Assumptions underlying two models of collective nuclear motion

Robert Gilmore and Da Hsuan Feng
Department of Physics and Atmospheric Science, Drexel University, Philadelphia, Pennsylvania 19104
(Received 13 May 1982)

The assumptions underlying the liquid drop model and the interacting boson model are compared. These two models are inequivalent. However, only minor modifications in the original assumptions are required to make these models different representations of the same underlying physical processes.

[NUCLEAR STRUCTURE Liquid drop model, interacting boson model, underlying Hilbert space, quantum-classical correspondence.]

There are now two seemingly complementary models of collective nuclear structure: the geometrical liquid drop model (LDM) presented in its phenomenological form by Bohr and Mottelson in 1952 (Ref. 1) (which constituted the basic philosophical underpinning of the entire study of collective motion in nuclei) and the algebraic interacting boson model (IBM).[2] It is natural to ask whether these models are closely related, even identical, at a fundamental level. Several authors have argued that the two models are identical or at least closely related.[3–8] Others have argued that the two models are totally unrelated.[9] Furthermore, comparisons have been made also between the IBM and the quantum mechanical five-dimensional oscillator [with dynamical group IU(5)] (Ref. 10) and the particle-hole (phonon number nonconserving) model of Janssen, Jolos, and Donau [also with dynamical group SU(6)].[11,12] The distinction between particle-hole and particle-particle collective modes of excitation is by no means clear or rigorous and has attracted some interest lately.[13]

At present, there is no clear-cut agreement on the relation between the LDM and IBM. In order to resolve the relationship between these two collective models, we undertake here a detailed comparison between the *underlying input assumptions* at the phenomenological level. This comparison is facilitated by their remarkably similar logical structure: the underlying assumptions exist in 1-1 correspondence. A detailed comparison of the input assumptions reveals that these models cannot be different representations of one underlying nuclear model. However, it is possible to indicate precisely where and how these input assumptions differ, and how the assumptions of either the LDM or the IBM must be minimally modified so that the two minimally modified models may eventually be shown to be different representations of the same nuclear processes. This comparison modification is carried out below, and summarized in Table I.

(1) Physical basis. The LDM is based on the oscillations of a liquid drop around its equilibrium shape. The IBM is based on the residual interactions between pairs of highly correlated nucleons.

(2) Physical bias. The starting point for the LDM is classical continuum mechanics while that for the IBM is quantum mechanics.

(3) Mathematical bias. The LDM is palpably geometric; the IBM is unabashedly algebraic.

These three assumptions represent different preferences for the starting points of these two models. If the models can be made equivalent (assumptions 4–14 below), then the geometric and the algebraic descriptions are only manifestations of the representation chosen, and surface oscillations a manifestation of residual boson interactions.

The remaining assumptions are of a technical nature, and will be modified as necessary to bring the two models into conformity.

(4) Shell model inputs. Our current understanding of the "Bohr-Mottelson model" of collective nuclear motion certainly includes shell model inputs to the Bohr collective Hamiltonian. These are discussed extensively in Ref. 14. The original phenomenological model, the LDM, used to derive the collective Hamiltonian, did not.[1] It is this model that we are comparing to the IBM.

The IBM has two shell model inputs: (a) the nonvalence nucleons occur in an inert closed core; and (b) the valence nucleons in even-even nuclei form highly correlated pairs ("Cooper pairs"). We shall modify the LDM by assuming that the liquid drop contains a rigid spherical core of radius R_c ("tidal planet model").

(5) Starting point. The radius of the liquid drop is defined by the equation in row 5, column A, in Table I. Nucleons in the IBM pair according to 5B in the table. This describes the pairing of nucleons of angular momentum j,j' to form a boson of angular momentum J. We modify the LDM by replacing the inequality $R(\theta,\phi) > 0$ by $R(\theta,\phi) \geq R_c$.

TABLE I. Logical inputs to the Bohr-Mottelson liquid drop model of collective nuclear motion and the Arima-Iachello interacting boson model of collective nuclear states.

