PHYSICAL REVIEW C VOLUME 26, NUMBER 2 AUGUST 1982

Microscopic calculation of the effects of the g boson on the interacting boson model Hamiltonian

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(Received 1 March 1982)

Second-order perturbation theory is used to examine the effect of excluding the g boson from the model space of the interacting boson model. Perturbative corrections are calculated in a paired fermion space (in a single j-shell approximation) and are mapped onto the interacting boson model Hamiltonian using the imaging techniques of Otsuka, Arima, and Iachello. The resulting renormalization of the boson parameters of the Hamiltonian to account for effects of the g boson depends strongly upon the numbers of valence protons and neutrons, and remains significant throughout the half shell.

[NUCLEAR STRUCTURE Interacting boson model, g boson, microscopic calculation using perturbation theory, renormalization of boson parameters.]

I. INTRODUCTION

The phenomenological interacting boson model (IBM) has been quite successful in describing spectral properties of low-lying collective states in many nuclei.[1] The usual model space of the IBM includes only s ($L=0$) and d ($L=2$) bosons; this so-called s-d dominance has been closely scrutinized in recent literature.[2,3] While the IBM bosons are presumed to arise from correlated pairs of fermions, the microscopic origin of the IBM is, nevertheless, not yet well understood.

Otsuka *et al.*[4] recently described a possible microscopic, fermion pair origin for the s and d bosons. A mapping technique was subsequently developed by Otsuka, Arima, and Iachello[5] (OAI) which related matrix elements of boson operators to matrix elements of fermion operators in a paired-fermion space; this mapping procedure forms the basis of other calculations which attempt to connect the IBM to some underlying fermionic shell model.[6,7] Ginocchio and Talmi[8] have also discussed the correspondence between boson and fermion states and operators. Other approaches seeking to justify and/or criticize the assumptions of the IBM have employed quasiparticle formalism,[9] boson expansion techniques,[10] perturbation theory,[11,12] and shell model calculations in severely truncated paired fermion bases.[13–15]

In this paper we use second order Rayleigh-Schrödinger perturbation theory to investigate, from a microscopic perspective, the renormalization of IBM parameters arising from the effects of g bosons. This use of perturbation theory is consistent with the assumption of s-d dominance; it is also equivalent, in spirit, to the perturbative calculation of Otsuka.[12] We see consequential effects, due to the renormalization, in all cases considered. Furthermore, we reinforce the conclusions of McGrory[13] and Otsuka[14] that the influence of the g boson decreases with increasing boson number.

II. EFFECTS OF STATES EXCLUDED FROM THE IBM MODEL SPACE

A. Background

The usual microscopic interpretation of the IBM assumes that the proton-proton and neutron-neutron interactions, $H_{\pi\pi}$ and $H_{\nu\nu}$, respectively, dominate the proton-neutron interaction, $H_{\pi\nu}$, so that proton pairs and neutron pairs are first formed, and these pairs subsequently interact via $H_{\pi\nu}$. By relating a particle pair to an IBM boson, one is led immediately to write the following Hamiltonian H_B for a system of interacting proton and neutron valence bosons:

$$H_B = H_{\pi\pi} + H_{\nu\nu} + H_{\pi\nu} \ , \tag{1}$$

$$H_{\rho\rho} = \epsilon_{s_\rho} \hat{n}_{s_\rho} + \epsilon_{d_\rho} \hat{n}_{d_\rho} + \sum_{L=0,2,4} C_{L_\rho} \tfrac{1}{2} (2L+1)^{1/2} [(d_\rho^\dagger d_\rho^\dagger)^{(L)} (\tilde{d}_\rho \tilde{d}_\rho)^{(L)}]^{(0)} \quad \rho = \pi, \nu \ , \tag{2}$$

$$H_{\pi\nu} = \kappa Q_\pi \cdot Q_\nu \ , \tag{3}$$

$$Q_\rho = (d_\rho^\dagger s_\rho + s_\rho^\dagger \tilde{d}_\rho)^{(2)} + \chi_\rho (d_\rho^\dagger \tilde{d}_\rho)^{(2)} \quad \rho = \pi, \nu \ , \tag{4}$$

where the brackets and parentheses denote angular momentum coupling with Clebsch-Gordan coefficients, e.g.,

$$(d_\alpha^\dagger d_\beta^\dagger)_M^{(L)} = \sum_{m_\alpha m_\beta} (22 m_\alpha m_\beta \,|\, LM) d_{\rho m_\alpha}^\dagger d_{\rho m_\beta}^\dagger \ .$$

