_{(\nu)}\cdot\vec{p}_e+3k_{(\nu)}\cdot p_e)=\left[3-\frac{\vec{p}_e\cdot\vec{k}_{(\nu)}}{E_eE_{(\nu)}}\right].$$

The $\mathcal{M}_{\mu\nu}\mathcal{L}_1^{\mu\nu}$ product can then be written as

$$\mathcal{M}_{\mu\nu}\mathcal{L}_1^{\mu\nu}\approx\Big\{[(\psi_f^\dagger\psi_i)_{\vec{r}}(\psi_f^\dagger\psi_i)_{\vec{r}\,'}^\dagger+\kappa^2(\psi_f^\dagger\vec{\sigma}\psi_i)_{\vec{r}}\cdot(\psi_f^\dagger\vec{\sigma}\psi_i)_{\vec{r}\,'}^\dagger]$$
$$+[(\psi_f^\dagger\psi_i)_{\vec{r}}(\psi_f^\dagger\psi_i)_{\vec{r}\,'}^\dagger-\tfrac{1}{3}\kappa^2(\psi_f^\dagger\vec{\sigma}\psi_i)_{\vec{r}}\cdot(\psi_f^\dagger\vec{\sigma}\psi_i)_{\vec{r}\,'}^\dagger]\frac{\vec{p}_e\cdot\vec{k}_{(\nu)}}{E_eE_{(\nu)}}\Big\}\,. \tag{33}$$

2. *Spin term*

From Eqs. (25) and (28), $\mathcal{L}_2^{\mu\nu}$ is the sum of two parts

$$\mathcal{L}_2^{\mu\nu}=\mathcal{L}_{2a}^{\mu\nu}+\mathcal{L}_{2b}^{\mu\nu}\,,$$

where

$$\mathcal{L}_{2a}^{\mu\nu}=\frac{1}{4E_eE_{(\nu)}}\left[\frac{e}{2p_e\cdot k}\right]^2\mathrm{Tr}[\not{A}\not{k}\gamma^\mu\not{k}_{(\nu)}\gamma^\nu\not{k}\not{A}'\not{p}_e]\,, \tag{34}$$

$$\mathcal{L}_{2b}^{\mu\nu}=-\frac{1}{4E_eE_{(\nu)}}\left[\frac{e}{2p_e\cdot k}\right]^2\mathrm{Tr}[\not{A}\not{k}\gamma^\mu\gamma^5\not{k}_{(\nu)}\gamma^\nu\not{k}\not{A}'\not{p}_e]\,. \tag{35}$$

Some reduction in these forms follows immediately from using the light cone condition $k^2=0$, and the transversality condition $k\cdot A=k\cdot A'=0$. A little Dirac matrix algebra gives

$$\not{k}\not{A}'\not{p}_e\not{A}\not{k}=-2p_e\cdot k\not{A}'\not{A}\not{k}\,. \tag{36}$$

With Eq. (36), Eq. (34) involves evaluating the trace of a product of six Dirac matrices. The result is

$$\mathcal{L}_{2a}^{\mu\nu}=-\frac{e^2}{2p_e\cdot k}\frac{1}{E_eE_{(\nu)}}[A\cdot A'(k^\mu k_{(\nu)}^\nu+k^\nu k_{(\nu)}^\mu-g^{\mu\nu}k\cdot k_{(\nu)})$$
$$+k_{(\nu)}\cdot A(k^\mu A'^\nu-k^\nu A'^\mu)+k_{(\nu)}\cdot A'(k^\nu A^\mu-k^\mu A^\nu)+k_{(\nu)}\cdot k(A'^\mu A^\nu-A^\mu A'^\nu)]\,. \tag{37}$$

The trace appearing in Eq. (35) contains a γ^5 factor, which involves a product of four Dirac matrices through its definition

$$\gamma^5=\frac{i}{4!}\epsilon_{\mu\nu\rho\lambda}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda .$$

Thus, even after reduction with the aid of Eq. (36), the trace of a product of ten Dirac matrices must be accomplished. The result is

$$\begin{aligned}\mathscr{L}_{2b}^{\mu\nu}=-\frac{e^2}{2p_e\cdot k}\frac{i}{E_eE_{(\nu)}}[&\epsilon_{\alpha\beta\gamma\delta}g^{\mu\nu}A^\alpha A'^\beta k_{(\nu)}^\gamma k^\delta-\epsilon^{\mu\nu}{}_{\alpha\beta}k\cdot k_{(\nu)}A'^\alpha A^\beta\\ &-\epsilon^{\mu\nu}{}_{\alpha\beta}A\cdot A'k_{(\nu)}^\alpha k^\beta+\epsilon^{\mu\nu}{}_{\alpha\beta}(k_{(\nu)}\cdot AA'^\alpha-k_{(\nu)}\cdot A'A^\alpha)k^\beta\\ &+(\epsilon^\mu{}_{\alpha\beta\gamma}k^\nu-\epsilon^\nu{}_{\alpha\beta\gamma}k^\mu)k_{(\nu)}^\alpha A'^\beta A^\gamma-(\epsilon^\mu{}_{\alpha\beta\gamma}k_{(\nu)}^\nu+\epsilon^\nu{}_{\alpha\beta\gamma}k_{(\nu)}^\mu)A^\alpha A'^\beta k^\gamma\\ &+\epsilon^\mu{}_{\alpha\beta\gamma}(A^\nu A'^\alpha-A'^\nu A^\alpha)k_{(\nu)}^\beta k^\gamma-\epsilon^\nu{}_{\alpha\beta\gamma}(A^\mu A'^\alpha-A'^\mu A^\alpha)k_{(\nu)}^\beta k^\gamma] .\end{aligned} \tag{38}$$

These expressions for $\mathscr{L}_2^{\mu\nu}$ are to be combined with the squared nuclear transition current $\mathscr{M}_{\mu\nu}$ as given in Eqs. (22) and (31). The result is of the form of Eq. (32), but with $\mathscr{L}_2^{\mu\nu}$ components in place of $\mathscr{L}_1^{\mu\nu}$. A gauge for the electromagentic field will be used uniformly hereafter in which

$$\epsilon^\mu=(0,\vec{\epsilon}) , \tag{39}$$

that is, the scalar potential associated with the plane wave vanishes. The terms needed for the $\mathscr{L}_2^{\mu\nu}$ analog of Eq. (32) are, from Eqs. (37) and (38),

$$\begin{aligned}\mathscr{L}_{2a00}&=\frac{e^2}{2p_e\cdot k}\frac{\omega}{E_e}\vec{A}\cdot\vec{A}'\left[1+\frac{\vec{k}\cdot\vec{k}_{(\nu)}}{\omega E_{(\nu)}}\right] ,\\ \mathscr{L}_{2ajj}&=\frac{e^2}{2p_e\cdot k}\frac{3\omega}{E_e}\vec{A}\cdot\vec{A}'\left[1-\frac{\vec{k}\cdot\vec{k}_{(\nu)}}{3\omega E_{(\nu)}}\right] ,\\ \mathscr{L}_{2b00}&=\frac{ie^2}{2p_e\cdot k}\frac{\omega}{E_e}\left[\frac{\vec{k}}{\omega}\cdot\vec{A}\times\vec{A}'-\frac{\vec{k}_{(\nu)}}{E_{(\nu)}}\cdot\vec{A}\times\vec{A}'\right] ,\\ \mathscr{L}_{2bjj}&=\frac{ie^2}{2p_e\cdot k}\frac{3\omega}{E_e}\left[\frac{\vec{k}}{\omega}\cdot\vec{A}\times\vec{A}'+\frac{\vec{k}_{(\nu)}}{3E_{(\nu)}}\cdot\vec{A}\times\vec{A}'\right] .\end{aligned} \tag{40}$$

When $\vec{A}$ is defined by a combination of Eqs. (16) and (39), then the overall squared transition current for the spin term is

$$\mathscr{M}_{\mu\nu}\mathscr{L}_2^{\mu\nu}=\frac{2\eta\omega}{E_e}\cos(k\cdot x+\rho)\cos(k\cdot x'+\rho)\mathscr{A}_2 , \tag{41}$$

where

$$\mathscr{A}_2=(\psi_f^\dagger\psi_i)_{\vec{r}}(\psi_f^\dagger\psi_i)_{\vec{r}'}^\dagger\left[1+\frac{\vec{k}\cdot\vec{k}_{(\nu)}}{\omega E_{(\nu)}}\right]+\kappa^2(\psi_f^\dagger\vec{\sigma}\psi_i)_{\vec{r}}\cdot(\psi_f^\dagger\vec{\sigma}\psi_i)_{\vec{r}'}^\dagger\left[1-\frac{\vec{k}\cdot\vec{k}_{(\nu)}}{3\omega E_{(\nu)}}\right] . \tag{42}$$

Equation (42) has exactly the same form as the right-hand side of Eq. (33), except for the replacement of $\vec{p}_e\cdot\vec{k}_{(\nu)}/E_eE_{(\nu)}$ by $\vec{k}\cdot\vec{k}_{(\nu)}/\omega E_{(\nu)}$. Subsequently, Eq. (33) will be written as

$$\mathscr{M}_{\mu\nu}\mathscr{L}_1^{\mu\nu}=\mathscr{A}_1 . \tag{43}$$

3. Interference term

As in the preceding cases, $\mathscr{L}_3^{\mu\nu}$ is divided into parts without and with a γ^5 by setting

$$\mathscr{L}_{3a}^{\mu\nu}=\frac{1}{4E_eE_{(\nu)}}\left[-\frac{e}{2p_e\cdot k}\right]\mathrm{Tr}[\gamma^\mu \not k_{(\nu)}\gamma^\nu \not k \not A' \not p_e+\gamma^\mu \not k_{(\nu)}\gamma^\nu \not p_e \not A \not k]\,,$$
$$\mathscr{L}_{3b}^{\mu\nu}=\frac{1}{4E_eE_{(\nu)}}\frac{e}{2p_e\cdot k}\mathrm{Tr}[\gamma^\mu\gamma^5 \not k_{(\nu)}\gamma^\nu \not k \not A' \not p_e+\gamma^\mu\gamma^5 \not k_{(\nu)}\gamma^\nu \not p_e \not A \not k]\,. \tag{44}$$