	(a)	(b)	(c)
ASSUMPTION	BOHR-MOTTELSON LIQUID DROP MODEL OF NUCLEAR MOTION	ARIMA-IACHELLO INTERACTING BOSON MODEL OF NUCLEAR STATE	MODIFICATIONS AND REMARKS
1. Physical Basis	Liquid drop motion	Cooper pairing	--
2. Physical Bias	Classical mechanics	Quantum mechanics	--
3. Mathematical Bias	Geometry	Algebra	--
4. Shell Model	--	a. Inert closed core b. Residual nucleon-nucleon interaction	LDM: inert closed core of radius R_c
5. Starting Point	$R(\Theta,\Phi)=\sum_{LM} a^L_M Y^L_M(\Theta,\Phi)>0$	$\lvert J\rangle=\sum_{jj'} \alpha_{jj'} \lvert (jj')J\rangle$	LDM: $R(\Theta,\Phi)\geq R_c$
6. Approximations J = 0	Pairing is strong	S Boson strongly bound	Include
J = 2	Quadrupole deformation is important	D Boson is important	Include
J = 4,6, . . .	Much less important than L = 0, 2 terms	Weakly bound or unbound	Usually neglect
J = 1	Represents nuclear displacement	Requires different oscillator shells	Usually neglect
J = 3,5, . . .		Requires different oscillator shells	Usually neglect
7. Technical Condition	$a^L_M=(-)^M a^{L*}_{-M}$ --	-- $s^\dagger s+d^\dagger d=N$	IBM: See 11 LDM: $\lvert a^0_0\rvert^2+\sum_M \lvert a^2_M\rvert^2=1$
8. Dynamics	$H=f(a^0_0,a^2_M,\dot{a}^0_0,\dot{a}^2_M)$	$H=g(s,s^\dagger,d,d^\dagger)$	Functions are rotationally invariant
9. Dynamical Variables	$a^2_M,\ \dot{a}^2_M$	$Z,Z^* \in SU(6)/U(5)$	
10. Phase Space	$a^2_M,\ \dot{a}^2_M$	$Z,Z^* \in SU(6)/U(5)$	$\{a^2_M,\dot{a}^2_M\}$ Describe a rank 1 Riemannian symmetric space = SU(6)/U(5)
11. Configuration Space	a^2_M	--	IBM: Choose a Lagrangian submanifold C^5_{LM}, with coordinates obeying condition 7A
12. Underlying Hilbert Space	$\mathfrak{L}^2(C^5,\sqrt{\lvert g\rvert})$	$\mathfrak{L}^2[SU(6)/U(5);\sqrt{\lvert G\rvert},N]$	LDM: $\mathfrak{L}^2(C^5,\sqrt{\lvert g\rvert};\mathcal{V})$ IBM: $\mathfrak{L}^2(C^5_{LM},\sqrt{\lvert g\rvert};N)$
13. Quantum-Classical Correspondence	$[f,g]=\frac{\hbar}{i}\{f,g\}_{SU(6)}$ $\{f,g\}_{SU(6)}=\sum_{ijk}\frac{\partial f}{\partial v^i}\frac{\partial g}{\partial v^j}C_{ij}{}^k v^k$	$\mathcal{H}\longrightarrow\langle {}^N_Z\lvert\mathcal{H}\rvert{}^N_Z\rangle$	Duality
14. Method of Solution	Solve Schrödinger-like equation	Diagonalize matrix	Construct transformation theory between two representations

(6) Approximations. In the LDM, it is customary to retain the quadrupole mode in 5A and sometimes the monopole ($L=0$) mode as well. The dipole mode represents displacement of the drop's center of mass and is generally neglected. Terms in 5A with $L>2$ "can be shown to be of much less importance"[1] than the monopole and quadrupole modes, and are generally neglected. In the IBM, odd J terms can arise only from nucleons in adjacent opposite parity harmonic oscillator levels. Such terms are much more highly excited than the even J bosons constructed from nucleons within a single oscillator level, and are generally neglected. Nucleon pairs are strongly bound in the $J=0$ state, and more weakly bound in the $J=2$ state. States with $J=4$ are unbound or very weakly bound; those with $J=6, 8, \ldots$ are unbound.[15] It should be noted that if one considers the so-called "two fluid model" for the nucleus (neutron and proton fluids) the $L=1$ mode of excitation is important (giant dipole resonance). However, this mode lies outside the scope of the original IBM where the distinction between protons and neutrons are not made.

(7) Technical conditions. In the LDM, the c-number amplitudes obey the reality condition 7A. In the IBM, the s- and d-boson operators obey a unitary condition 7B, where N is half the number of valence nucleons. These conditions are not comparable. The LDM must obey some kind of unitary constraint while the IBM must obey some kind of reality constraints. The unitary condition is imposed on the LDM by requiring the liquid drop oscillations to be volume (density) preserving. This places one condition on the six amplitudes: $f(a_0^0, a_M^2)=0$. If the liquid drop radius is written in the form $R=K+a_0^0+a_M^2 Y_M^2$, then in the second order the amplitudes a_0^0, a_M^2 obey the constraint given in 7C, and the liquid drop volume is conserved to second order[16] in the amplitudes a_0^0, a_M^0, rather than the first order as in the original liquid drop model where $a_0^0=0$.[1] This is essentially the classical limit[17] of the IBM unitarity constraint 7B. The condition on the IBM parameters analogous to the LDM reality constraint 7A will be discussed under point 11 below.

