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Half-shell scattering by a screened Coulomb potential

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We investigate the physical half-shell T matrix for scattering by a smoothly screened Coulomb potential. In particular, we study the Kowalski-Noyes half-shell extension function $F^{l}_{\rm hs}(p;k)$ in the limit that the screening radius a tends to infinity. For the Coulomb potential the half-shell scattering cross section is a discontinuous (step) function of the off-shell momentum p at the on-shell point $p=k$. For the *screened* Coulomb potential (with screening of the Hulthén form, for $l=0$) we find that the half-shell cross section exhibits *almost* a step behavior near $p=k$: Its magnitude varies dramatically in an interval, the width of which is of the order of the inverse screening radius a. We discuss physical implications of this result.

[NUCLEAR REACTIONS Half-shell scattering, Coulomb potential, screening effects, Kowalski-Noyes half-shell extension function.]

I. INTRODUCTION

In charged-particle scattering the long range of the Coulomb interaction is a source of special difficulties. These difficulties can be recognized already in classical scattering. At positive energies classical particles subject to a Coulomb or Newton force follow hyperbolic Kepler orbits. At large distances and large times, $t\rightarrow\pm\infty$, their trajectories differ in an essential manner from the trajectories (straight lines) of free particles with the same energy and the same asymptotic velocity. This is clear from the presence of a term logarithmic in t in the position vector of a particle subject to the Coulomb or Newton force. This term is due to the $1/r$ behavior of the potential at large distances. Such a term is not present for potentials which behave as $r^{-\alpha}$, $\alpha>1$, or potentials which are exponentially bounded for $r\rightarrow\infty$.

In quantum mechanics the situation is similar. When expressed in the coordinate representation the scattering wave functions for charged-particle scattering (i.e., at positive energies) exhibit an essentially different behavior at large distances, compared to scattering wave functions from short-range potentials. This deviation from the short-range case is seen more clearly if one works in the momentum representation. In particular, the off-shell transition (T) matrix has a branch-point type singularity in the on-shell point. Among others, this means that the off-shell T-matrix elements have no on-shell limits.[1] By the introduction of suitably defined Coulombian asymptotic states one nevertheless can define a physical scattering amplitude.[2] If only the Coulomb potential is acting this is the Rutherford amplitude f^c. Also, one can define a physical half-shell scattering amplitude. It has a singularity in its on-shell point, such that its on-shell limit does not exist. Yet, the on-shell limit of its modulus can be defined, but its value depends on whether the off-shell momentum p approaches the on-shell momentum k from above or from below. The corresponding two limits can be very different.[3] Relatively little attention has been paid to this fact. We note that this singular behavior occurs both in the full T matrix, and in its partial-wave (p.w.) projections.

In contradistinction, the T-matrix elements for short-range potentials are continuous functions of the off-shell momenta in the on-shell point (both for the full and the p.w. case). This fact is basic for the so-called Kowalski-Noyes (KN) method.[4] Using Muskhelishvili techniques, this method reduces the singular p.w. Lippmann-Schwinger (LS) equation for T at positive energies to a nonsingular integral equation of the Fredholm type. A key ingredient (in each p.w. space l) is the KN half-shell extension function $F^{l}_{\rm hs}(p;k)$. It is defined as the ratio of the half-shell and on-shell p.w. T-matrix elements. The KN method is convenient to calculate scattering phase shifts.

In the present paper we shall study a smoothly screened Coulomb potential. For finite values of the screening radius a the KN half-shell extension function is well defined. We investigate the limit of unscreening, i.e., $a\rightarrow\infty$. We shall use the particular type of screening provided by the Hulthén potential. Its radial dependence $1/[\exp(r/a)-1]$ allows the investigation for S waves to be carried out largely by analytic means.

In Sec. II we discuss scattering by the pure Coulomb potential, acting in all partial waves, and the singular behavior of various quantities at the on-shell point. Section III briefly recalls how, in the case of a short-range potential, Jost functions, Jost solutions, and the KN half-shell extension function are defined, in one partial-wave state l. The pure Coulomb case in one partial-wave state is discussed in Sec. IV. In Sec. V we have collected a number of remarkable inequalities which involve matrix elements of the transition operators T_c and T_{cl} for the Coulomb interaction. These inequalities take an elegant form when we make a comparison with the corresponding matrix elements of the Coulomb potential V_c. The KN half-shell extension function cannot be defined in the usual manner for the Coulomb potential. In Sec. VI we consider some related half-shell ratios for this interaction. The connection between the KN half-shell extension function for our *screened* Coulomb potential, and the half-shell ratio introduced in Sec. VI for one particular l, is investigated in detail in Sec. VII. There the limit that the screening parameter a tends to infinity is considered. Section VIII concludes this paper with a discussion.

II. THE PURE COULOMB CASE

We shall use the notations and conventions which have been used and developed in Refs. 2 and 3. In particular, we shall use the so-called (Coulombian) asymptotic states $|\vec{k}\infty\pm\rangle$ and the partial-wave projected asymptotic states $|kl\infty\pm\rangle$. Often we suppress the symbols l and $+$ in this notation. Our units are such that $\hbar=1=2m$. The Coulomb potential V is

$$V_c(r)=Ze^2/r=-2s/r\equiv 2k\gamma/r\ . \tag{2.1}$$

The constant $s\equiv-k\gamma$ is real, $s>0$ corresponds to attraction, for $s<0$ we have repulsion, and $|s|^{-1}$ is the Bohr radius. The energy-dependent dimensionless parameter γ is Sommerfeld's parameter, and k is related to the energy E by the relation $E\equiv(k+i\epsilon)^2$, $\epsilon\downarrow 0$.

The physical on-shell (Coulomb) T matrix is

$$\begin{aligned}\langle\vec{k}'\infty-|T_c|\vec{k}\infty\rangle&=\langle\vec{k}'\infty-|V_c|\vec{k}+\rangle_c\\ &=-\frac{1}{2\pi^2}f^c(\hat{k}\cdot\hat{k}'),\ \ \hat{k}'\neq\hat{k},\ \ k'=k\in\mathrm{R}^+\ ,\\ &=-\frac{1}{2\pi^2}\cdot\frac{-\gamma}{2k}e^{2i\sigma_0}\left[\frac{1-\hat{k}\cdot\hat{k}'}{2}\right]^{-1-i\gamma}\\ &=\frac{k\gamma}{\pi^2Q^2}e^{2i\sigma_0}\left[\frac{4k^2}{Q^2}\right]^{i\gamma}.\end{aligned}\tag{2.2}$$

Throughout, we suppress the energy dependence of the T operator $T(E)$. Furthermore, we have introduced the momentum transfer $\vec{Q}$,

$$\vec{Q}\equiv\vec{k}'-\vec{k},\ \ Q=|\vec{k}'-\vec{k}|,\ \ 0\le Q\le 2k\ . \tag{2.3}$$

In Eq. (2.2) f^c is the Coulomb scattering amplitude, and σ_0 is the Coulomb phase shift defined by

$$\Gamma(l+1+i\gamma)/\Gamma(l+1-i\gamma)=\exp(2i\sigma_l)\ . \tag{2.4}$$

Cross sections σ and amplitudes f are connected by $\sigma(x)=f^*(x)f(x)$. Amplitudes are $-2\pi^2$ times the corresponding T-matrix element.