After evaluation of the traces, the results are

$$\begin{aligned}\mathscr{L}_{3a}^{\mu\nu}=\frac{1}{E_eE_{(\nu)}}\left[-\frac{e}{2p_e\cdot k}\right][&p_e\cdot A(k^\mu k_{(\nu)}^\nu+k^\nu k_{(\nu)}^\mu)+p_e\cdot k_{(\nu)}(A^\mu k^\nu-A^\nu k^\mu)\\&+k_{(\nu)}\cdot A(k^\mu p_e^\nu-k^\nu p_e^\mu)-k\cdot k_{(\nu)}(g^{\mu\nu}p_e\cdot A-p_e^\mu A^\nu+A^\mu p_e^\nu)\\&-p_e\cdot k(k_{(\nu)}^\mu A^\nu-g^{\mu\nu}k_{(\nu)}\cdot A+A^\mu k_{(\nu)}^\nu)+p_e\cdot A'(k^\mu k_{(\nu)}^\nu+k^\nu k_{(\nu)}^\mu)\\&-p_e\cdot k_{(\nu)}(A'^\mu k^\nu-A'^\nu k^\mu)-k_{(\nu)}\cdot A'(k^\mu p_e^\nu-k^\nu p_e^\mu)\\&-k\cdot k_{(\nu)}(g^{\mu\nu}p_e\cdot A'+p_e^\mu A'^\nu-A'^\mu p_e^\nu)-p_e\cdot k(k_{(\nu)}^\mu A'^\nu-g^{\mu\nu}k_{(\nu)}\cdot A'+A'^\mu k_{(\nu)}^\nu)]\,,\end{aligned}$$

$$\begin{aligned}\mathscr{L}_{3b}^{\mu\nu}=\frac{i}{E_eE_{(\nu)}}\left[-\frac{e}{2p_e\cdot k}\right][&\epsilon_{\alpha\beta\gamma\delta}k^\alpha k_{(\nu)}^\beta p_e^\gamma A^\delta g^{\mu\nu}+\epsilon^{\mu\nu}{}_{\alpha\beta}k^\alpha A^\beta k_{(\nu)}\cdot p_e\\&-\epsilon^{\mu\nu}{}_{\alpha\beta}k^\alpha p_e^\beta k_{(\nu)}\cdot A+\epsilon^{\mu\nu}{}_{\alpha\beta}k^\alpha k_{(\nu)}^\beta p_e\cdot A-\epsilon^{\mu\nu}{}_{\alpha\beta}p_e^\alpha A^\beta k\cdot k_{(\nu)}+\epsilon^{\mu\nu}{}_{\alpha\beta}k_{(\nu)}^\alpha A^\beta p_e\cdot k\\&-(\epsilon^\mu{}_{\alpha\beta\gamma}k_{(\nu)}^\nu+\epsilon^\nu{}_{\alpha\beta\gamma}k_{(\nu)}^\mu)k^\alpha A^\beta p_e^\gamma+(\epsilon^\mu{}_{\alpha\beta\gamma}p_e^\nu-\epsilon^\nu{}_{\alpha\beta\gamma}p_e^\mu)k^\alpha k_{(\nu)}^\beta A^\gamma\\&+(\epsilon^\mu{}_{\alpha\beta\gamma}A^\nu-\epsilon^\nu{}_{\alpha\beta\gamma}A^\mu)k^\alpha p_e^\beta k_{(\nu)}^\gamma+(\epsilon^\mu{}_{\alpha\beta\gamma}k^\nu-\epsilon^\nu{}_{\alpha\beta\gamma}k^\mu)k_{(\nu)}^\alpha p_e^\beta A^\gamma\\&-\epsilon_{\alpha\beta\gamma\delta}k^\alpha k_{(\nu)}^\beta p_e^\gamma A'^\delta g^{\mu\nu}-\epsilon^{\mu\nu}{}_{\alpha\beta}k^\alpha A'^\beta k_{(\nu)}\cdot p_e+\epsilon^{\mu\nu}{}_{\alpha\beta}k^\alpha p_e^\beta k_{(\nu)}\cdot A'\\&+\epsilon^{\mu\nu}{}_{\alpha\beta}k^\alpha k_{(\nu)}^\beta p_e\cdot A'+\epsilon^{\mu\nu}{}_{\alpha\beta}p_e^\alpha A'^\beta k\cdot k_{(\nu)}+\epsilon^{\mu\nu}{}_{\alpha\beta}k_{(\nu)}^\alpha A'^\beta p_e\cdot k\\&-(\epsilon^\mu{}_{\alpha\beta\gamma}k_{(\nu)}^\nu+\epsilon^\nu{}_{\alpha\beta\gamma}k_{(\nu)}^\mu)k^\alpha A'^\beta p_e^\gamma+(\epsilon^\mu{}_{\alpha\beta\gamma}p_e^\nu-\epsilon^\nu{}_{\alpha\beta\gamma}p_e^\mu)k^\alpha k_{(\nu)}^\beta A'^\gamma\\&+(\epsilon^\mu{}_{\alpha\beta\gamma}A'^\nu-\epsilon^\nu{}_{\alpha\beta\gamma}A'^\mu)k^\alpha p_e^\beta k_{(\nu)}^\gamma+(\epsilon^\mu{}_{\alpha\beta\gamma}k^\nu-\epsilon^\nu{}_{\alpha\beta\gamma}k^\mu)k_{(\nu)}^\alpha p_e^\beta A'^\gamma]\,.\end{aligned} \tag{45}$$

When combined with $\mathscr{M}^{\mu\nu}$ as in Eq. (32), and with the electromagnetic field potential prescribed by Eqs. (16) and (39), the resultant interference term squared transition current is

$$\mathscr{M}_{\mu\nu}\mathscr{L}_3^{\mu\nu}=-\frac{ea\omega}{2p_e\cdot k}[\cos(k\cdot x+\rho)+\cos(k\cdot x'+\rho)]\mathscr{A}_3\,, \tag{46}$$

where

$$\mathscr{A}_3=(\psi_f^\dagger\psi_i)_{\vec{r}}(\psi_f^\dagger\psi_i)^\dagger_{\vec{r}'}\left[-\frac{\vec{p}_e\cdot\vec{\epsilon}}{E_e}\left[1+\frac{\vec{k}\cdot\vec{k}_{(\nu)}}{\omega E_{(\nu)}}\right]-\left[1-\frac{\vec{k}}{\omega}\cdot\frac{\vec{p}_e}{E_e}\right]\frac{\vec{k}_{(\nu)}\cdot\vec{\epsilon}}{E_{(\nu)}}\right.$$
$$\left.+i\frac{\vec{k}}{\omega}\times\frac{\vec{p}_e\cdot\vec{\epsilon}}{E_e}-i\frac{\vec{p}_e\times\vec{\epsilon}}{E_e}\cdot\frac{\vec{k}_{(\nu)}}{E_{(\nu)}}+i\frac{\vec{k}\times\vec{\epsilon}}{\omega}\cdot\frac{\vec{k}_{(\nu)}}{E_{(\nu)}}\right]$$
$$+\kappa^2(\psi_f^\dagger\vec{\sigma}\psi_i)_{\vec{r}}(\psi_f^\dagger\vec{\sigma}\psi_i)^\dagger_{\vec{r}'}\left[-\frac{\vec{p}_e\cdot\vec{\epsilon}}{E_e}\left[1-\frac{\vec{k}\cdot\vec{k}_{(\nu)}}{\omega E_{(\nu)}}\right]+\frac{1}{3}\left[1-\frac{\vec{k}}{\omega}\cdot\frac{\vec{p}_e}{E_e}\right]\frac{\vec{k}_{(\nu)}\cdot\vec{\epsilon}}{E_{(\nu)}}\right.$$
$$\left.+i\frac{\vec{k}}{\omega}\times\frac{\vec{p}_e\cdot\vec{\epsilon}}{E_e}-\frac{i}{3}\frac{\vec{p}_e\times\vec{\epsilon}}{E_e}\cdot\frac{\vec{k}_{(\nu)}}{E_{(\nu)}}+\frac{i}{3}\frac{\vec{k}\times\vec{\epsilon}}{\omega}\cdot\frac{\vec{k}_{(\nu)}}{E_{(\nu)}}\right]. \quad (47)$$

C. Transition probability per unit time

1. Direct term

The squared S-matrix element associated with the direct term follows from replacing the squared transition current in Eq. (19) by Eq. (33), or its equivalent from Eq. (43). This leads to

$$|S_{fi}|_1^2=\frac{G^2}{V^2}\int dt\int dt'\exp[i(-E_0+E_e+E_{(\nu)}+\eta\omega)(t-t')]$$
$$\times\int d^3r\int d^3r'\mathscr{A}_1\exp[-ie\vec{A}\cdot\vec{r}+ie\vec{A}'\cdot\vec{r}'-i(\vec{k}_e+\vec{k}_{(\nu)}+\eta\vec{k})\cdot(\vec{r}-\vec{r}')]$$
$$\times\exp[i(\zeta\sin(k\cdot x+\rho)-\zeta\sin(k\cdot x'+\rho)$$
$$+\tfrac{1}{2}\eta\sin2(k\cdot x+\rho)-\tfrac{1}{2}\eta\sin2(k\cdot x'+\rho))]\;. \quad (48)$$

Two remarks about Eq. (48) are needed. One is that the notation

$$E_0\equiv E_i-E_f$$

has been introduced. The other concerns the difference between final and initial reduced charges. From Eqs. (5) or (6) it follows for β^- decay ($Z_f=Z_i+1, N_f=N_i-1$) that

$$\tilde{e}_f-\tilde{e}_i=e\frac{A_{\rm core}}{A}\approx e\;, \quad (49)$$

where A is the total nuclear mass number, and $A_{\rm core}$ refers to the number of nucleons in the core alone. For most cases of interest,

$$A_{\rm core}/A\approx 1\;.$$

The result (49) is incorporated in Eq. (48).