The proton-neutron interaction $H_{\pi\nu}$ should in general contain terms other than $Q_\pi \cdot Q_\nu$; we discuss the general form of $H_{\pi\nu}$ in Sec. III. Here we include only the quadrupole-quadrupole part of $H_{\pi\nu}$, the standard assumption of the phenomenological IBM, and the form used by Otsuka.[12] The only nonzero commutation relations between the s and d creation and annihilation operators are

$$[d_{\rho\mu}, d_{\rho'\mu'}^\dagger] = \delta_{\rho\rho'} \delta_{\mu\mu'} \ [s_\rho, s_{\rho'}^\dagger] = \delta_{\rho\rho'} \quad \rho, \rho' = \pi \text{ or } \nu \tag{5}$$

and $\mu, \mu' = -2, -1, 0, 1,$ or 2. The quantities ϵ_{s_ρ}, ϵ_{d_ρ}, c_{L_ρ}, χ_ρ, and κ in Eqs. (1)–(4) are variable parameters of the phenomenological IBM Hamiltonian. [In practice, only $\epsilon_{d_\pi} = \epsilon_{d_\nu}$, χ_π, χ_ν, κ, c_{L_π}, and $c_{L_\nu}(L = 0,2,4)$ are varied in fitting experimental spectra.] The operators $\hat{n}_{s_\rho}$ and $\hat{n}_{d_\rho}$ count the numbers of s and d bosons of type ρ (proton or neutron).

The boson Hamiltonian in Eq. (1) is formulated under the assumption that the important degrees of freedom in low-lying collective states can be described using only s and d bosons, leading to a dynamical symmetry (in the case of exact symmetries) governed by the group $U(6) \times U(2)$.[1] If we allow for the existence of proton and neutron g bosons, we must expand this group structure to $U(15) \times U(2)$. The number of boson-boson interaction terms in the Hamiltonian which would then be possible for the s-d-g system leads to intractable calculations, except in some group-theoretical limits.[6] We would like to account for the effects of g bosons without including these degrees of freedom. In the subsequent sections we apply second-order perturbation theory to the calculation of renormalized matrix elements of a fermionic proton-neutron interaction connecting states made up only of $L = 0$ and $L = 2$ pairs of protons and neutrons, to states which contain one or more $L = 4$ proton or neutron pairs (G pairs). In Sec. III we discuss the renormalization of the IBM Hamiltonian to take these extra-model space effects into account. Section IV contains our conclusions and some comparisons to other relevant calculations.

B. OAI imaging technique

The calculation of the spectrum for a medium mass nucleus with several valence protons and neutrons, in a full shell model basis, is intractable due to the dimensionality of the Hamiltonian matrix which must be diagonalized. The number of basis states involved in even the lowest-lying collective states can easily exceed a trillion.[17] A shell model calculation for such a nucleus requires a *severe* truncation of the basis to include a manageable number of states. The effects of the states left out of the model space on the states inside the model space must be included. We have used perturbation theory through second order to study the coupling of excluded states to the states in a paired-fermion model space. Prior to a description of that calculation, we present a brief explanation of the mapping techniques of OAI.

The OAI method is a procedure for determining the form and parametrization of boson operators through the calculation of matrix elements of fermion operators in a correlated fermion pair model space. Otsuka *et al.*[5] first build a model space using only S ($L=0$) and D ($L=2$) correlated pairs of protons and pairs of neutrons. The states in this model space are obtained by acting on the paired-fermion vacuum (the inert core of the nucleus) with products of $S^\dagger$ and $D^\dagger$ creation operators, where

$$S^\dagger=\frac{1}{\sqrt{2}}\sum_j \alpha_j[a_j^\dagger a_j^\dagger]^{(0)}\,,$$
$$D_\mu^\dagger=\sum_{jj'}\frac{\beta_{jj'}}{\sqrt{1+\delta_{jj'}}}[a_j^\dagger a_{j'}^\dagger]_\mu^{(2)}\,. \tag{6}$$

The sums over j and j' extend over all active valence orbitals.