The physical half-shell T matrix is $\langle \vec{p} \mid T_c \mid \vec{k}\infty \rangle$. By using

$$T_c \mid \vec{k}\infty \rangle = V_c \mid \vec{k}+\rangle_c$$

it can also be written as

$$\langle \vec{p} \mid T_c \mid \vec{k}\infty \rangle = \langle \vec{p} \mid V_c \mid \vec{k}+\rangle_c \ , \tag{2.5}$$

where $\mid \vec{k}+\rangle_c$ is the (Coulomb) scattering state at energy $k^2>0$. It has the simple form

$$\langle \vec{p} \mid T_c \mid \vec{k}\infty \rangle = \frac{k\gamma}{\pi^2} e^{-(1/2)\pi\gamma}\Gamma(1+i\gamma)q^{-2-2i\gamma} \times \lim_{\epsilon\downarrow 0}[p^2-(k+i\epsilon)^2]^{i\gamma} \ , \quad p\neq k \ . \tag{2.6}$$

Here the momentum transfer $\vec{q}$ has been introduced,

$$\vec{q} \equiv \vec{p}-\vec{k}, \quad q=\mid \vec{p}-\vec{k} \mid \ , \\ \mid p-k \mid \leq q \leq p+k \ . \tag{2.7}$$

In the following we shall take the directions of $\vec{p}$ and $\vec{k}'$ to be the same ($\hat{p}=\hat{k}'$). Their magnitudes, in general, will be different. The potential-matrix element

$$\langle \vec{p} \mid V_c \mid \vec{k} \rangle = k\gamma\pi^{-2}q^{-2}, \quad \vec{p}\neq\vec{k} \ , \tag{2.8}$$

is real for $p>0$, $k>0$.

Intimately connected to the long range of the Coulomb potential are certain singularities of the half-shell and off-shell Coulomb T matrix. These singularities are branch points of the expression $(p-k)^{i\gamma}$. They occur when the half-shell and off-shell variables take their on-shell value. In fact, the "on-shell limit" of the physical half-shell T-matrix element $\langle \vec{p} \mid T_c \mid \vec{k}\infty \rangle$ does not equal the physical on-shell matrix element, in both cases $p\uparrow k$ and $p\downarrow k$. Both of these "limits" do not exist. The on-shell limits for $p\uparrow k$ and for $p\downarrow k$ of the *modulus* of the physical half-shell T matrix *do* exist. However, these two limits differ from each other, and they also differ from the modulus of the physical on-shell T matrix. Consequently, the two on-shell limits ($p\uparrow k$ and $p\downarrow k$) of the half-shell scattering *cross section* are not equal to each other, and neither of them is equal to the physical on-shell scattering cross section. The limits are readily calculated. We choose the following convenient notation. Because of the $i\epsilon$ prescription (k stands for $k+i\epsilon$, $\epsilon\downarrow 0$), we have

$$(p^2-k^2)^{i\gamma} = e^{\pi\gamma}(k^2-p^2)^{i\gamma} \ , \quad \text{if } 0<p<k \ . \tag{2.9}$$

It is useful to introduce the step function $\mathcal{S}$,

$$\mathcal{S} \equiv \begin{cases} 1 & \text{for } p>k>0 \ , \\ \exp(\pi\gamma) & \text{for } 0<p<k \ . \end{cases} \tag{2.10}$$

Furthermore, $C_0{}^2$ is the familiar Coulomb penetrability factor

$$C_0{}^2 \equiv \frac{2\pi\gamma}{\exp(2\pi\gamma)-1} \ . \tag{2.11}$$

For the pure Coulomb case, Eqs. (2.2) and (2.6) give

$$\sigma_{\rm half}^{\rm Coul}(\hat{p}\cdot\hat{k}) = \sigma_{\rm on}^{\rm Coul}(\hat{k}'\cdot\hat{k})C_0{}^2\mathcal{S}^2Q^4/q^4 \ . \tag{2.12}$$

Because $q\to Q$ for $p\to k$, we have

$$\lim_{p\downarrow k}\sigma_{\rm half} = \exp(-2\pi\gamma)\lim_{p\uparrow k}\sigma_{\rm half} \\ = C_0{}^2\sigma_{\rm on} \ . \tag{2.13}$$

In Eq. (2.13) we have deleted the superscript Coul, because it holds for any potential $V=V_c+V_s$, where V_s is a short-range potential.[3]

In Fig. 1, C_0 and $C_0\exp(\pi\gamma)$ are plotted as a function of k. For Coulomb repulsion ($k\gamma>0$) the $p\downarrow k$ limit of $\sigma_{\rm half}$ is smaller than $\sigma_{\rm on}$, whereas the $p\uparrow k$ limit is larger than $\sigma_{\rm on}$. For Coulomb attraction the reverse is true. Equation (2.12) shows that the ratio of the half-shell and on-shell Coulomb scattering cross sections consists of two factors, $C_0{}^2\mathcal{S}^2$ and Q^4/q^4. Only the first factor survives in

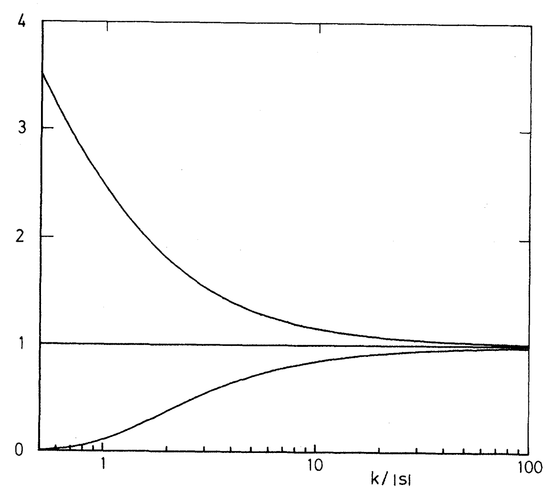


FIG. 1. Energy dependence of C_0 and $C_0\exp(\pi\gamma)$: The upper curve gives, as a function of k, (i) C_0 for the case of Coulomb attraction; (ii) $C_0\mathcal{S}$ for Coulomb repulsion as long as $k>p$; and (iii) $C_0\mathcal{S}$ for Coulomb attraction as long as $k<p$. The straight line marks the value 1. The lower curve gives, as a function of k, (i) C_0 for the case of Coulomb repulsion; (ii) $C_0\mathcal{S}$ for Coulomb repulsion as long as $k<p$; and (iii) $C_0\mathcal{S}$ for Coulomb attraction as long as $k>p$.