The last exponential in Eq. (48) can be written in terms of a particular transcendental function introduced previously,[28] which is a generalization of the ordinary Bessel function. It may be defined for integer order by the integral expression

$$J_n(u,v)=\frac{1}{2\pi}\times\int_{-\pi}^{\pi}d\theta\exp[i(u\sin\theta+v\sin2\theta-n\theta)]\;, \quad (50)$$

or, for arbitrary order, by the series representation

$$J_n(u,v)=\sum_{k=-\infty}^{\infty}J_{n-2k}(u)J_k(v)\;.$$

This is not a standard transcendental function. Some of its basic properties are listed in Appendices B and C of Ref. 28. Particular properties needed here are

$$J_n(-u,v)=(-)^n J_n(u,v)\ , \tag{51}$$

$$J_n(u,-v)=(-)^n J_{-n}(u,v)\ , \tag{52}$$

$$\exp[i(u\sin\theta+v\sin 2\theta)]=\sum_{n=-\infty}^{\infty} e^{in\theta}J_n(u,v)\ . \tag{53}$$

The last exponential in Eq. (48) can be expressed in terms of the generalized Bessel function. Equation (53) gives

$$\exp[i(\zeta\sin(k\cdot x+\rho)+\tfrac{1}{2}\eta\sin 2(k\cdot x+\rho))]=\sum_{l=-\infty}^{\infty} J_l(\zeta,\tfrac{1}{2}\eta)e^{il(k\cdot x+\rho)}=\sum_{l=-\infty}^{\infty} J_l(-\zeta,-\tfrac{1}{2}\eta)e^{-il(k\cdot x+\rho)}\ , \tag{54}$$

where the second form (which is the more convenient to use) follows from Eqs. (51) and (52). In like fashion, the expression

$$\exp[-i(\zeta\sin(k\cdot x'+\rho)+\tfrac{1}{2}\eta\sin 2(k\cdot x'+\rho))]=\sum_{n=-\infty}^{\infty} J_n(-\zeta,-\tfrac{1}{2}\eta)e^{in(k\cdot x'+\rho)} \tag{55}$$

is obtained. Analogous expansions in terms of the ordinary Bessel functions give

$$\exp(-ie\vec{A}\cdot\vec{r})=\exp[-ie\vec{a}\cdot\vec{r}\cos(k\cdot x+\rho)]=\sum_{j=-\infty}^{\infty} J_j(e\vec{a}\cdot\vec{r})e^{-ij(k\cdot x+\rho)}\ , \tag{56}$$

$$\exp(ie\vec{A}'\cdot\vec{r}')=\exp[ie\vec{a}\cdot\vec{r}'\cos(k\cdot x'+\rho)]=\sum_{m=-\infty}^{\infty} J_m(e\vec{a}\cdot\vec{r}')e^{im(k\cdot x'+\rho)}\ , \tag{57}$$

where $\vec{a}=a\vec{\epsilon}$, and $\vec{A}$ is as given by Eqs. (16) and (39).

The direct term squared matrix element is now

$$\begin{aligned}|S_{fi}|_1^2=\frac{G^2}{V^2}\sum_l\sum_n\sum_j\sum_m J_l(-\zeta,-\tfrac{1}{2}\eta)J_n(-\zeta,-\tfrac{1}{2}\eta)\\ \times\int dt\int dt'\exp[i(-E_0+E_e+E_{(\nu)}+\eta\omega)(t-t')-il\omega t+in\omega t'-ij\omega t+im\omega t']\\ \times\exp[i(-l+n-j+m)\rho]\\ \times\int d^3r\int d^3r'\mathscr{A}_1 J_j(e\vec{a}\cdot\vec{r})J_m(e\vec{a}\cdot\vec{r}')\\ \times\exp[-i(\vec{k}_e+\vec{k}_{(\nu)}+\eta\vec{k})\cdot(\vec{r}-\vec{r}')+il\vec{k}\cdot\vec{r}-in\vec{k}\cdot\vec{r}'\\ +ij\vec{k}\cdot\vec{r}-im\vec{k}\cdot\vec{r}']\ .\end{aligned} \tag{58}$$

The integrations over t and t' can be performed, with the t integration giving rise to

$$\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-l\omega-j\omega)\ ,$$

and the t' integration yielding

$$\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-n\omega-m\omega)\ .$$

The product of the two delta functions will give a zero result unless their zeroes are concurrent, which gives the condition

$$l+j=n+m\ . \tag{59}$$

Equation (59) eliminates all dependence on the phase ρ from $|S_{fi}|_1^2$, and will also be used to eliminate the sum over l.

To calculate the transition probability per unit time, the limit

$$w_1 = \lim_{T\to\infty} \frac{1}{T} |S_{fi}|_1^2$$

will be taken. This can be accomplished by using the device that

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \cdots \int_{-T/2}^{T/2} dt' \cdots = 2\pi\delta(-E_0 + E_e + E_{(\nu)} + \eta\omega - n\omega - m\omega) .$$

The transition probability per unit time is thus, from Eqs. (58) and (59),

$$w_1 = 2\pi \frac{G^2}{V^2} \sum_n \sum_j \sum_m J_n(-\zeta, -\tfrac{1}{2}\eta) J_{n-j+m}(-\zeta, -\tfrac{1}{2}\eta) \delta(-E_0 + E_e + E_{(\nu)} + \eta\omega - n\omega - m\omega)$$
$$\times \int d^3r \int d^3r' \mathscr{A}_1 J_j(e\vec{a}\cdot\vec{r}) J_m(e\vec{a}\cdot\vec{r}') \exp[-i(\vec{k}_e + \vec{k}_{(\nu)} + \eta\vec{k} - n\vec{k} - m\vec{k})\cdot(\vec{r} - \vec{r}')] . \tag{60}$$

The long-wavelength approximation will now be introduced. As is always done in the theory of allowed beta decay, the electron and antineutrino three-momentum contributions $\vec{k}_e \cdot \vec{r}$ and $\vec{k}_{(\nu)} \cdot \vec{r}$ will be neglected. Electromagnetic field terms, as expressed in $m\vec{k}\cdot\vec{r}$, are certainly negligible, since it will appear shortly that m is a small integer. The remaining electromagnetic field contribution, $(\eta - n)\vec{k}\cdot\vec{r}$, can be the largest of the terms involved, but it is also small, and will be neglected. Equation (60) will thus be used with its final exponential factor replaced by unity.

The total transition probability per unit time is calculated by integrating w_1 over the final states available to the emitted particles, or

$$W_1 = \int \frac{V p_e^2 dp_e d\Omega_e}{(2\pi)^3} \int \frac{V k_{(\nu)}^2 dk_{(\nu)} d\Omega_{(\nu)}}{(2\pi)^3} w_1 . \tag{61}$$

In the solid angle integration for the neutrino, the only angular dependence is from the $\vec{p}_e \cdot \vec{k}_{(\nu)}$ term in $\mathscr{A}_1$. Since the $\vec{p}_e$ vector can be considered fixed for this integration, then

$$\int d\Omega_{(\nu)} \vec{p}_e \cdot \vec{k}_{(\nu)} = 0 , \tag{62}$$

and so

$$\mathscr{A}_1 \to \mathscr{B}_1 ,$$

where

$$\mathscr{B}_1 \equiv (\psi_f^\dagger \psi_i)_{\vec{r}} (\psi_f^\dagger \psi_i)^\dagger_{\vec{r}'} + \kappa^2 (\psi_f^\dagger \vec{\sigma} \psi_i)_{\vec{r}} \cdot (\psi_f^\dagger \vec{\sigma} \psi_i)^\dagger_{\vec{r}'} \tag{63}$$

and

$$\int d\Omega_{(\nu)} \mathscr{B}_1 = 4\pi \mathscr{B}_1 .$$

The neutrino is taken to be massless, so that $|\vec{k}_{(\nu)}| = E_{(\nu)}$, and

$$\int d^3k_{(\nu)} \to 4\pi \int E_{(\nu)}^2 dE_{(\nu)} .$$

The energy delta function can be used to accomplish the integration over $E_{(\nu)}$, which leads to

$$E_{(\nu)} = E_0 - E_e - \eta\omega + n\omega + m\omega .$$

Then, after a change of integration variable from p_e to E_e, the total transition probability per unit time is

$$W_1 = \frac{2G^2}{(2\pi)^4} \sum_n \sum_j \sum_m \int dE_e d\Omega_e E_e (E_e^2 - m^2)^{1/2} (E_0 - E_e - \eta\omega + n\omega + m\omega)^2$$
$$\times J_n(-\zeta, -\tfrac{1}{2}\eta) J_{n-j+m}(-\zeta, -\tfrac{1}{2}\eta) \int d^3r \int d^3r' \mathscr{B}_1 J_j(e\vec{a}\cdot\vec{r}) J_m(e\vec{a}\cdot\vec{r}') . \tag{64}$$

Equation (64) differs from the transition probability expression for an allowed beta decay by the presence of the ordinary and generalized Bessel functions, and the summations over their indices.

2. Spin term

The squared S-matrix element associated with the spin term follows from replacing the squared transition current in Eq. (19) by Eq. (41). To simplify matters, long-wavelength-approximation terms of the type that were dropped from the direct term following Eq. (60) will now be neglected from the outset. Equation (49) will also be incorporated. Thus the squared S-matrix element is

$$|S_{fi}|^2=\frac{G^2}{V^2}\frac{2\eta\omega}{E_e}\int dt\int dt'\exp[i(-E_0+E_e+E_{(\nu)}+\eta\omega)(t-t')]$$
$$\times\int d^3r\int d^3r'\mathscr{A}_2\exp(-ie\vec{A}\cdot\vec{r}+ie\vec{A}'\cdot\vec{r})$$
$$\times\exp[i(\zeta\sin(\omega t+\rho)-\zeta\sin(\omega t'+\rho)+\tfrac{1}{2}\eta\sin 2(\omega t+\rho)$$
$$-\tfrac{1}{2}\eta\sin 2(\omega t'+\rho))]\cos(\omega t+\rho)\cos(\omega t'+\rho)\ . \qquad (65)$$

Generalized Bessel functions are introduced as in Eqs. (54) and (55), and ordinary Bessel functions as in Eqs. (56) and (57). The result is

$$|S_{fi}|^2=\frac{G^2}{V^2}\frac{2\eta\omega}{E_e}\sum_l\sum_n\sum_j\sum_m J_l(-\zeta,-\tfrac{1}{2}\eta)J_n(-\zeta,-\tfrac{1}{2}\eta)$$
$$\times\int dt\int dt'\exp[i(-E_0+E_e+E_{(\nu)}+\eta\omega)(t-t')$$
$$-i(l+j)\omega t+i(n+m)\omega t']\exp[i(-l-j+n+m)\rho]$$
$$\times\int d^3r\int d^3r'\mathscr{A}_2 J_j(e\vec{a}\cdot\vec{r})J_m(e\vec{a}\cdot\vec{r}')\tfrac{1}{4}$$
$$\times[e^{i(\omega t+\omega t'+2\rho)}+e^{i(\omega t-\omega t')}+e^{-i(\omega t-\omega t')}+e^{-i(\omega t+\omega t'+2\rho)}]\ . \qquad (66)$$