The resulting states are then put into a one-to-one correspondence with states in the boson space, constructed using the same number of proton and neutron s and d bosons as the numbers of S and D proton and neutron fermion pairs. In practice, the $L=2$ operators $D_\mu^\dagger$ and D_μ are complicated by the presence of a seniority projection operator, included in order to obtain an orthogonal set of paired-fermion states. The projection operator is the unit operator if only one D pair of each type (π or ν) is present. For a given fermion operator O_F, the corresponding boson operator O_B is determined by first expanding O_B as a linear combination of products of boson creation and annihilation operators, $s^\dagger$, s, $d_\mu^\dagger$, $\tilde{d}_\mu=(-)^\mu d_{-\mu}$; for example, a boson quadrupole operator can be expanded as

$$q_B=\alpha_\pi(d_\pi^\dagger s_\pi+s_\pi^\dagger\tilde{d}_\pi)^{(2)}+\beta_\pi(d_\pi^\dagger\tilde{d}_\pi)^{(2)}$$
$$+\alpha_\nu(d_\nu^\dagger s_\nu+s_\nu^\dagger\tilde{d}_\nu)^{(2)}+\beta_\nu(d_\nu^\dagger\tilde{d}_\nu)^{(2)}\,, \tag{7}$$

plus the terms of higher order in s and d operators. The parameters α_π, β_π, α_ν, and β_ν in Eq. (7) can be found by requiring that matrix elements of q_B between given boson states be *equal* to matrix elements of the corresponding fermion quadrupole operator q_F, taken between paired-fermion states corresponding to the boson states:

$$((n_\pi,n_\nu)\gamma J\,|\,q_B\,|\,(n_\pi,n_\nu)\gamma' J')$$
$$=\langle(\tfrac{1}{2}N_\pi,\tfrac{1}{2}N_\nu)\Gamma J\,|\,Q_F\,|\,(\tfrac{1}{2}N_\pi,\tfrac{1}{2}N_\nu)\Gamma' J'\rangle\,, \tag{8}$$

where n_π (n_ν) is the number of proton (neutron) bosons and $N_\pi/2$ ($N_\nu/2$) is the number of proton (neutron) fermion pairs. The additional quantum numbers γ,γ' and Γ,Γ' may be necessary to uniquely specify the boson and paired-fermion states, respectively. OAI have investigated a "zeroth-order" approximation in which the particle rank of the boson operator is the same as the particle rank of the fermion operator; they have found that for number-conserving operators, the zeroth-order approximation for the boson operator yields satisfactory agreement between the matrix elements in Eq. (8). OAI assert that, in the case of number nonconserving operators, the equality expressed in Eq. (9) can be maintained if terms of higher boson rank are included in the boson operator. The OAI method is a powerful tool which can be used to predict the values of IBM parameters from a microscopic many-fermion calculation.[6] Ginocchio and Talmi[8] have investigated the OAI method, as well as other correspondences between boson and paired-fermion systems, in detail.

Because phenomenological IBM calculations have been performed for many rare-earth nuclei spanning the mass region with proton number $Z=50-82$ and neutron numbers $N=50-82$ and $N=82-126$ (as well as several actinide nuclei), there has accumulated a great deal of information on trends in the empirical parameters of the IBM Hamiltonian. Efforts to relate these trends to the underlying fermionic shell structure using the OAI method have been somewhat successful,[6,7] and improvements in the fermionic models employed in these imaging calculations are expected to produce better results.[18]

These investigations of the microscopic basis of the IBM have progressed to the point that studies of extra-model space effects have recently appeared in the literature.[11,12,19] One procedure that has been used to examine the effects of boson degrees of freedom excluded from the s-d boson space is to treat this problem in the paired-fermion space (e.g., in a shell model calculation) and then map the result into the boson space using OAI imaging techniques.[11,12] For example, in predicting the parameters of the IBM Hamiltonian with the OAI method, the matrix elements of some fermion Hamiltonian are calculated; these matrix elements (in a paired-fermion space) can be corrected perturbatively to account for states outside the paired-fermion model space. The corrected matrix elements can then be used to renormalize the parameters of the boson Hamiltonian. In this manner, the question of excluding states from the boson space becomes a problem of excluding states from the paired-fermion model space. In the remainder of this section we discuss the coupling of states in an S-D paired-fermion space to states which lie outside this model space.