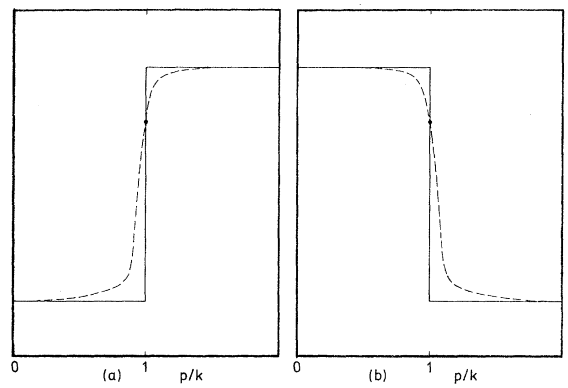


FIG. 2. (a) The step function $\mathscr{S}$ (full line) as a function of the off-shell momentum in the case of Coulomb attraction, and (b) the case of Coulomb repulsion. Also sketched (broken line) is the pseudostep function that we expect to represent the case of a smoothly screened Coulomb potential (cf. Secs. VII and VIII).

the on-shell limit. For $p \neq k$ it is independent of p, i.e., it is not dependent on how far one is off shell. Instead, as shown in Fig. 1, it is highly energy dependent. Its (rather trivial) dependence on p is shown in Fig. 2.

The other factor, Q^4/q^4, lies between 0 and $(\frac{1}{2}+\frac{1}{2}p/k)^{-4}$ for all $p,k>0$. For fixed $\hat{k}\cdot\hat{k}'$ it depends on the off-shell variable p/k only. For $p>k$ it represents a suppression factor (between 0 and 1) which is particularly effective in the forward directions. Note that if we consider the cross sections as functions of energy and *momentum transfer* (instead of energy and *scattering angle)* the following relation between $\sigma_{\rm half}^{\rm Coul}(E,q^2)$ and $\sigma_{\rm on}^{\rm Coul}(E,Q^2)$ holds:

$$\sigma_{\rm half}^{\rm Coul}(E,Q^2)=\sigma_{\rm on}^{\rm Coul}(E,Q^2)C_0{}^2\mathscr{S}^2 \ . \tag{2.14}$$

Note also that in half-shell scattering the magnitude of the momentum transfer $q=|\vec{p}-\vec{k}|$ can take values larger than $2k$, if $p>k$. Such large values of q are inaccessible in on-shell scattering.

III. THE SHORT-RANGE CASE IN ONE PARTIAL-WAVE STATE

The Jost[5,6] solution $f_l(k,r)$ is that solution of the radial Schrödinger equation,

$$\left[k^2+\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-V(r)\right]f_l(k,r)=0 \ , \tag{3.1}$$

which satisfies the asymptotic condition (in the convention of Ref. 6)

$$\lim_{r\to\infty} f_l(k,r)e^{-ikr}=1 \ . \tag{3.2}$$

The Jost *function* is defined by[6]

$$f_l(k)\equiv \lim_{r\to 0} f_l(k,r)(-2ikr)^l l!/(2l)! \ . \tag{3.3}$$

The *off-shell* Jost solution $f_l(k,p,r)$ and the *off-shell* Jost function $f_l(k,p)$ have been introduced by Fuda and Whiting[7]: $f_l(k,p,r)$ is that solution of the so-called inhomogeneous Schrödinger equation

$$\left[k^2+\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-V(r)\right]f_l(k,p,r) =(k^2-p^2)i^l prh_l^{(+)}(pr) \ , \tag{3.4}$$

which satisfies the asymptotic condition

$$\lim_{r\to\infty} f_l(k,p,r)e^{-ipr}=1 \ . \tag{3.5}$$

Here $h_l^{(+)}$ is the spherical Hankel function (in the convention of Messiah, see Ref. 7). The off-shell Jost function is defined by

$$f_l(k,p)\equiv \lim_{r\to 0} f_l(k,p,r)(-2ipr)^l l!/(2l)! \ . \tag{3.6}$$

For short-range potentials one has[7,8]

$$\lim_{p\to k} f_l(k,p,r)=f_l(k,r) \ , \tag{3.7}$$

$$\lim_{p\to k} f_l(k,p)=f_l(k) \ , \tag{3.8}$$

$$f_l(k,-p)=f_l^*(k,p) \quad (k \text{ and } p \text{ real}) \ , \tag{3.9}$$

and

$$\langle p\,|\,T_l\,|\,k\rangle=\left[\frac{k}{p}\right]^l \frac{f_l(k,p)-f_l(k,-p)}{i\pi p f_l(k)} \ . \tag{3.10}$$

As a consequence the KN half-shell extension function[4]

$$F_{\rm hs}^l(p;k)\equiv\frac{\langle p\,|\,T_l\,|\,k\rangle}{\langle k\,|\,T_l\,|\,k\rangle} \tag{3.11}$$

is real for p and k real,

$$F_{\rm hs}^l(p;k)=\left[\frac{k}{p}\right]^{l+1}\frac{f_l(k,p)-\text{c.c.}}{f_l(k)-\text{c.c.}} \ , \quad (p \text{ and } k \text{ real}) \ , \tag{3.12}$$

and normalized in such a way that

$$F_{\rm hs}^l(k;k)=1 \ . \tag{3.13}$$

Furthermore, we recall the parametrization of $\langle k\,|\,T_l\,|\,k\rangle$ in terms of the scattering phase shift δ_l,

$$\langle k \mid T_l \mid k \rangle = \frac{i}{\pi k}(e^{2i\delta_l}-1) = \frac{-2}{\pi k}e^{i\delta_l}\sin\delta_l \ . \quad (3.14)$$

IV. THE PURE COULOMB CASE IN ONE PARTIAL-WAVE STATE

When $V(r)$ in Eq. (3.1) has a $1/r$ behavior for $r \to \infty$ the definition of the Jost solution has to be modified. For the Coulomb potential it can be taken, according to Ref. 6,

$$\lim_{r\to\infty} f_{c,l}(k,r)\exp[-ikr + i\gamma\ln(2kr)] = 1 \ . \quad (4.1)$$

Explicit expressions for $f_{c,l}(k,r)$ are known.[1,8] For the Coulomb potential, the off-shell Jost solution $f_{c,l}(k,p,r)$ can be defined using the same definition as in the short-range case, i.e., through Eqs. (3.4) and (3.5). Also the Coulomb Jost function $f_{c,l}(k)$ and the off-shell Coulomb Jost function $f_{c,l}(k,p)$ can be defined using the same definitions as in the short-range case, i.e., through Eqs. (3.3) and (3.6), respectively.