The sum of four terms in the final square bracket in Eq. (66) gives four different delta function behaviors when the t and t' integrations are performed. Designate the four terms in succession as term (a), term (b), term (c), and term (d). Term (a) has the product of delta functions

$$\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-l\omega-j\omega+\omega)\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-n\omega-m\omega-\omega)\ ,$$

which gives the condition

$$l=n-j+m+2\ . \qquad (67a)$$

Equation (67a) causes all dependence on the phase ρ to vanish from term (a). Term (b) has the delta function product

$$\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-l\omega-j\omega+\omega)\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-n\omega-m\omega+\omega)\ ,$$

which gives

$$l=n-j+m\ , \qquad (67b)$$

and eliminates dependence on ρ from term (b). Term (c) has

$$\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-l\omega-j\omega-\omega)\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-n\omega-m\omega-\omega)\ ,$$

which gives

$$l=n-j+m\ , \tag{67c}$$

and eliminates ρ from term (c). Term (d) has

$$\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-l\omega-j\omega-\omega)\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-n\omega-m\omega+\omega)\ ,$$

which gives

$$l=n-j+m-2\ , \tag{67d}$$

and eliminates ρ. With the l sum gone as a consequence of Eqs. (67a)—(67d), and with the transition probability per unit time introduced by

$$w_2=\lim_{T\to\infty}\frac{1}{T}\,|\,S_{fi}\,|_2^2\ ,$$

Eq. (66) leads to

$$\begin{aligned} w_2=2\pi\frac{G^2}{V^2}\frac{\eta\omega}{2E_e}\sum_n\sum_j\sum_m J_n(-\zeta,-\tfrac{1}{2}\eta) \\ \times\int d^3r\int d^3r'\mathscr{A}_2 J_j(e\vec{a}\cdot\vec{r})J_m(e\vec{a}\cdot\vec{r}\,') \\ \times\{\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-n\omega-m\omega-\omega) \\ \times[J_{n-j+m+2}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-j+m}(-\zeta,-\tfrac{1}{2}\eta)] \\ +\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-n\omega-m\omega+\omega) \\ \times[J_{n-j+m}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-j+m-2}(-\zeta,-\tfrac{1}{2}\eta)]\}\ . \end{aligned} \tag{68}$$

There are now two delta functions. To obtain a single common delta function, shift the origin of the n sum in the first term so that $n+1\to n$, and in the second term so that $n-1\to n$. As a consequence, Eq. (68) becomes

$$\begin{aligned} w_2=2\pi\frac{G^2}{V^2}\frac{\eta\omega}{2E_e}\sum_n\sum_j\sum_m \delta(-E_0+E_e+E_{(\nu)}+\eta\omega-n\omega-m\omega) \\ \times\int d^3r\int d^3r'\mathscr{A}_2 J_j(e\vec{a}\cdot\vec{r})J_m(e\vec{a}\cdot\vec{r}\,')[J_{n+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-1}(-\zeta,-\tfrac{1}{2}\eta)] \\ \times[J_{n-j+m+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-j+m-1}(-\zeta,-\tfrac{1}{2}\eta)]\ . \end{aligned} \tag{69}$$

Passage to the total transition probability per unit time is accomplished as in Eq. (61). Also, Eq. (62) has its analog with the $\vec{k}\cdot\vec{k}_{(\nu)}$ terms in $\mathscr{A}_2$, so the antineutrino solid angle integration leads to

$$\mathscr{A}_2\to\mathscr{B}_1\ ,$$

where $\mathscr{B}_1$ is given in Eq. (63). As with W_1, integration over $E_{(\nu)}$ is achieved with the energy delta function, and the integral over p_e is converted to an E_e integration. These steps lead to

$$W_2=\frac{2G^2}{(2\pi)^4}\sum_n\sum_j\sum_m\int dE_e d\Omega_e E_e(E_e{}^2-m^2)^{1/2}(E_0-E_e-\eta\omega+n\omega+m\omega)^2$$
$$\times\frac{\eta\omega}{2E_e}[J_{n+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-1}(-\zeta,-\tfrac{1}{2}\eta)]$$
$$\times[J_{n-j+m+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-j+m-1}(-\zeta,-\tfrac{1}{2}\eta)]$$
$$\times\int d^3r\int d^3r'\mathscr{B}_1 J_j(e\vec{a}\cdot\vec{r})J_m(e\vec{a}\cdot\vec{r}')\ . \tag{70}$$

3. Interference term

The procedure to be followed for the interference term follows the lines already established with the direct term and spin term. The squared S-matrix element comes from Eqs. (19) and (46). With the long-wavelength approximation introduced from the beginning, the squared S-matrix element is

$$|S_{fi}|_3{}^2=\frac{G^2}{V^2}\left[-\frac{ea\omega}{2p_e\cdot k}\right]\int dt\int dt'\exp[i(-E_0+E_e+E_{(\nu)}+\eta\omega)(t-t')]$$
$$\times\int d^3r\int d^3r'\mathscr{A}_3\exp(-ie\vec{A}\cdot\vec{r}+ie\vec{A}'\cdot\vec{r}')$$
$$\times\exp[i(\zeta\sin(\omega t+\rho)-\zeta\sin(\omega t'+\rho)$$
$$+\tfrac{1}{2}\eta\sin 2(\omega t+\rho)-\tfrac{1}{2}\eta\sin 2(\omega t'+\rho))]$$
$$\times[\cos(\omega t+\rho)+\cos(\omega t'+\rho)]\ . \tag{71}$$

Next, the representations in Eqs. (54)–(57) are introduced, and the trigonometric terms in the final square bracket in Eq. (71) are expressed in exponential terms. As was the case with the spin part, there are four terms resulting, with different products of delta functions implying a constraint on the l summation index in terms of the n, m, and j indices. When the l sum is removed, w_3 introduced by

$$w_3=\lim_{T\to\infty}\frac{1}{T}|S_{fi}|_3{}^2\ ,$$

and a single delta function factor achieved by appropriate shifts in the origin of the n sum, the resulting expression is

$$w_3=2\pi\frac{G^2}{V^2}\left[-\frac{ea\omega}{4p_e\cdot k}\right]\sum_n\sum_j\sum_m\delta(-E_0+E_e+E_{(\nu)}+\eta\omega-n\omega-m\omega)$$
$$\times\int d^3r\int d^3r'\mathscr{A}_3 J_j(e\vec{a}\cdot\vec{r})J_m(e\vec{a}\cdot\vec{r}')$$
$$\times\{J_n(-\zeta,-\tfrac{1}{2}\eta)[J_{n-j+m+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-j+m-1}(-\zeta,-\tfrac{1}{2}\eta)]$$
$$+J_{n-j+m}(-\zeta,-\tfrac{1}{2}\eta)[J_{n+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-1}(-\zeta,-\tfrac{1}{2}\eta)]\}\ .$$

Total transition probability per unit time is calculated as in Eq. (61). When the antineutrino solid angle integration is performed, those terms in $\mathscr{A}_3$ [Eq. (47)] containing $\vec{k}\cdot\vec{k}_{(\nu)}$, $\vec{\epsilon}\cdot\vec{k}_{(\nu)}$, $(\vec{p}_e\times\vec{\epsilon})\cdot\vec{k}_{(\nu)}$, and $(\vec{k}\times\vec{\epsilon})\cdot\vec{k}_{(\nu)}$

will vanish. Then W_3 is

$$W_3=\frac{G^2}{(2\pi)^4}\sum_n\sum_j\sum_m\int dE_e d\Omega_e E_e(E_e{}^2-m^2)^{1/2}(E_0-E_e-\eta\omega+n\omega+m\omega)^2\left[-\frac{ea\omega}{2p_e\cdot k}\right]$$
$$\times\{J_n(-\zeta,-\tfrac{1}{2}\eta)[J_{n-j+m+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-j+m-1}(-\zeta,-\tfrac{1}{2}\eta)]$$
$$+J_{n-j+m}(-\zeta,-\tfrac{1}{2}\eta)[J_{n+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-1}(-\zeta,-\tfrac{1}{2}\eta)]\}$$
$$\times\int d^3r\int d^3r'\mathscr{B}_3 J_j(e\vec{a}\cdot\vec{r})J_m(e\vec{a}\cdot\vec{r}\,'),\qquad(72)$$

$$\mathscr{B}_3=\mathscr{B}_1\left[-\frac{\vec{p}_e\cdot\vec{\epsilon}}{E_e}+i\frac{\vec{k}}{\omega}\times\frac{\vec{p}_e}{E_e}\cdot\vec{\epsilon}\right],\qquad(73)$$

with $\mathscr{B}_1$ as given in Eq. (63).

IV. VERY HIGH INTENSITY, VERY LOW FREQUENCY CASE

A. Orders of magnitude

The transition probability expressions in Eqs. (64), (70), and (72) are complete within the context of the present work, but they are difficult to evaluate. Significant simplification in these expressions can be accomplished if certain special cases are considered. One obvious limit is the case where the field amplitude goes to zero. Then W_2 and W_3 vanish, and W_1 reduces to the transition probability for an allowed beta decay without Coulomb corrections. Of far more physical interest for this investigation are the cases where the order of magnitude of either of the two field intensity parameters that occur in the results are in the neighborhood of unity.

Because of the finite range of the nuclear wave functions in Eqs. (64), (70), and (72), the arguments of the Bessel functions appearing therein are limited to values of the order of eaR_0, where R_0 is the nuclear radius. This suggests the introduction of the intensity parameter z, defined by

$$z\equiv(eaR_0)^2\,.\qquad(74)$$

This quantity is typical of intensity parameters which arise in all bound-state intense-field problems.[29] For specific intense-field phenomena to occur through the nucleus-field interaction, it is generally true that z must be roughly of order unity. Values of z much larger than unity will cause a decline in transition probability as intensity increases, in contrast to the common behavior of the increase in probability with intensity.

The second intensity parameter which arises in this problem is that normally associated with free electrons in an electromagnetic field, and is defined by[29,30]

$$z_f\equiv\frac{e^2a^2}{2m^2}\,.\qquad(75)$$

It is related to the field-nucleus intensity parameter z by

$$z_f=\frac{1}{2(mR_0)^2}z\approx(3\times10^3)z\,.\qquad(76)$$

The numerical relation given in Eq. (76) comes from the estimate

$$R_0\approx5\times10^{-13}\text{ cm}\,,\qquad(77)$$

which is typical of a broad range of light-to-medium nuclei. When the order of magnitude of z_f is in the neighborhood of unity, Eq. (76) shows that z is small. Field-electron effects will then be more important than field-nucleus effects.[31] In the remainder of this section, attention will be focused on the case where the order of magnitude of z is in the neighborhood of unity. This is the case where the field is most effective in overcoming forbiddenness in beta decay. Equation (76) then implies that z_f is large.

The easiest way to achieve z in the neighborhood of order unity in practice is to employ a very low frequency field. The energy flux in the field necessary to achieve a given value of z is expressed by

$$P=10^{-7}\frac{\pi}{2}\frac{z}{\alpha_0\lambda^2R_0{}^2}\,,\qquad(78)$$

in units of W/cm^2. The factor 10^{-7} is for conversion from ergs to joules, α_0 is the fine structure constant, and λ is the wavelength of the applied field. The inverse-square dependence on wavelength in Eq. (78) means that a given z value can be achieved much more readily at long wavelengths than at short wavelengths.