(8) Dynamics. The most general LDM Hamiltonian function is a superposition of functions constructed in a rotationally invariant way from $a_0^0, a_M^2, \dot{a}_0^0, \dot{a}_M^2$. At the operational level, such a general function $f(a_0^0, a_M^2, \dot{a}_0^0, \dot{a}_M^2)$ is usually truncated beyond terms of degree 4. The most general IBM Hamiltonian operator is a superpositon of operators constructed in a rotationally invariant way from $s, s^\dagger, d_\mu, d_\mu^\dagger$. At the operational level, such a general operator $g(s, s^\dagger, d_\mu, d_\mu^\dagger)$ is usually truncated beyond terms of degree 4. Despite these suggestive similarities in construction, the *dynamics* of the LDM and the IBM cannot be compared until the equivalence of the underlying kinematics has been established. Once done, the secondary issue of related comparison of the dynamics can be considered.

(9) Dynamical variables. The coordinates for the LDM are the six complex amplitudes a_0^0, a_M^2 subject to the reality condition 7A and the normalization condition 7C. There are thus five independent real coordinates. The five corresponding velocities round out the ten dynamical variables for the LDM. The IBM, as an algebraic model, was originally devoid of dynamical variables. However, the presence of the dynamical group SU(6) together with the occurrence of only the fully symmetric representation $\{N, \dot{0}\}$ of SU(6), requires that the associated geometric space be the coset SU(6)/U(5),[18] a compact ten-dimensional space whose coordinates parametrize the SU(6) coherent states[19] and are the dynamical variables for the IBM.

(10) Phase space. For the LDM, this is the compact ten-dimensional space described above. This Riemannian space is symmetric, has rank 1 (by permutation symmetry),[15] and has a symplectic and an imaginary structure. The compact ten-dimensional Riemannian symmetric space of rank 1 is uniquely SU(6)/U(5).[18] For the IBM, Feng and Gilmore[20] have shown that the SU(6) coherent state parameters obey Hamilton's equations of motion under the quantum-classical mapping.

(11) Configuration space. For the LDM the five independent coordinates parametrize configuration space. For the IBM, since the coherent state parameters obey canonical equations of motion, any Lagrangian submanifold[21] C_{LM}^5 may be chosen as configuration space. The particular choice of Lagrangian submanifold will be governed by the form of the IBM Hamiltonian and the requirement that in its classical limit no higher than quadratic momentum terms occur. Coordinates for this Lagrangian manifold can be chosen to satisfy a reality condition of the form 7A.

(12) Underlying Hilbert space. The set of square integrable ($\mathcal{L}^2$) functions on the compact configuration space C^5 with five independent parameters a_0^0, a_M^2 constitutes the usual Hilbert space $\mathcal{L}^2(C^5; \sqrt{|g|})$ for the LDM. The measure $\sqrt{|g|}$ on this space is determined from the Riemannian metric g_{ij} on C^5. For the IBM, the Hilbert space consists of the square integrable functions on SU(6)/U(5) with respect to the natural measure $\sqrt{|G|}$, where $G_{\mu\nu}$ is the metric induced on SU(6)/U(5) from the Haar measure[18] on SU(6) for the fully symmetric representation $\{N, \dot{0}\}$ of SU(6): $\mathcal{L}^2[\mathrm{SU}(6)/\mathrm{U}(5); \sqrt{|G|}; N]$. Hilbert spaces for both models are modified as follows. For the LDM the parameter ν is introduced. This is the ratio of fluid outside the solid core to the liquid drop volume. The parameter ν stratifies the original LDM Hilbert space. The invariant subspace of interest is $\mathcal{L}^2(C^5; \sqrt{|g|}; \nu)$. For the IBM, consideration is restricted to square integrable functions which are nonzero only on the Lagrangian submanifold C_{LM}^5 of

SU(6)/U(5) with respect to the metric g_{ij} on C^5_{LM} induced from $G_{\mu\nu}$ on SU(6)/U(5):$\mathcal{L}^2(C^5_{LM};\sqrt{|g|};N)$. It is natural to make the identification $\nu = 2N/A$.