The Coulomb Jost function is known,

$$f_{c,l}(k) = e^{(1/2)\pi\gamma}\Gamma(l+1)/\Gamma(l+1+i\gamma) \ . \quad (4.2)$$

Furthermore the off-shell Coulomb Jost function for $l=0$ is extremely simple[8]

$$f_{c0}(k,p) = \left[\frac{p+k}{p-k}\right]^{i\gamma} . \quad (4.3)$$

The on-shell limits ($p \to k$) of the off-shell Jost function and the off-shell Jost solution do not exist. We have, for $l=0$,

$$\lim_{p\to k}\left[\frac{p-k}{p+k}\right]^{i\gamma}\frac{e^{(1/2)\pi\gamma}}{\Gamma(1+i\gamma)}f_{c0}(k,p) = f_{c0}(k) \ . \quad (4.4)$$

Many limiting relations involving Jost functions and solutions, the limits $p \to k$, $r \to 0$, $r \to \infty$, and the unscreening limit $a \to \infty$ (cf. Sec. VII), are given in Ref. 8. For the pure Coulomb potential the physical half-shell Coulomb T matrix for $l=0$ is

$$\langle p \mid T_{c0} \mid k\,\infty \rangle = \langle p \mid V_{c0} \mid k+ \rangle_c = \lim_{\epsilon\downarrow 0}\frac{f_{c0}(k,p+i\epsilon) - f_{c0}^{*}(k,p+i\epsilon)}{i\pi p f_{c0}(k)} \ . \quad (4.5)$$

This leads to the following expressions,

$$\langle p \mid T_{c0} \mid k\,\infty \rangle = \frac{2\gamma}{\pi p}e^{(1/2)\pi\gamma}\mathscr{S}Q_0^{i\gamma}\left[\frac{p^2+k^2}{2pk}\right] = \frac{-2}{\pi p}C_0\mathscr{S}e^{i\sigma_0}\sin\left[\gamma\ln\left|\frac{p-k}{p+k}\right|\right]$$

$$(p>0,\ \ k>0,\ \ p\neq k) \ . \quad (4.6)$$

Here $Q_0^{i\gamma}$ is the Legendre function of the second kind.

V. EQUALITIES AND INEQUALITIES FOR SOME RATIOS INVOLVING T_c AND V_c

In Ref. 9 we introduced the ratio $\mathscr{R}$ of the off-shell Coulomb T matrix and the off-shell Coulomb potential matrix

$$\mathscr{R} \equiv \frac{\langle \vec{p}\,' \mid T_c \mid \vec{p} \rangle}{\langle \vec{p}\,' \mid V_c \mid \vec{p} \rangle}\ , \ \ p' \neq k,\ \ p \neq k \ . \quad (5.1)$$

For all physical values of $\hat{p}\,'\cdot\hat{p}$ this ratio satisfies

$$0 \leq |\mathscr{R}| \leq 1,\ \ p'>0,\ \ p>0,\ \ k>0 \ , \quad (5.2)$$

in case of Coulomb repulsion, and

$$\begin{aligned} &0 \leq |\mathscr{R}| \leq 1 \ , \\ &\quad 0<p'<k<p \text{ or } 0<p<k<p' \ , \\ &1 \overset{?}{\leq} |\mathscr{R}| \leq C_0{}^2 \ , \\ &\quad p'>k>0\ ,\ \ p>k>0 \\ &\quad \text{or } 0<p'<k,\ \ 0<p<k \ , \end{aligned} \quad (5.3)$$

in case of Coulomb attraction. (Inequalities marked with a question mark are conjectures.)

By rewriting Eq. (2.6), thereby introducing the ratio $\mathscr{R}_{\rm hs}$ of the physical half-shell T matrix and the half-shell Coulomb-potential matrix, we find

$$\mathscr{R}_{\rm hs} \equiv \frac{\langle \vec{p} \mid T_c \mid \vec{k}\,\infty \rangle}{\langle \vec{p} \mid V_c \mid \vec{k} \rangle} = C_0 e^{i\sigma_0}\left[\frac{p^2-k^2}{q^2}\right]^{i\gamma}\ ,\ \ p \neq k \ . \quad (5.4)$$

The modulus of $\mathscr{R}_{\rm hs}$ satisfies

$$|\mathscr{R}_{\rm hs}| = C_0\mathscr{S},\ \ p>0,\ \ k>0,\ \ p\neq k \ . \quad (5.5)$$

For completeness we mention the well-known result that the ratio $\mathscr{R}_{\rm ons}$ of the physical on-shell Coulomb T matrix and the on-shell Coulomb-potential matrix satisfies [cf. Eqs. (2.2) and (2.8)]

$$|\mathscr{R}_{\rm ons}|=1, \quad k>0 . \tag{5.6}$$

In Ref. 9, also

$$\mathscr{R}_l \equiv \frac{\langle p' \,|\, T_{cl} \,|\, p \rangle}{\langle p' \,|\, V_{cl} \,|\, p \rangle} \tag{5.7}$$

has been considered. For all physical l, one has

$$0 \le \mathscr{R}_l \le 1, \quad p'>0, \quad p>0, \quad k>0 \tag{5.8}$$

for Coulomb repulsion, and

$$0 \le |\mathscr{R}_l| \le 1, \quad 0<p'<k<p \text{ or } 0<p<k<p' ,$$

$$\left.\begin{array}{l} 0 \le |\mathscr{R}_{l=0}| \le C_0{}^2 \\ 0 \le |\mathscr{R}_l| \overset{?}{\le} C_0{}^2 \left| \binom{l+i\gamma}{l} \right| \end{array}\right| p'>k>0, \quad p>k>0 \text{ or } 0<p'<k, \quad 0<p<k , \tag{5.9}$$

for Coulomb attraction. From Sec. IV it follows that $\mathscr{R}_{{\rm hs},l=0}$, defined by

$$\mathscr{R}_{{\rm hs},l} \equiv \frac{\langle p \,|\, T_{cl} \,|\, k \infty \rangle}{\langle p \,|\, V_{cl} \,|\, k \rangle} , \quad p \ne k , \tag{5.10}$$

where

$$\langle p \,|\, V_{cl} \,|\, k \rangle = \frac{2\gamma}{\pi p} Q_l \left[\frac{p^2+k^2}{2pk} \right] , \tag{5.11}$$

satisfies

$$-\mu C_0 \mathscr{S} \le \mathscr{R}_{{\rm hs},l=0} \exp(-i\sigma_0) \le C_0 \mathscr{S} ,$$
$$p>0, \quad k>0, \quad p \ne k . \tag{5.12}$$

Here $\mu \equiv -\nu^{-1}\sin\nu = -\cos\nu$, where ν is the first positive zero of the spherical Bessel function j_1, hence $\nu \approx 4.4934$ and $\mu \approx 0.217$.