The fact that field-nucleus effects are dominant in a field intensity domain where z_f is large means that the transition probabilities stated in Eqs. (64), (70), and (72) can be simplified. The quantity z_f occurs only in the generalized Bessel functions in these equations, and $z_f \gg 1$ makes it appropriate to use an asymptotic form for the generalized Bessel functions. Because these functions depend on three quantities (order and two arguments), there are a number of different asymptotic forms possible depending on the relative magnitudes of these parameters. The magnitudes of the physical parameters which arise in the present problem will now be explored to demonstrate first that an asymptotic form of the generalized Bessel function is really appropriate. A particular asymptotic case is generated, and it will then be shown to be the applicable one for the present work.

An immediate consequence of $z=O(1)$ is that large values of the j and m indices in Eqs. (64), (70), and (72) are of no importance. Since the magnitude of the ω energy is very much less than an electron rest energy, the term $m\omega$ as it appears, e.g., in the energy delta function in Eq. (60), can be neglected as compared to the other terms.

With $m\omega$ neglected in the energy delta function, the physical condition $E_{(\nu)} \geq 0$ leads directly to

$$n \geq \eta-(E_0-E_e)/\omega=\eta-(Q-T_e)/\omega \ . \tag{79}$$

The last expression in Eq. (79) comes from setting $E_0=m+Q$, where Q is the usual Q_{β^-} of beta decay (and m is now the electron mass, not the summation index), and setting $E_e=m+T_e$, where T_e is the electron kinetic energy. One immediate implication of Eq. (79) is that there is a lower limit on the n index. From the definition of η in Eq. (18), its order of magnitude is

$$\eta=O\left[\frac{e^2a^2}{4m\omega}\right] . \tag{80}$$

This means that

$$\frac{\eta}{Q/\omega}=O\left[\frac{e^2a^2}{4mQ}\right]=O\left[\frac{e^2a^2}{4m^2}\right]=O\left[\frac{z}{(2mR_0)^2}\right] . \tag{81}$$

For a typical nuclear radius, one has

$$mR_0 \approx \tfrac{1}{80} \tag{82}$$

[which is consistent with Eq. (77)], so that $z=O(1)$ employed in Eq. (81) gives

$$\eta \gg Q/\omega \ . \tag{83}$$

Furthermore, since $Q=O(m)$ typically, and $m/\omega \gg 1$, then

$$\eta \gg 1 \ . \tag{84}$$

Also, Eqs. (83) and (79) then imply that

$$n \gg 1 \ . \tag{85}$$

Finally, from the definition, Eq. (18), ζ has the order of magnitude

$$|\zeta|=O(ea/\omega)=O(z^{1/2}/\omega R_0) \ . \tag{86}$$

One can write

$$\omega R_0=(\omega/m)(mR_0) \approx (\tfrac{1}{80})(\omega/m) \ .$$

But ω/m is many orders of magnitude less than unity, so it follows that $\omega R_0 \ll 1$, and then Eq. (86) gives

$$|\zeta| \gg 1 \ . \tag{87}$$

Equations (84), (85), and (87) are the basic conditions making it appropriate to employ an asymptotic form of the generalized Bessel function.

B. Asymptotic generalized Bessel function

The generalized Bessel functions in Eqs. (64), (70), and (72) will be written here as $J_n(-\zeta,-\frac{1}{2}\eta)$. Equations (84), (85), and (87) show that η, $|\zeta|$, and n are all much greater than unity. (Note that ζ can be either positive or negative.)

From Eq. (50), $J_n(-\zeta,-\frac{1}{2}\eta)$ can be written

$$J_n(-\zeta,-\tfrac{1}{2}\eta)=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{\eta f(\theta)} ,$$

where

$$f(\theta)=-i(4b\sin\theta+\tfrac{1}{2}\sin 2\theta+\nu\theta) , \tag{88}$$

with

$$b \equiv \frac{\zeta}{4\eta}, \quad \nu \equiv \frac{n}{\eta} \ . \tag{89}$$

The saddle points θ_0 of $f(\theta)$ in the complex θ plane are found from $f'(\theta)=0$ to be located at

$$\cos\theta_0=-b \pm c , \tag{90}$$

where

$$c \equiv [b^2+\tfrac{1}{2}(1-\nu)]^{1/2} \ . \tag{91}$$

Constraints that arise from physical considerations (discussed below) are that

$$|b| \ll 1,\ b^2+\tfrac{1}{2}(1-\nu)\ge 0,\ (1-\nu)\ll 1\ . \tag{92}$$

Hence, $\cos\theta_0$ is real, and four saddle points occur on the real θ axis in the interval $-\pi\le\theta\le\pi$. These saddle points are designated θ_j, $j=1,2,3,4$, and are listed in Table I.

The real axis is a level surface for $f(\theta)$ in the complex plane, and all paths of steepest descent (and ascent) cross the real axis at a 45° angle. For evaluation of $J_n(-\zeta,-\frac{1}{2}\eta)$ by the steepest descent method, the path of integration is deformed as shown in Fig. 1. Saddle point locations are shown in Fig. 1 by the small circles. Steepest-descent evaluation of $J_n(-\zeta,-\frac{1}{2}\eta)$, gives, in general

$$J_n(-\zeta,-\tfrac{1}{2}\eta)\approx\frac{1}{\sqrt{2\pi\eta}}\sum_{j=1}^{4}\frac{e^{\eta f_j}}{(-f_j'')^{1/2}}\ , \tag{93}$$

where $f_j=f(\theta_j)$, $f_j''=f''(\theta_j)$ are found by substituting the solution (90) into (88) and its second derivative. Equation (93) leads to

$$J_n(-\zeta,-\tfrac{1}{2}\eta)\approx\frac{1}{\sqrt{2\pi\eta}}\left[\frac{2}{(|f_1''|)^{1/2}}\cos\phi_1+\frac{2}{(|f_2''|)^{1/2}}\cos\phi_2\right]\ , \tag{94}$$

where

$$\begin{aligned}\phi_1&=n\theta_1+\eta(3b+c)[1-(c-b)^2]^{1/2}-\frac{\pi}{4}\ ,\\ \phi_2&=n\theta_2+\eta(3b-c)[1-(c+b)^2]^{1/2}+\frac{\pi}{4}\ ,\\ f_1''&=4ic[1-(c-b)^2]^{1/2}\ ,\\ f_2''&=-4ic[1-(c+b)^2]^{1/2}\ .\end{aligned} \tag{95}$$

The inequalities (92) give

$$|f_1''|\approx|f_2''|\approx 4c\ ,$$

so the result is then

TABLE I. Location of saddle points in the asymptotic evaluation of the generalized Bessel function.

θ_0	$\cos\theta_0$	$\sin\theta_0$
1	$c-b$	$[1-(c-b)^2]^{1/2}$
2	$-c-b$	$[1-(c+b)^2]^{1/2}$
3	$-c-b$	$-[1-(c+b)^2]^{1/2}$
4	$c-b$	$-[1-(c-b)^2]^{1/2}$

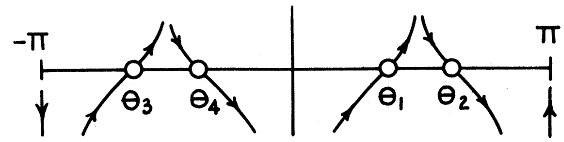

FIG. 1. Path of integration for evaluation of the asymptotic generalized Bessel function.

$$J_n(-\zeta,-\tfrac{1}{2}\eta)\approx\frac{1}{(2\pi\eta c)^{1/2}}(\cos\phi_1+\cos\phi_2)\ . \tag{96}$$

In addition to the generalized Bessel function of order n, certain other contiguous, or nearly contiguous, orders are also needed. A change of n to $n+1$ changes the parameter ν to $\nu+1/\eta$, so the amplitude factors in Eq. (94) or (96) are scarcely affected. However, the phases ϕ_1, ϕ_2 change significantly when n changes by even one integer. This change is found by differentiating Eqs. (95) with respect to ν to find

$$\frac{\partial\phi_{1,2}}{\partial\nu}=\eta\theta_{1,2}\ ,$$

which gives

$$\phi_{1,2}\left[\nu+\frac{1}{\eta}\right]\approx\phi_{1,2}(\nu)+\theta_{1,2}\ . \tag{97}$$

Then, for example, if q is a small integer,

$$J_{n+2q}(-\zeta,-\tfrac{1}{2}\eta)\approx\frac{1}{(2\pi\eta c)^{1/2}}[\cos(\phi_1+2q\theta_1)+\cos(\phi_2+2q\theta_2)]\ . \tag{98}$$

The inequalities (92) imply that θ_1 and θ_2 are both nearly $\pi/2$, so then

$$J_{n+2q}(-\zeta,-\tfrac{1}{2}\eta)\approx(-)^q J_n(-\zeta,-\tfrac{1}{2}\eta)\ . \tag{99}$$

In the following, the sum of generalized Bessel functions of orders $n+1$ and $n-1$ will occur. The approximation involved in Eq. (99) is then not adequate, since it would indicate a spurious zero result. The true result is small, but not zero, so a form analogous to Eq. (98) must be used. The required expression is

$$J_{n+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-1}(-\zeta,-\tfrac{1}{2}\eta)\approx\frac{1}{(2\pi\eta c)^{1/2}}[\cos(\phi_1+\theta_1)+\cos(\phi_1-\theta_1)+\cos(\phi_2+\theta_2)+\cos(\phi_2-\theta_2)]\ .$$

The trigonometric terms can be combined to yield

$$J_{n+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-1}(-\zeta,-\tfrac{1}{2}\eta) \approx \frac{2}{(2\pi\eta c)^{1/2}}[(c-b)\cos\phi_1-(c+b)\cos\phi_2] . \quad (100)$$

The particular conditions of Eq. (92) associated with the final form of the asymptotic generalized Bessel function will now be explored. From Eqs. (80) and (86), the parameter $|b|$ has the magnitude

$$|b| = |\zeta/4\eta| = O(m/ea) = O(mR_0/z^{1/2}) \ll 1 , \quad (101)$$

as long as $z^{1/2} \gg \frac{1}{80}$. Equation (101) is the first of the conditions (92). For the third condition in Eq. (92), one has

$$1-\nu = \frac{\eta-n}{\eta} \le \frac{(Q-T_e)/\omega}{\eta} \le \frac{Q/\omega}{\eta} \ll 1 . \quad (102)$$

Equations (79) and (83) have been used to arrive at (102). Finally, Eq. (92) states the condition

$$b^2+(1-\nu)/2>0 .$$

When the definitions in Eqs. (89) and (18) are substituted, this condition can be written as

$$(n-\eta)\omega \le \frac{(p_e\cdot\epsilon)^2}{2(E_e-|\vec{p}_e|)} \approx \frac{|\vec{p}_e|^2}{2m} = T_e . \quad (103)$$

From the energy delta function, it can be seen that $(n-\eta)\omega$ is the amount of energy contributed by the electromagnetic field. In view of the low frequency and consequent low energy density of the applied fields, it is presumed here that no substantial amount of energy can be extracted from the field. This will be stated as

$$\eta \ge n . \quad (104)$$

Not only does Eq. (104) automatically satisfy Eq. (103), it is also a conservative assumption in that it truncates the sum over n and omits a portion of the transition probability.