(13) Quantum-classical correspondence. At this point the parallelism exhibited in the preceding points breaks down, and is here replaced by a duality. In order for both models to have a classical and a quantum counterpart, the IBM must have a classical limit while the LDM must be quantized. The classical limit[19, 20] of the IBM is obtained by taking its expectation value ($\langle N,Z|\mathcal{H}|N,Z\rangle$) in the SU(6) coherent state representation.[20] Quantization of the LDM with pairing plus quadrupole interactions proceeds as follows. The dynamical group G[SU(6)] has Lie algebra g[SU(6)] which is spanned by basis vectors e_i with commutation relations $[e_i,e_j] = C^k_{ij}e_j$. Let v^i be local variables on the manifold G dual to the generators e_i. Then the quantization condition is given in 13A of Table I.[22]

In the usual quantization procedure[23] the classical kinetic energy is replaced by the differential kernel $|g|^{-1/2}\partial_\mu g^{\mu\nu}|g|^{1/2}\partial_\nu$ (Laplace-Beltrami operator).[17] *The validity of this quantization prescription has been questioned by Kumar and Baranger,[24] since it was developed by Pauli[21] for curvilinear coordinates on Euclidean spaces, and the pairing plus quadropole model requires a quantization procedure valid for curves spaces. However, the result remains valid for curved spaces also, since the Laplace-Beltrami operator described above is the unique second order partial differential operator in a Riemannian space which is invariant under all isometries.*[18]

It should be pointed out that in comparing two quantum mechanical models, it is not sufficient to establish equivalence at the operator (commutation relation) level. It is necessary also to establish equivalence at the Hilbert space (representation) level.[6] The present and previous points establish these two equivalences.

(14) Method of solution. The eigenvalues and eigenvectors for the IBM are easily determined by matrix diagonalization.[25] The LDM is solved by quantizing the classical Hamiltonian and solving the resulting Schrödinger-like differential equation.

Under the changes proposed in the table, the two models are different representations of the same underlying physical processes. They are related to each other by a similarity transformation of a type previously proposed,[5(b)] where the domain of integration is restricted to the embedded Lagrangian submanifold.

We thank F. Iachello and M. Vallieres for useful discussions. This work was partially supported by the National Science Foundation under Grant No. PHY-8102977.

[1]A. Bohr and B. R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1975), Vol. 2.

[2]A. Arima and F. Iachello, Ann. Phys. (N.Y.) 99, 253 (1976); 111, 201 (1978); 123, 468 (1979).

[3]A. E. L. Dieperink, O. Scholten, and F. Iachello, Phys. Rev. Lett. 44, 1747 (1980).

[4]J. N. Ginocchio and M. W. Kirson, Phys. Rev. Lett. 44, 1744 (1980).

[5](a) D. H. Feng, R. Gilmore, and S. R. Deans, Phys. Rev. C 23, 1254 (1981); (b) R. Gilmore and D. H. Feng, in *Interacting Bose-Fermi System,* edited by F. Iachello (Plenum, New York, 1981).

[6]O. Castanos, A. Frank, P. Hess, and M. Moshinsky, Phys. Rev. C 24, 1367 (1981).

[7]P. van Isacker and J. Q. Chen, Phys. Rev. C 24, 684 (1981).

[8]R. Hatch and S. Levit, Phys. Rev. C 25, 614 (1982).

[9]A. Bohr and B. R. Mottelson, Phys. Scr. 22, 468 (1980).

[10]A. Klein and M. Vallieres, Phys. Rev. Lett. 46, 586 (1981).

[11]D. Janssen, R. V. Jolos, and F. Donau, Nucl. Phys. A224, 93 (1974).

[12]G. Kyrchev, Nucl. Phys. A349, 416 (1980); V. Paar, in *Interacting Bosons in Nuclear Physics,* edited by F. Iachello (Plenum, New York, 1979).

[13]A. B. Johnson and C. M. Vincent, Phys. Rev. C 25, 1595 (1982); A. Arima, in *Proceedings of the International School of Nuclear Physics, 1982* (6th Course: Collective Bands in Nuclei), edited by. D. Wilkinson (Pergamon Press, London, 1982).

[14]P. Ring and P. Schuck, *Nuclear Many-Body Problem* (Springer-Verlag, New York, 1980); D. H. Feng and R. Gilmore (unpublished).

[15]J. P. Schiffer and W. W. True, Rev. Mod. Phys. 48, 191 (1976).

[16]J. Eisenburg and W. Griener, *Collective Models of Nuclei* (North-Holland, Amsterdam, 1969).

[17]R. Gilmore, J. Math Phys. 20, 891 (1979).

[18]R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications* (Wiley, New York, 1974).

[19]R. Gilmore, Ann. Phys. (N.Y.) 74, 391 (1972).

[20]D. H. Feng and R. Gilmore, Phys. Lett. 90B, 327 (1980).

[21]V. I. Arnol'd, *Mathematical Methods of Classical Mechanics* (Springer, Berlin, 1978).

[22]D. V. Chudnovsky and G. V. Chudnousky, Phys. Rev. Lett. 47, 1093 (1981).

[23]W. Pauli, in *Handbuch der Physik* (Springer, Berlin, 1933), Vol. 24, p. 120.

[24]K. Kumar and M. Baranger, Nucl. Phys. A92, 608 (1967).

[25]O. Scholten, Program PHINT (unpublished).