VI. EQUALITIES AND INEQUALITIES INVOLVING SOME HALF-SHELL RATIOS

We shall compare the physical half-shell T matrix with the physical on-shell T matrix, for the same k, $\hat{k}$, and $\hat{k}' \equiv \hat{p}$, for the full Coulomb potential. Their ratio, $F_{c,{\rm hs}}$,

$$F_{c,{\rm hs}} \equiv \frac{\langle p\hat{k}' \,|\, T_c \,|\, \vec{k} \infty \rangle}{\langle k\hat{k}' \infty - \,|\, T_c \,|\, \vec{k} \infty \rangle}$$
$$= C_0 e^{-i\sigma_0} \frac{Q^2}{q^2} \left[\frac{p^2-k^2}{4k^2} \frac{Q^2}{q^2} \right]^{i\gamma} \tag{6.1}$$

has no on-shell limit. We note that its modulus,

$$|F_{c,{\rm hs}}| = C_0 \mathscr{S} Q^2/q^2, \quad p \ne k , \tag{6.2}$$

again exhibits the familiar jump at the on-shell point $p=k$, cf. Eq. (2.10).

The p.w. projections of the Coulomb scattering amplitude

$$f^c = -2\pi^2 \langle \vec{k}' \infty - \,|\, T_c \,|\, \vec{k} \infty \rangle$$

and of $\langle \vec{k}' \,|\, V_c \,|\, \vec{k} \rangle$ are not well defined, because of the nonintegrability of these objects in the forward direction, $\hat{k}' \cdot \hat{k} = 1$, for $k'=k$. These difficulties are well understood,[10,11] and can be circumvented, as was shown by Taylor,[10] by the introduction of suitable test functions. We then may use

$$\langle k \infty - \,|\, T_{cl} \,|\, k \infty \rangle \sim \frac{i}{\pi k} e^{2i\sigma_l} . \tag{6.3}$$

[In comparison with the middle member of Eq. (3.14) we notice the absence of the term -1.] We shall introduce for each physical l value the half-shell ratio

$$F^l_{c,{\rm hs}} \equiv \frac{\langle p \,|\, T_{cl} \,|\, k \infty \rangle}{\dfrac{i}{\pi k} \exp(2i\sigma_l)} . \tag{6.4}$$

For $l=0$ we find the half-shell ratio

$$F^{l=0}_{c,{\rm hs}} = e^{-i\sigma_0} \frac{k}{p} C_0 \mathscr{S} 2i \sin\left[\gamma \ln\left| \frac{p-k}{p+k} \right| \right] ,$$
$$p>0, \quad k>0, \quad p \ne k . \tag{6.5}$$

It has a fixed phase (modulo π), determined essentially by σ_0. It satisfies

$$-C_0 \mathscr{S} \frac{2k}{p} \le i F^{l=0}_{c,{\rm hs}} \exp(i\sigma_0) \le C_0 \mathscr{S} \frac{2k}{p} . \tag{6.6}$$

Owing to the sine function in the right member of Eq. (6.5), its modulus fluctuates more and more rapidly between 0 and $2C_0\mathscr{S}k/p$ when p approaches k. In Sec. VII we shall consider a smoothly screened Coulomb potential. For such an interaction the half-shell function $F^l_{\rm hs}$ is well defined [by Eq. (3.11)]. We shall investigate this function (for $l=0$) in the limit of unscreening. In particular, we shall investigate the limiting behavior of $|F^{l=0}_{\rm hs}|^2$

which is the ratio of the p.w. half-shell cross section and the p.w. on-shell cross section.

VII. THE SCREENED COULOMB POTENTIAL IN ONE PARTIAL-WAVE STATE

We shall consider the Hulthén potential

$$V_H(r)=\frac{2k\gamma/a}{\exp(r/a)-1} \tag{7.1}$$

for angular momentum $l=0$. We often suppress l. For $a\to\infty$, V_H goes over into V_c. Its shape is given in Fig. 3 for various values of a. The off-shell and on-shell Jost functions (and solutions) are known.[7] We shall need

$$f_H(k,p)=\frac{\Gamma(1+\sigma)\Gamma(1+A+B+\sigma)}{\Gamma(1+A+\sigma)\Gamma(1+B+\sigma)}\ , \tag{7.2}$$

$$f_H(k)=\frac{\Gamma(1+A+B)}{\Gamma(1+A)\Gamma(1+B)}\ , \tag{7.3}$$

where we have adopted the notation of Refs. 7 and 8:

$$\begin{aligned}A&\equiv ika[-1+(1+2\gamma/ka)^{1/2}]\ ,\\ B&\equiv ika[-1-(1+2\gamma/ka)^{1/2}]\ ,\\ \sigma&\equiv -i(p-k)a\ .\end{aligned} \tag{7.4}$$

We study the limit of unscreening, $a\to\infty$, for fixed k. For $ka\to\infty$ we have

$$\begin{aligned}A+\sigma&=-i(p-k)a+i\gamma+O(1/ka)\ ,\\ B+\sigma&=-i(p+k)a-i\gamma+O(1/ka)\ ,\\ A+B+\sigma&=-i(p+k)a\ .\end{aligned} \tag{7.5}$$

The (real) KN half-shell extension function, given by Eq. (3.12), is

$$F_{H,\mathrm{hs}}^{l=0}(p;k)=\frac{k}{p}H^{1/2}\frac{2i\sin\varphi}{2i\sin\delta_H}\ , \tag{7.6}$$

where

$$H^{1/2}=\frac{\left[\dfrac{\sigma(A+B+\sigma)}{(A+\sigma)(B+\sigma)}\,\dfrac{\sin\pi(A+\sigma)\sin\pi(B+\sigma)}{\sin\pi\sigma\sin\pi(A+B+\sigma)}\right]^{1/2}}{\left[\dfrac{A+B}{AB}\,\dfrac{\sin\pi A\sin\pi B}{\pi\sin\pi(A+B)}\right]^{1/2}}\ , \tag{7.7}$$

$$\varphi=-\arg f_H(k,p),\quad \delta_H=-\arg f_H(k)\ . \tag{7.8}$$

Note that δ_H is the S-wave scattering phase shift for the Hulthén potential. Using $\Gamma(z^*)=\Gamma^*(z)$, and

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}=z^{\alpha-\beta}[1+O(z^{-1})],\qquad z\to\infty,\quad |\arg z|<\pi\ , \tag{7.9}$$

one easily derives [note that $\sigma_0=\arg\Gamma(1+i\gamma)$]:

$$\delta_H\sim\sigma_0-\gamma\ln(2ka),\quad ka\to\infty\ . \tag{7.10}$$

In the limit of unscreening, for p outside the interval $(k-\Delta,\ k+\Delta)$, Δ fixed positive and arbitrarily small, one easily proves

$$\lim_{a\to\infty}H^{1/2}=C_0\mathscr{E}\ ,\qquad p>0,\ k>0,\ p\notin(k-\Delta,k+\Delta)\ . \tag{7.11}$$