C. Transition probability per unit time

1. Direct term

The asymptotic form for the generalized Bessel function arrived at in Eq. (96) is now to be employed in Eq. (64). One of the two generalized Bessel functions in Eq. (64) is exactly in the terminology given in Eq. (96), whereas the other function can be written as

$$J_{n+m-j}(-\zeta,-\tfrac{1}{2}\eta) = \frac{1}{(2\pi\eta c)^{1/2}}\left\{\cos\left[\phi_1+(m-j)\frac{\pi}{2}\right]+\cos\left[\phi_2+(m-j)\frac{\pi}{2}\right]\right\} . \quad (105)$$

The result (105) follows from Eq. (95), along with the knowledge that $\phi_1,\phi_2 \approx \pi/2$. This last item is inferred from Eqs. (91) and (92), and Table I. Equation (105) can be simplified to a form depending on whether $m-j$ is an even or an odd number. Suppose first that $m-j$ is even, or $m-j=2q$, where q is an integer. It follows that

$$J_{n+m-j}(-\zeta,-\tfrac{1}{2}\eta) = (-)^q J_n(-\zeta,-\tfrac{1}{2}\eta) . \quad (106)$$

With the knowledge that only small values of the index m will contribute, the term m in the

$$(E_0-E_e-\eta\omega+n\omega+m\omega)^2$$

factor of Eq. (64) can be dropped, and then the threefold sum over the n, j, and m indices in Eq. (64) involves only the ordinary and generalized Bessel functions. This threefold sum is

$$\sum_n\sum_j\sum_m J_n(-\zeta,-\tfrac{1}{2}\eta)J_{n-j+m}(-\zeta,-\tfrac{1}{2}\eta)J_j(e\vec{a}\cdot\vec{r})J_m(e\vec{a}\cdot\vec{r}\,')$$

$$\approx \sum_n\sum_m\sum_q \frac{1}{2\pi\eta c}(-)^q(\cos\phi_1+\cos\phi_2)^2 J_{m-2q}(e\vec{a}\cdot\vec{r})J_m(e\vec{a}\cdot\vec{r}\,') , \quad (107)$$

where Eqs. (106) and (96) have been used. The quantities ϕ_1,ϕ_2 depend directly on n, and n is always a very large number. Hence

$$(\cos\phi_1+\cos\phi_2)^2=1+\tfrac{1}{2}\cos2\phi_1+\tfrac{1}{2}\cos2\phi_2+\cos(\phi_1+\phi_2)+\cos(\phi_1-\phi_2) \tag{108}$$

can be replaced by its average value of unity because of the rapid oscillation of the trigonometric functions as n changes. The j and m sums in Eq. (107) can then be carried out. From Neumann's addition theorem,[32] one has

$$\sum_m J_{m-2q}(e\vec{a}\cdot\vec{r})J_m(e\vec{a}\cdot\vec{r}')=J_{2q}(e\vec{a}\cdot\vec{r}-e\vec{a}\cdot\vec{r}') , \tag{109}$$

which then also makes possible the sum over q in Eq. (107), to give[33]

$$\sum_q (-)^q J_{2q}(e\vec{a}\cdot\vec{r}-e\vec{a}\cdot\vec{r}')=\cos(e\vec{a}\cdot\vec{r}-e\vec{a}\cdot\vec{r}') . \tag{110}$$

The above results pertain to the case where the difference of indices $m-j$, which appears in Eq. (105), is an even number. When $m-j$ is odd, the cosine functions in Eq. (105) become sine functions. This has the essential consequence that the square of the sum of cosines in Eqs. (107) and (108) is replaced by

$$(\cos\phi_1+\cos\phi_2)(\sin\phi_1+\sin\phi_2)=\tfrac{1}{2}\sin2\phi_1+\tfrac{1}{2}\sin2\phi_2+\sin(\phi_1+\phi_2) . \tag{111}$$

When Eq. (111) is averaged over its oscillating trigonometric functions, the outcome is zero. Hence, the contribution of odd $m-j$ differences is heavily dominated by the contribution of even $m-j$ differences, and so only the even case will be retained.

With Eqs. (109) and (110) employed in Eq. (107), only a single sum remains. When all of the above results are incorporated into Eq. (64) for the total transition probability per unit time, one has

$$W_1=\frac{2G^2}{(2\pi)^5}\int dE_e\int d\Omega_e E_e(E_e{}^2-m^2)^{1/2}\sum_n (E_0-E_e-\eta\omega+n\omega)^2\frac{1}{\eta c}\times\int d^3r\int d^3r'\cos(e\vec{a}\cdot\vec{r}-e\vec{a}\cdot\vec{r}')\mathscr{B}_1 . \tag{112}$$

The twofold spatial integration in Eq. (112) combines with the definition of $\mathscr{B}_1$ in Eq. (63) to give

$$\int d^3r\int d^3r'(\cos e\vec{a}\cdot\vec{r}\cos e\vec{a}\cdot\vec{r}'+\sin e\vec{a}\cdot\vec{r}\sin e\vec{a}\cdot\vec{r}')\mathscr{B}_1 = |\cos(ear\cos\theta)|_{fi}{}^2+|\sin(ear\cos\theta)|_{fi}{}^2+\kappa^2|\cos(ear\cos\theta)\vec{\sigma}|_{fi}{}^2+\kappa^2|\sin(ear\cos\theta)\vec{\sigma}|_{fi}{}^2 . \tag{113}$$

Equations (112) and (113) yield

$$W_1=\frac{2G^2}{(2\pi)^5}\int dE_eE_e(E_e{}^2-m^2)^{1/2}\int d\Omega_e\sum_n (E_0-E_e-\eta\omega+n\omega)^2\frac{1}{\eta c}\times\left[\left|\cos\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\right|_{fi}^2+\left|\sin\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\right|_{fi}^2+\kappa^2\left|\cos\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\vec{\sigma}\right|_{fi}^2+\kappa^2\left|\sin\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\vec{\sigma}\right|_{fi}^2\right] . \tag{114}$$

The sum over index n will now be accomplished. Since n is always very large, the parameter $\nu=n/\eta$ [see Eq. (89)] will be viewed as a continuous variable. The sum over n is then replaced by an integration over ν in the form

$$\sum_{n=\eta-(E_0+E_e)/\omega}^{\eta}\frac{(E_0-E_e-\eta\omega+n\omega)^2}{\eta c}=(\eta\omega)^2\int_{1-(E_e-E_e)/\eta\omega}^{1}d\nu\frac{\left[\dfrac{E_0-E_0}{\eta\omega}-1+\nu\right]^2}{[b^2+\frac{1}{2}(1-\nu)]^{1/2}} , \tag{115}$$

where the limits on the sum come from Eqs. (79) and (104). With the notation

$$\alpha=\frac{E_0-E_e}{\eta\omega} \tag{116}$$

and change of integration variable

$$\beta=1-\nu ,$$

Eq. (115) becomes

$$\sum_n \frac{(E_0-E_e-\eta\omega+n\omega)^2}{\eta c}=2^{1/2}(\eta\omega)^2\int_0^\alpha d\beta\frac{(\alpha-\beta)^2}{(2b^2+\beta)^{1/2}}$$
$$=\frac{2^{3/2}}{15}(\eta\omega)^2[8(2b^2+\alpha)^{5/2}-8(2b^2)^{5/2}-20(2b^2)^{3/2}\alpha-15(2b^2)^{1/2}\alpha^2] . \tag{117}$$

The solid angle integral in Eq. (114) will now be treated. It is convenient to introduce a dimensionless integration variable for the energy integral by setting

$$\epsilon_e=E_e/m . \tag{118}$$

With the further definitions

$$\epsilon_0=E_0/m,\ \ \rho_e=p_e/m , \tag{119}$$

and with z_f as given in Eq. (75) or (76), the parameter α of Eq. (116) is

$$\alpha=\frac{2}{z_f}(\epsilon_0-\epsilon_e)(\epsilon_e-\rho_e\sin\theta_e\cos\phi_e) ; \tag{120}$$

the parameter $2b^2$ is, from Eqs. (89) and (18),

$$2b^2=\frac{{\rho_e}^2}{z_f}\cos^2\theta_e ; \tag{121}$$

and η in the combination $\eta\omega/m$ is

$$\frac{\eta\omega}{m}=\frac{z_f}{2}\frac{1}{(\epsilon_0-\rho_e\sin\theta_e\cos\phi_e)} . \tag{122}$$

Implicit in Eqs. (120)–(122) is a selection of the axes of the spherical polar coordinates p_e, θ_e, and ϕ_e. Setting $\vec{p}_e\cdot\vec{\epsilon}=p_e\cos\theta_e$ as in Eq. (121) means that the polar axis is along the $\vec{\epsilon}$ direction; and setting

$$\vec{p}_e\cdot\vec{k}=p_e\omega\sin\theta_e\cos\phi_e$$

as in Eqs. (120) and (122) means that the x axis is along $\vec{k}$.[34]