Similarly,

$$\lim_{a\to\infty}\varphi=\gamma\ln\left|\frac{p-k}{p+k}\right|\ ,\qquad p>0,\ k>0,\ p\notin(k-\Delta,k+\Delta)\ , \tag{7.12}$$

follows from Eqs. (7.2), (7.8), and (7.9). Hence, by Eq. (6.5),

$$\lim_{a\to\infty}2i\sin\delta_H F_{H,\mathrm{hs}}^{l=0}=e^{i\sigma_0}F_{c,\mathrm{hs}}^{l=0}\ ,\qquad p>0,\ k>0,\ p\notin(k-\Delta,k+\Delta)\ . \tag{7.13}$$

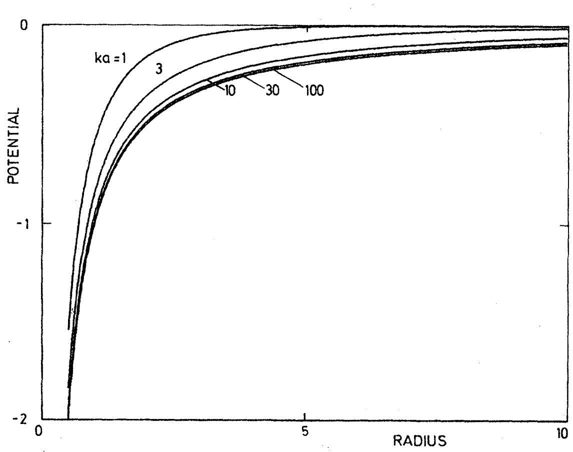


FIG. 3. Shape of the smoothly screened Coulomb potential (Hulthén potential) for various indicated values of ka. Also shown is the Coulomb potential itself. On the scale of the figure it coincides with the curve labeled $ka=100$.

For $p=k$, the KN half-shell extension function satisfies the normalization condition (3.13), $F^{l}_{H,\mathrm{hs}}(k;k)=1$. For $p\neq k$, $F^{l=0}_{H,\mathrm{hs}}$ can take large values, when $\sin\delta_H$ is close to zero. For fixed k the phase shift δ_H is a decreasing (increasing) function of a for large a in case of repulsion, $\gamma>0$ (attraction, $\gamma<0$), cf. Eq. (7.10). For $\delta_H=\pm n\pi$, $n=0,1,\ldots$, $F^{l=0}_{H,\mathrm{hs}}$ is unbounded for $p\neq k$. This happens at values of a which grow as const. $\times\exp(n\pi/|\gamma|)$. Physically, this corresponds to a vanishing on-shell cross section (Ramsauer-Townsend effect) and a nonvanishing half-shell cross section.

In Figs. 4(a), (b), and (c) we show $(p/k\cdot F^{l=0}_{H,\mathrm{hs}})^2$, which is essentially the ratio of the half-shell and the on-shell cross section for the S-wave projected Hulthén potential, for $\gamma=0.1$, 1, and 10, respectively, for various indicated values of ka. In all three cases the abscissa p/k ranges between 0 and 2. Note, however, that the (logarithmic) vertical scales are different. The limiting behavior of the plots for large a clearly obeys Eqs. (7.11) and (7.12). In all three cases ($\gamma=0.1$, 1, and 10) we have considered a repulsive potential. For an attractive potential one obtains very similar results: The major difference is that $H^{1/2}$ *increases* (rather than *decreases*) by roughly a factor $\exp(-\pi\gamma)$ around the value $p=k$.

We observe that the number of zeros of $F^{l=0}_{H,\mathrm{hs}}$ to the left of $p=k-\Delta$ equals the number of zeros to the right of $p=k+\Delta$, and that it is independent of a, for large a. It is easily found to be $\pi^{-1}\gamma\ln(2k/\Delta)$.

It is interesting to study the behavior of $2i\sin\delta_H F^{l=0}_{H,\mathrm{hs}}$ near the on-shell point $p=k$, as a function of p, for large a. Equation (7.6) shows that it comprises a positive factor $H^{1/2}$, and a factor $\sin\varphi$. In Table I we have collected a few values of $H^{1/2}$ and φ on the small interval $(k-|\gamma|/a, k+|\gamma|/a)$.

The sine in the corresponding Coulomb expression, Eq. (6.5), obviously has an infinite number of zeros near $p=k$. The number of zeros of $\sin\varphi$ for the *screened* Coulomb potential is limited. On the interval $(k,k+\gamma/a)$, φ is (to good approximation) symmetric about $p=k+\frac{1}{2}\gamma/a$. Its variation left and right of this point is

$$\arg\Gamma(1+i\gamma)-2\arg\Gamma(1+\tfrac{1}{2}i\gamma)\ .$$

Hence, the number of zeros of $\sin\varphi$ in each half of the interval is

$$[\pi^{-1}\arg\Gamma(1+i\gamma)-2\pi^{-1}\arg\Gamma(1+\tfrac{1}{2}i\gamma)]\ ,$$

i.e., independent of a. For large γ this number is well approximated by $\gamma\pi^{-1}\ln2+\frac{1}{4}$.

For large a values φ behaves as $\varphi\sim\sigma_0-\gamma\ln(2ka)$, cf. Table I. Because $p=k+\frac{1}{2}\gamma/a$ is a stationary point of φ we observe, with increasing a, the "birth" of two zeros of $\sin\varphi$. This occurs at values of a which grow as const. $\times\exp(n\pi/|\gamma|)$. This is clearly illustrated in Fig. 5, for the particular case $\gamma=10$. We show the interval $(k,k+\gamma/a)$, for various indicated values of ka. Plotted for each ka is the function

$$(k/p\cdot F^{l=0}_{H,\mathrm{hs}})^2=H\sin^2\varphi/\sin^2\delta_H\ .$$

TABLE I. Behavior of various quantities for large values of the screening radius a, for various values of p near the on-shell value $p=k$. Note that σ_0 stands for the S-wave Coulomb scattering phase shift, $\sigma_0=\arg\Gamma(1+i\gamma)$, whereas σ stands for the off-shell variable defined in Eq. (7.4).