The solid angle integration indicated by Eq. (114) with Eq. (117) is very complicated. It can be simplified by the approximation that those quantities in the integrand which are proportional to $\cos\phi_e$ will average essentially to zero when integrated over ϕ_e from $-\pi$ to π. The remaining integration over $\cos\theta_e$ can then be done in closed form to yield

$$\int d\Omega_e\left[\frac{\eta\omega}{m}\right]^2[8(2b^2+\alpha)^{5/2}-8(2b^2)^{5/2}-20(2b^2)^{3/2}\alpha-15(2b^2)^{1/2}\alpha^2]$$
$$\approx\frac{5\pi}{2}\frac{1}{z_f^{1/2}{\epsilon_e}^2}\left[\frac{\sigma^3}{|\rho_e|}\ln\left[\frac{(\sigma+{\rho_e}^2)^{1/2}+|\rho_e|}{\sigma^{1/2}}\right]+\tfrac{8}{15}(\sigma+{\rho_e}^2)^{5/2}+\tfrac{2}{3}\sigma(\sigma+{\rho_e}^2)^{3/2}\right.$$
$$\left.+\sigma^2(\sigma+{\rho_e}^2)^{1/2}-\tfrac{8}{15}|\rho_e|^5-2\sigma|\rho_e|^3-3\sigma^2|\rho_e|\right] , \tag{123}$$

where

$$\sigma \equiv 2\epsilon_e(\epsilon_0-\epsilon_e) . \tag{124}$$

When Eq. (123) is substituted in Eq. (114), the result is of the form

$$W_1 = \frac{G^2 m^5}{2\pi^3} f_1 \, |M_{\rm ind}|^2 . \tag{125}$$

The quantities in Eq. (125) are defined so that $|M_{\rm ind}|^2$ contains the entire nuclear matrix element for induced beta decay, including all intensity dependence; and f_1 is the spectral integral for the emitted electron. Specifically, the definitions are introduced that the squared nuclear matrix element is

$$|M_{\rm ind}|^2 = \frac{1}{4\pi(2z_f)^{1/2}} \left[\left| \cos\left[z^{1/2}\frac{r}{R_0}\cos\theta \right] \right|_{fi}^2 + \left| \sin\left[z^{1/2}\frac{r}{R_0}\cos\theta \right] \right|_{fi}^2 + \kappa^2 \left| \cos\left[z^{1/2}\frac{r}{R_0}\cos\theta \right] \vec{\sigma} \right|_{fi}^2 + \kappa^2 \left| \sin\left[z^{1/2}\frac{r}{R_0}\cos\theta \right] \vec{\sigma} \right|_{fi}^2 \right] , \tag{126}$$

and the spectral integral is

$$f_1(\epsilon_0) = \int_1^{\epsilon_0} d\epsilon_e h_1(\epsilon_0,\epsilon_e) , \tag{127}$$

where $h_1(\epsilon_0,\epsilon_e)$ is the spectral function

$$h_1(\epsilon_0,\epsilon_e) = \frac{1}{3}\frac{\sigma^3}{\epsilon_e}\ln\left[\frac{(\sigma+\rho_e^2)^{1/2}+|\rho_e|}{\sigma^{1/2}} \right] + \frac{(\sigma+\rho_e^2)^{1/2}|\rho_e|}{45\epsilon_e}[8(\sigma+\rho_e^2)^2+10\sigma(\sigma+\rho_e^2)+15\sigma^2] - \frac{\rho_e^2}{45\epsilon_e}(8\rho_e^4+30\sigma\rho_e^2+45\sigma^2) . \tag{128}$$

The form (125) corresponds to the standard form for allowed beta decay, where

$$W_0 = \frac{G^2 m^5}{2\pi^3} f_0 \, |M_0|^2 ,$$

with

$$|M_0|^2 = |1|_{fi}^2 + \kappa^2 |\vec{\sigma}|_{fi}^2 ,$$

and, when Coulomb corrections are neglected, as they are in the present work, the spectral integral is

$$f_0(\epsilon_0) = \int_1^{\epsilon_0} d\epsilon_e h_0(\epsilon_0,\epsilon_e); \quad h_0(\epsilon_0,\epsilon_e) = \epsilon_e(\epsilon_e^2-1)^{1/2}(\epsilon_0-\epsilon_e)^2 .$$

Were a complete expression for W_1 retained, it would reduce to a form like W_0 in the limit $z \to 0$ (whereas $W_2 \to 0$, $W_3 \to 0$ as $z \to 0$). However, the results obtained for W_1 as given in Eqs. (125)–(128) derive from the asymptotic form for $J_n(-\zeta,-\frac{1}{2}\eta)$, and $|\zeta| \gg 1$, $\eta \gg 1$ are incompatible with $z \to 0$. Thus, the zero intensity limit of W_1 is not directly accessible from Eqs. (125)–(128). One must return to Eq. (64) for the low intensity limit.

2. Spin term

The generalized Bessel functions in Eq. (70) will be replaced by their asymptotic forms. Equation (100) gives

$$J_{n+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-1}(-\zeta,-\tfrac{1}{2}\eta) \approx \frac{2}{(2\pi\eta c)^{1/2}}[(c-b)\cos\phi_1-(c+b)\cos\phi_2] ,$$

$$J_{n-j+m+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-j+m-1}(-\zeta,-\tfrac{1}{2}\eta) \approx \frac{(-)^q 2}{(2\pi\eta c)^{1/2}}[(c-b)\cos\phi_1-(c+b)\cos\phi_2] ,$$

where, as in the W_1 case, $m-j=2q$ gives the dominant contribution, and odd-integer $m-j$ values are unimportant. As before, the energy contribution $m\omega$ can be neglected as compared to the other energies, and so the threefold sum over n,j,m indices involves only ordinary and generalized Bessel functions. The sums are

$$\sum_n \sum_j \sum_m [J_{n+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-1}(-\zeta,-\tfrac{1}{2}\eta)][J_{n-j+m+1}(-\zeta,-\tfrac{1}{2}\eta) + J_{n-j+m-1}(-\zeta,-\tfrac{1}{2}\eta)]J_j(e\vec{a}\cdot\vec{r})J_m(e\vec{a}\cdot\vec{r}\,') \approx \sum_n \sum_m \sum_q \frac{(-)^q 2}{\pi\eta c}[(c-b)\cos\phi_1-(c+b)\cos\phi_2]^2 J_{m-2q}(e\vec{a}\cdot\vec{r})J_m(e\vec{a}\cdot\vec{r}\,') \ . \quad (129)$$

When the squared term containing the phases ϕ_1,ϕ_2 is averaged because of the rapid oscillations of these phases, the result is just c^2+b^2. The sums over m and q can then be done exactly as in Eqs. (109) and (110). These results employed in Eq. (129), plus the procedure shown in Eq. (113), take Eq. (70) to the form

$$W_2=\frac{4G^2}{(2\pi)^5}\int dE_e(E_e{}^2-m^2)^{1/2}\int d\Omega_e \eta\omega \sum_n (E_0-E_e-\eta\omega+n\omega)^2\frac{(c^2+b^2)}{\eta c} \times\left[\left|\cos\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\right|_{fi}^2+\left|\sin\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\right|_{fi}^2 +\kappa^2\left|\cos\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\vec{\sigma}\right|_{fi}^2+\kappa^2\left|\sin\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\vec{\sigma}\right|_{fi}^2\right] . \quad (130)$$

As was done in the direct term, the sum over n will be performed by treating $\nu(=n/\eta)$ as a continuous variable. The sum to be done is

$$\sum_{n=\eta-(E_0+E_e)/\omega}^{\eta}(E_0-E_e-\eta\omega+n\omega)^2\frac{(c^2+b^2)}{\eta c}=\frac{(\eta\omega)^2}{2^{1/2}}\int_0^\alpha d\beta\frac{(\alpha-\beta)^2(4b^2+\beta)}{(2b^2+\beta)^{1/2}}$$

with α defined as in Eq. (116), and $\beta=1-\nu$. The outcome of the β integral is

$$\sum_n (E_0-E_e-\eta\omega+n\omega)^2\frac{(c^2+b^2)}{\eta c}=\frac{2^{5/2}}{105}(\eta\omega)^2[2(2b^2+\alpha)^{7/2}+14(2b^2)(2b^2+\alpha)^{5/2} -9(2b^2)^{7/2}+28(2b^2)^{5/2}(2b^2+\alpha)-35(2b^2)^{3/2}(2b^2+\alpha)^2] . \quad (131)$$

When the extra $\eta\omega$ factor appearing in Eq. (130) is incorporated with Eq. (131), the solid angle integration that must be performed is

$$\int d\Omega_e\left[\frac{\eta\omega}{m}\right]^3[2(2b^2+\alpha)^{7/2}+14(2b^2)(2b^2+\alpha)^{5/2}-35(2b^2)^{3/2}(2b^2+\alpha)^2+28(2b^2)^{5/2}(2b^2+\alpha)-9(2b^2)^{7/2}] \approx\frac{5\pi}{2}\frac{1}{z_f^{1/2}\epsilon_e{}^3}\left[\frac{\sigma^4}{|\rho_e|}\ln\left[\frac{(\sigma+\rho_e{}^2)^{1/2}+|\rho_e|}{\sigma^{1/2}}\right]+\tfrac{2}{5}(\sigma+\rho_e{}^2)^{7/2}-|\rho_e|^3(\tfrac{7}{4}\sigma^2+\tfrac{7}{5}\sigma\rho_e{}^2+\tfrac{2}{5}\rho_e{}^4)\right] , \quad (132)$$

where definitions introduced in Eqs. (118), (119), and (124) are employed; and where it is again assumed that quantities in the integrand which are proportional to $\cos\phi_e$ will average to zero when integrated over ϕ_e from $-\pi$ to π.

The spin part of the total transition probability per unit time is of the form

$$W_2=\frac{G^2m^5}{2\pi^3}f_2|M_{\text{ind}}|^2 , \quad (133)$$

exactly as in Eq. (125), where $|M_{\text{ind}}|^2$ is given in Eq. (126). In analogy with Eq. (127), the spectral integral is

$$f_2(\epsilon_0)=\int_1^{\epsilon_0}d\epsilon_e h_2(\epsilon_0,\epsilon_e) , \quad (134)$$

where Eqs. (130) and (132) give the spectral function as

$$h_2(\epsilon_0,\epsilon_e)=\frac{4}{21}\frac{\sigma^4}{\epsilon_e{}^3}\ln\left[\frac{(\sigma+\rho_e{}^2)^{1/2}+|\rho_e|}{\sigma^{1/2}}\right]+\frac{8}{105}\frac{1}{\epsilon_e{}^3}[\,|\rho_e|(\sigma+\rho_e{}^2)^{7/2}-\rho_e{}^8]-\frac{\sigma\rho_e{}^4}{3\epsilon_e{}^3}(\sigma+\tfrac{4}{5}\rho_e{}^2)\ . \quad (135)$$

3. Interference term

When the generalized Bessel functions in Eq. (72) are treated according to Eqs. (96) and (100), and averages are taken in the resulting trigonometric functions, the consequences are

$$J_n(-\zeta,-\tfrac{1}{2}\eta)[J_{n-j+m+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-j+m-1}(-\zeta,-\tfrac{1}{2}\eta)]\approx-\frac{(-)^q b}{\pi\eta c}\ ,$$

$$J_{n-j+m}(-\zeta,-\tfrac{1}{2}\eta)[J_{n+1}(-\zeta,-\tfrac{1}{2}\eta)+J_{n-1}(-\zeta,-\tfrac{1}{2}\eta)]\approx-\frac{(-)^q b}{\pi\eta c}\ ,$$

where again $m-j=2q$ and odd $m-j$ differences can be neglected. Sums over m and j are done as in Eqs. (109) and (110), the definitions of Eqs. (73) and (63) are employed, and the outcome is that Eq. (72) becomes

$$W_3=\frac{G^2}{(2\pi)^5}\int dE_eE_e(E_e{}^2-m^2)^{1/2}\int d\Omega_e\frac{ea\omega}{2p_e\cdot k}\left[-\frac{\vec{p}_e\cdot\vec{\epsilon}}{E_e}+i\frac{\vec{k}}{\omega}\times\frac{\vec{p}_e}{E_e}\cdot\vec{\epsilon}\right]\sum_n(E_0-E_e-\eta\omega+n\omega)^2\frac{4b}{\eta c}$$
$$\times\left[\left|\cos\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\right|_{fi}^2+\left|\sin\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\right|_{fi}^2\right.$$
$$\left.+\kappa^2\left|\cos\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\vec{\sigma}\right|_{fi}^2+\kappa^2\left|\sin\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\vec{\sigma}\right|_{fi}^2\right]\ . \quad (136)$$