		$p=k-\gamma/a$	$p=k$ (on shell)	$p=k+\gamma/2a$	$p=k+\gamma/a$
σ	$=$	$i\gamma$	0	$-\frac{1}{2}i\gamma$	$-i\gamma$
$A+\sigma$	$\sim$	$2i\gamma$	$i\gamma$	$+\frac{1}{2}i\gamma$	$-i\gamma^2(2ka)^{-1}$
$A+B+\sigma$	$=$	$-2ika+i\gamma$	$-2ika$	$-2ika-\frac{1}{2}i\gamma$	$-2ika-i\gamma$
$B+\sigma$	$\sim$	$-2ika$	$-2ika-i\gamma$	$-2ika-\frac{3}{2}i\gamma$	$-2ika-2i\gamma$
$H^{1/2}$		$\sim\left[\dfrac{\pi\gamma}{\tanh\pi\gamma}\right]^{1/2}$	$=1$	$\sim\left[\dfrac{\pi\gamma}{\sinh\pi\gamma}\right]^{1/2}$	$\sim\dfrac{\pi\gamma}{\sinh\pi\gamma}$
φ			$\sim\sigma_0-\gamma\ln(2ka)$	$\sim2\arg\Gamma(1+\frac{1}{2}i\gamma)-\gamma\ln(2ka)$	$\sim\sigma_0-\gamma\ln(2ka)$
$\varphi-\delta_H$			$=0$	$\sim2\arg\Gamma(1+\frac{1}{2}i\gamma)-\arg\Gamma(1+i\gamma)$	~0

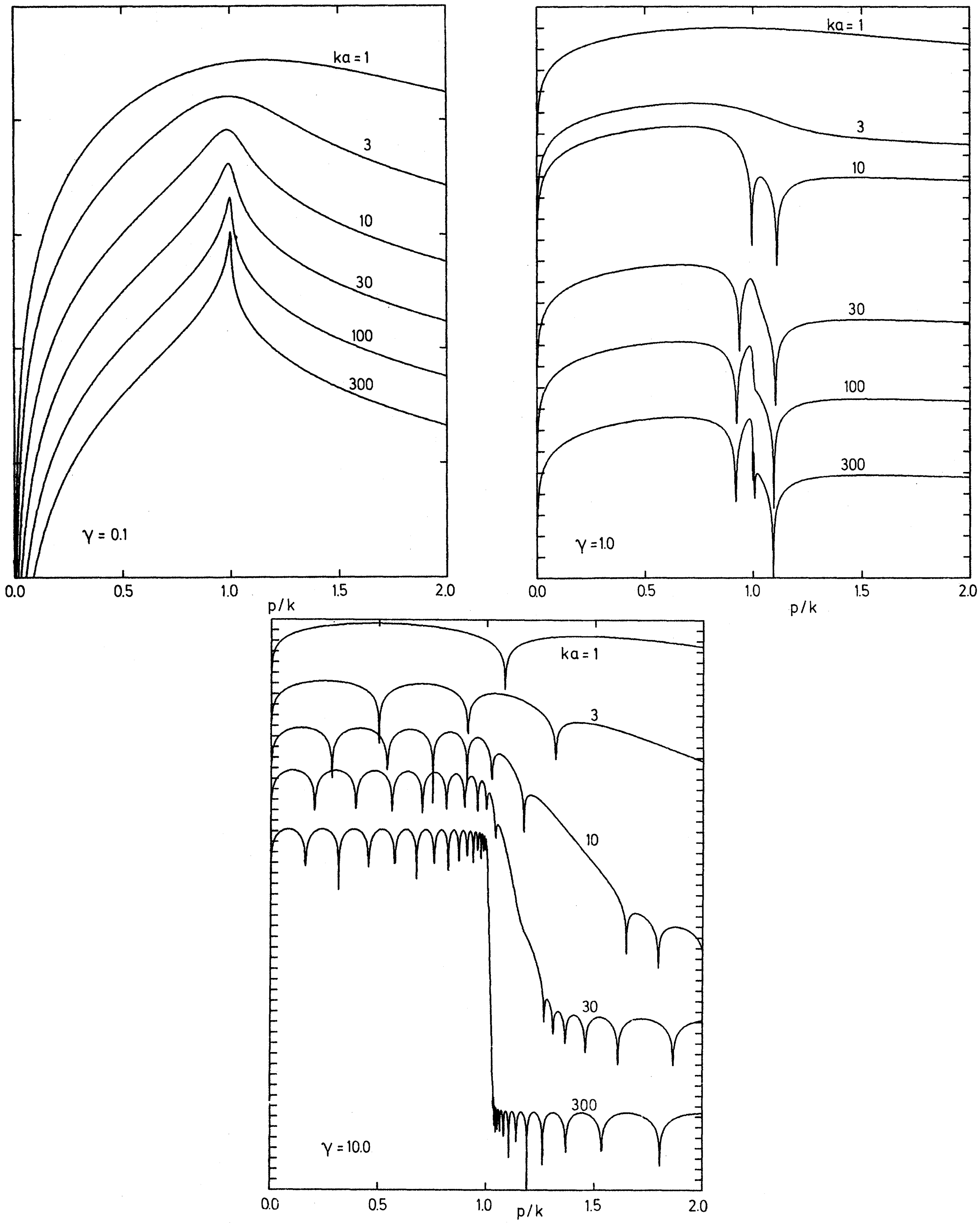
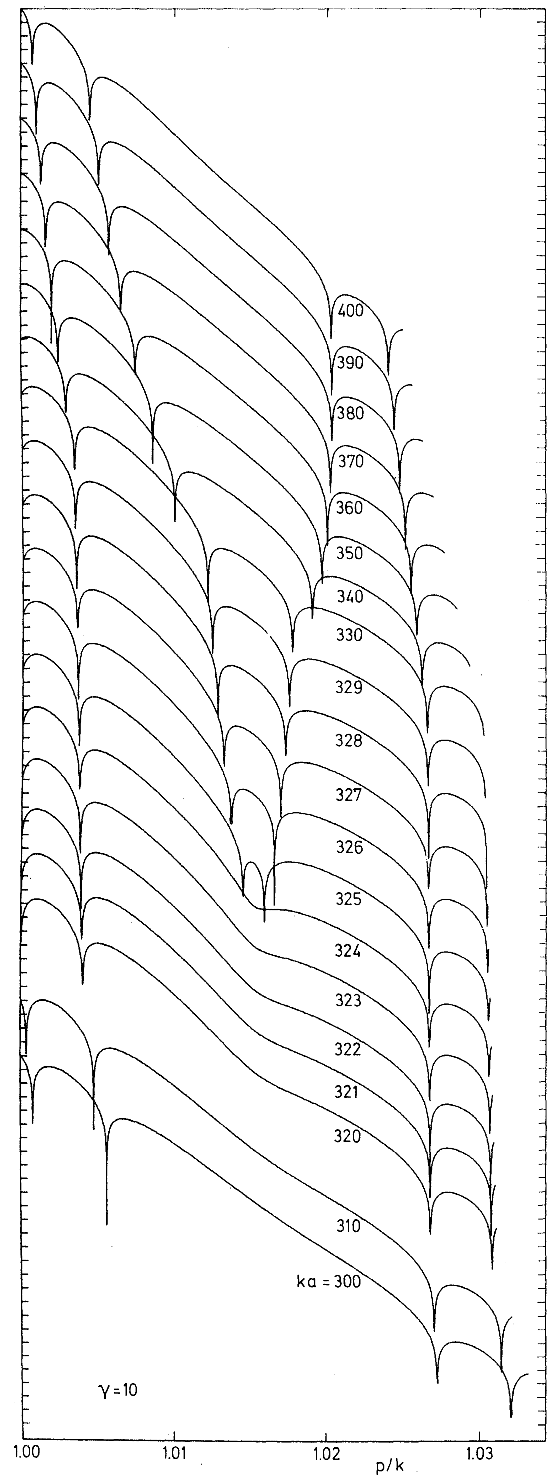