The sum over n that has to be done in Eq. (136) is the same as the one which appears in Eq. (112), and which is evaluated in Eq. (117). The philosophy for the solid angle integral is as before, with terms proportional to $\cos\phi_e$ or $\sin\phi_e$ neglected. As a result, the term $\vec{k}\times\vec{p}_e\cdot\vec{\epsilon}$ can be dropped in Eq. (136). The solid angle integral that arises from Eq. (136) with Eq. (117) is

$$\int d\Omega_e\frac{\omega\vec{p}_e\cdot\vec{\epsilon}}{p_e\cdot k}\left[-\frac{\vec{p}_e\cdot\vec{\epsilon}}{E_e}\right]\left[\frac{\eta\omega}{m}\right]^2[8(2b^2+\alpha)^{5/2}-8(2b^2)^{5/2}-20(2b^2)^{3/2}\alpha-15(2b^2)^{1/2}\alpha^2]$$
$$\approx\frac{5\pi}{16}\frac{1}{z_f^{1/2}\epsilon_e{}^4}\left\{\frac{\sigma^4}{|\rho_e|}\ln\left[\frac{(\sigma+\rho_e{}^2)^{1/2}+|\rho_e|}{\sigma^{1/2}}\right]+(\sigma+\rho_e{}^2)^{1/2}\right.$$
$$\left.\times[\sigma^3+\tfrac{2}{3}\sigma^2(\sigma+\rho_e{}^2)+\tfrac{8}{15}\sigma(\sigma+\rho_e{}^2)^2-\tfrac{16}{5}(\sigma+\rho_e{}^2)^3]+4|\rho_e|^3(3\sigma^2+\tfrac{8}{3}\sigma\rho_e{}^2+\tfrac{4}{5}\rho_e{}^4)\right\}\ . \quad (137)$$

As with the other parts, the interference part of the total transition probability per unit time is of the form

$$W_3=\frac{G^2m^5}{2\pi^3}f_3|M_{\rm ind}|^2\ , \quad (138)$$

with $|M_{\rm ind}|^2$ shown in Eq. (126), and with Eqs. (137) and (138) leading to the spectral integral

$$f_3(\epsilon_0)=\int_1^{\epsilon_0}d\epsilon_e h_3(\epsilon_0,\epsilon_e)\ , \quad (139)$$

involving the spectral function

$$h_3(\epsilon_0,\epsilon_e)=\frac{1}{48}\frac{\sigma^4}{\epsilon_e^3}\ln\left[\frac{(\sigma+\rho_e^2)^{1/2}+|\rho_e|}{\sigma^{1/2}}\right]-\frac{1}{48}\frac{|\rho_e|(\sigma+\rho_e^2)^{1/2}}{\epsilon_e^3}(\sigma^3+\tfrac{118}{45}\sigma^2\rho_e^2+\tfrac{136}{15}\sigma\rho_e^4+\tfrac{16}{5}\rho_e^6)$$
$$+\frac{1}{12}\frac{\rho_e^4}{\epsilon_e^3}(3\sigma^2+\tfrac{8}{3}\rho_e^2+\tfrac{4}{5}\rho_e^4)\,. \quad (140)$$

D. Nuclear parameters

Application of the foregoing formalism to practical calculations requires first that a determination be made of the appropriate separation of the nucleus into a core and a fragment. This can be done in terms of the standard single-particle shell model.[35]

A few examples of how fragment assignments are made are given here. For example, ${}^{113}_{48}\mathrm{Cd}_{65}$ has a single nucleon fragment. The core nucleus, ${}^{112}_{48}\mathrm{Cd}_{64}$, is a stable nuclide in nature with spin and parity of 0^+. By the usual single-particle model, this means that this "even-even" nuclide has the spins of all of its protons and of all of its neutrons antialigned in pairs to give pairwise and overall zero angular momentum. The odd neutron in ^{113}Cd has a shell model assignment of $s_{1/2}$, which should then determine the entire nuclear spin and parity to be $(\frac{1}{2})^+$, which is the case. Upon beta decay, the unpaired $s_{1/2}$ neutron becomes an unpaired $g_{9/2}$ proton, which then contributes the entire observed $(\frac{9}{2})^+$ spin and parity of the final ${}^{113}_{49}\mathrm{In}_{64}$ nucleus.

An example of a two-nucleon fragment is provided by ${}^{90}_{38}\mathrm{Sr}_{52}$. The core nucleus, ${}^{88}_{38}\mathrm{Sr}_{50}$, is the principal stable isotope of strontium. In particular, $N=50$ is a magic number for the neutron shell in ^{88}Sr, and $Z=38$ represents the closure of an $f_{5/2}$ shell for the protons, so ^{88}Sr is a clear case of a stable, relatively tightly bound core nucleus. The two neutrons in ^{90}Sr beyond the magic number of $N=50$ then constitute the fragment, one of whose two neutrons will undergo beta decay. They must be considered as a pair because initially they are angular-momentum coupled to 0^+, and it is impossible to say which of the two will decay. Finally, the remaining $d_{5/2}$ neutron will couple to the newly formed $p_{1/2}$ proton to give the 2^- state of the ${}^{90}_{39}\mathrm{Y}_{51}$ daughter nucleus.

${}^{87}_{37}\mathrm{Rb}_{50}$ is an example of a nuclide where the fragment must consist of three nucleons. The odd proton in ^{87}Rb must be part of the fragment because initially this $p_{3/2}$ particle accounts for the entire ^{87}Rb spin and parity of $(\frac{3}{2})^-$. The beta decay itself involves a neutron, not the odd proton, and since the beta decay neutron is initially paired with another to give 0^+, then both of these neutrons must also be assigned to the fragment. In the final state, the $g_{9/2}$ neutron which beta decays to a $p_{3/2}$ proton will couple to 0^+ with the initial odd proton, while the remaining $g_{9/2}$ neutron finds itself unpaired in the final state, and so accounts for the $(\frac{9}{2})^+$ spin and parity of the ^{87}Sr daughter nucleus.

E. Form of the nuclear matrix element

Total transition probability per unit time is the sum of the three contributions W_1, W_2, and W_3, given in Eqs. (125), (133), and (138). Each of the W_n contains the same squared transition matrix element $|M_{\text{ind}}|^2$, defined in Eq. (126). This will now be examined in more detail.

Equation (126) is expressed as the sum of four terms. The first pair of terms arises from the vector part of the beta decay interaction, and corresponds to the usual Fermi matrix element of beta decay theory. The second pair of terms (the ones containing the Pauli spin operators $\vec{\sigma}$) comes from the axial vector part of the beta decay interaction, and corresponds to the usual Gamow-Teller matrix element of beta decay theory. However, a simplification can be introduced from isospin considerations, which have not been placed in evidence in the above work. For Fermi matrix elements, the isospin conservation rule is $\Delta T=0$,[36,37] where T is the total isospin quantum number. This condition is not satisfied for most transitions involving forbidden decays. Then only the Gamow-Teller matrix elements need to be retained. In such cases, Eq. (126) is replaced by

$$|M_{\text{ind}}|^2=\frac{\kappa^2}{4\pi(2z_f)^{1/2}}\left[\left|\cos\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\vec{\sigma}\right|_{fi}^2+\left|\sin\left[z^{1/2}\frac{r}{R_0}\cos\theta\right]\vec{\sigma}\right|_{fi}^2\right]. \quad (141)$$

The terms in the square bracket in Eq. (141) are squared nuclear transition matrix elements, with the f and i subscripts referring to final and initial nuclear states. The coordinate r which occurs in the matrix elements refers to the position vector $\vec{r}$ of the nuclear fragment with respect to the nuclear core. In practical calculation of the nuclear matrix elements, one needs the coordinates of the separate nucleons contained in the fragment. The vector $\vec{r}$ gives the location of the c.m. of the fragment. Since each nucleon in the fragment can be taken to have the same mass M, then the position vector of the jth nucleon in the fragment (r_j) is related to r by

$$qM\vec{r}=\sum_{j=1}^{q} M\vec{r}_j \ ,$$

where q is the total number of nucleons in the fragment. Since only one of these q nucleons will undergo beta decay (say the jth one), then whenever $r\cos\theta$ appears in the matrix element, the replacement

$$r\cos\theta \rightarrow \frac{1}{q} r_j \cos\theta_j \tag{142}$$

should be used, where θ_j measures the angle between $\vec{r}_j$ and the polarization vector of the applied field.

Equation (141) can be stated in more detail as

$$|M_{\text{ind}}|^2=\frac{\kappa^2}{4\pi(2z_f)^{1/2}}(|\vec{M}_{fi}^{\cos}|^2+|\vec{M}_{fi}^{\sin}|^2) \ , \tag{143}$$

where

$$\vec{M}_{fi}^{\cos}=\frac{1}{(2j_i+1)}\sum_{m_i}\sum_{m_f}\left[\psi_f,\cos\left[\frac{z^{1/2}}{q}u_j\cos\theta_j\right]\vec{\sigma}\psi_i\right] \tag{144}$$

$$\vec{M}_{fi}^{\sin}=\frac{1}{(2j_i+1)}\sum_{m_i}\sum_{m_f}\left[\psi_f,\sin\left[\frac{z^{1/2}}{q}u_j\cos\theta_j\right]\vec{\sigma}\psi_i\right] . \tag{145}$$

In Eqs. (144) and (145), u_j is the dimensionless radial coordinate

$$u_j=r_j/R_0 \ ; \tag{146}$$

j_i is the total angular momentum of the initial state, so that $(2j_i+1)^{-1}$ times the sum over m_i is an average over orientations of initial angular momentum; and the sum over m_f is a sum over orientations of the final angular momentum. In practice, only one of the two terms in Eq. (143) will be nonzero. When ψ_f and ψ_i have the same parity, only $\vec{M}_{fi}^{\cos}$ will survive; and when they have opposite parity, only $\vec{M}_{fi}^{\sin}$ will survive. Practical calculational examples are given in a companion paper.

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