FIG. 4. Plots of $(p/k \cdot F^{l=0}_{H,\mathrm{hs}})^2$ as a function of the off-shell variable p/k. Apart from the factor p^2/k^2 this is the ratio of the half-shell and the on-shell cross section for the S-wave smoothly screened Coulomb potential. Values of the screening parameter ka are indicated. In (a), (b), and (c) we show the cases $\gamma=0.1$, 1, and 10, respectively. All curves are normalized to 1 at $p/k=1$. Note the different vertical scales; these scales are logarithmic; each horizontal marker indicates one decade.



VIII. CONCLUSION

For the pure Coulomb potential the on-shell limits ($p\uparrow k$ and $p\downarrow k$) of the physical half-shell T matrix do not exist, because of singularities of the type $(p-k)^{i\gamma}$. The on-shell limits of the *modulus* of the half-shell T matrix do exist. However, these limits (for $p\uparrow k$ and $p\downarrow k$) differ by a factor $\exp(\pi\gamma)$, and neither of the two equals the modulus of the physical on-shell T matrix.

These phenomena may seem somewhat confusing, and are certainly unfamiliar from the theory of scattering by short-range potentials. Indeed, the source of the difficulties lies in the long-range tail $(1/r)$ of the Coulomb potential. One may argue that in nature the pure Coulomb potential does not occur, because there always is screening. For nuclear interactions one may think of screening at atomic distances. A related question is raised by observing that in, for example, a nuclear scattering process some energy on the atomic or molecular level is always transferred so that one always measures half-shell (or even off-shell) scattering (rather than on-shell scattering), although the amount that one goes off shell is extremely small. Yet, in the first paragraph dramatic effects are recalled, no matter how far (how little) one goes off shell. We have studied this question by considering a smoothly screened Coulomb potential, and its unscreening.

Our paper has given arguments that, if one goes off shell on the typical nuclear scale, the screening effects on the atomic or molecular scale are immaterial: Strictly speaking one has a pseudostep function instead of the real step function $\mathfrak{F}$, but the "width" of the "step" will be negligibly small on the nuclear scale.

In Sec. V we summed up a number of equalities and inequalities for ratios of an element of the T matrix and an element of the V matrix, for the Coulomb case. We have not been able to find other potentials for which the same, or similar, elegant relations hold. Connected to these relations are the (in)equalities satisfied by certain Coulomb half-shell ratios, discussed in Sec. VI.

We have observed that the usual definition of the KN half-shell extension function is not applicable for interactions with a long-range tail, and the KN method has to be revised. For potentials with a

FIG. 5. On the interval $(k,k+\gamma/a)$ we show $(pF_{H,\mathrm{hs}}^{l=0}/k)^2$ as a function of p, for $\gamma=10$ (strong repulsion), and for a number of indicated values of ka. Note that this is essentially the ratio of the half-shell and the on-shell cross section, near the on-shell point $p=k$. The curves are normalized (by definition) to 1 at $p=k$. The horizontal markers on the vertical logarithmic scale each indicate one decade.

repulsive Coulomb tail this is not difficult, as will be shown in a subsequent paper.[12] In Sec. VII we studied the KN half-shell extension function for the S-wave Hulthén potential in the limit of unscreening, $a \to \infty$. The square of this (real) function gives the ratio of the half-shell and the on-shell (p.w.) cross section. For any $p \neq k$ the function contains a factor $1/\sin\delta_H$ which diverges for $a \to \infty$, cf. Eq. (7.10). It is very instructive to see the intricate mechanism by which the remaining constituents simulate and approach the half-shell ratio

$$F_{c,\mathrm{hs}}^{l=0} \exp(i\sigma_0) ,$$

defined for the Coulomb potential, see Eqs. (7.11)–(7.13) and Figs. 4 and 5. The Coulombic step function $\mathscr{B}$, defined in Eq. (2.10), is intimately connected to the singular behavior $(p-k)^{i\gamma}$. For a *screened* Coulomb potential we observe a *pseudostep* function, given essentially by Eq. (7.7). The width of the interval around $p=k$ in which this pseudostep function changes from one constant value to another constant is approximately equal to (and certainly of the order of) $|\gamma|/a$. This verifies the conjecture in Ref. 13, thereby providing a more precise estimate.

A characterization of the strength of the screened Coulomb potential is given by its number of bound states, when its sign is taken negative ($\gamma \to -|\gamma|$). This number is

$$n_B = [(2|\gamma| ka)^{1/2}] = [(2a|s|)^{1/2}] . \qquad (8.1)$$

It approaches ∞ when $a \to \infty$, in agreement with the infinite number of bound states for the Coulomb potential. Note that, although the zero-energy phase shift, in agreement with Levinson's theorem, increases as $(2|s|a)^{1/2}$, the phase shift at energy k^2 increases only as $|\gamma| \ln(2ka) + O(1)$, in agreement with Eq. (7.10).

We expect that the type of screening (Hulthén, exponential, or otherwise smooth) is rather immaterial in the limit of unscreening as long as the screening is smooth; see, for example, Ref. 14. Also, because the Coulombic singularity $(p-k)^{i\gamma}$ is present for all partial waves,[3] we expect pseudostep functions to build up similarly for angular momenta $l>0$, and for the case that the p.w. series are summed, cf. Eqs. (5.4) and (5.5).

Many experimental processes are described in terms of half-shell scattering amplitudes. Examples are knockout reactions and other quasifree processes in particle and nuclear physics. Also in bremsstrahlung processes half-shell scattering occurs, and the primary goal of many bremsstrahlung experiments has been to obtain information on the off-shell behavior of T matrices. In most of these practical cases one has $p>k$ and Coulomb repulsion. This means that half-shell scattering is suppressed, compared to the corresponding on-shell scattering processes, by a factor of $C_0{}^2$; cf. Eq. (2.15). This is a highly energy-dependent factor; see Fig. 1. Empirically one has found a similar energy dependence for the observed suppression: For quasifree scattering see, for example, Ref. 15, and for bremsstrahlung see Ref. 16, and a recent paper,[17] in which the factor $C_0{}^2$ was included to obtain agreement between theory and experiment.

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