C. Calculation of matrix elements

In order to investigate extra-model space effects in the paired-fermion space, we have calculated matrix elements of the proton-neutron interaction, $V_{\pi\nu}$, coupling S-D states to states which contain one or more G ($L=4$) pairs of protons or neutrons (S-D-G space). The interaction $V_{\pi\nu}$ is used here because matrix elements of the proton-proton (or neutron-neutron) interaction $V_{\pi\pi}$ ($V_{\nu\nu}$) between our states vanishes, e.g.,

$$\langle G_\pi D_\nu \mid V_{\pi\pi} \mid D_\pi D_\nu \rangle = 0 ,$$

as a result of the scalar rank of $V_{\pi\pi}$. Following Otsuka[12] we use an effective proton-neutron interaction

$$V_{\pi\nu} = -f C_\pi^{(2)} \cdot C_\nu^{(2)} , \tag{9}$$

where $C^{(2)}$ is the surface quadrupole operator and $f=1.5$ MeV, appropriate for the shell $Z=50-82$. The unperturbed S-D model states are

$$\begin{aligned}
|\psi_1\rangle &= |(S_\pi^{n_\pi}; S_\nu^{n_\nu})J=0\rangle , \\
|\psi_2\rangle &= |(S_\pi^{n_\pi-1} D_\pi; S_\nu^{n_\nu})J=2,M\rangle , \\
|\psi_3\rangle &= |(S_\pi^{n_\pi}; S_\nu^{n_\nu-1} D_\nu)J=2,M\rangle , \\
|\psi_4\rangle &= |(S_\pi^{n_\pi-1} D_\pi; S_\nu^{n_\nu-1} D_\nu)J=L,M\rangle , \\
& \qquad (L=0-4) .
\end{aligned} \tag{10}$$

These states are constructed using the paired-fermion operators of Eq. (6), e.g.,

$$|\psi_2\rangle = N_2{}^{-1}[((S_\pi^\dagger)^{n_\pi-1} D_\pi^\dagger)^{(2)} (S_\nu^\dagger)^{n_\nu}]_M^{(2)} |0\rangle . \tag{11}$$

The constant $N_2{}^{-1}$ in Eq. (11) is included for normalization of $|\psi_2\rangle$; furthermore, we have used the boson number notation (lower case n's) to suggest a correspondence to boson states with n_π proton bosons and n_ν neutron bosons. States of interest outside the S-D model space contain at least one G pair (S-D-G states):

$$\begin{aligned}
|\psi_5\rangle &= |(S_\pi^{n_\pi}; S_\nu^{n_\nu-1} G_\nu)J=4,M\rangle , \\
|\psi_6\rangle &= |(S_\pi^{n_\pi-1} G_\pi; S_\nu^{n} \nu)J=4,M\rangle , \\
|\psi_7\rangle &= |(S_\pi^{n_\pi-1} D_\pi; S_\nu^{n_\nu-1} G_\nu)JM\rangle , \\
|\psi_8\rangle &= |(S_\pi^{n_\pi-1} G_\pi; S_\nu^{n_\nu-1} D_\nu)JM\rangle , \\
|\psi_9\rangle &= |(S_\pi^{n_\pi-1} G_\pi; S_\nu^{n_\nu-1} G_\nu)JM\rangle ,
\end{aligned} \tag{12}$$

where the G pairs are created through the operation of $G_\mu^\dagger$

$$G_\mu^\dagger = \sum_{jj'} \frac{\gamma_{jj'}}{(1+\delta_{jj'})^{1/2}} [a_j^\dagger a_{j'}^\dagger]_\mu^{(4)} . \tag{13}$$

In the work presented here, we simplify our calculations by considering only single j shells, j_π and j_ν, for protons and neutrons, respectively. For the mass region of interest ($Z=50-82$ and $N=50-82$) $j_\pi = j_\nu = \frac{31}{2}$. In this single j-shell approximation, the pair creation operators become simply

$$\begin{aligned}
S^\dagger &= \frac{1}{\sqrt{2}} [a_j^\dagger a_j^\dagger]^{(0)} , \\
D_\mu^\dagger &= \frac{1}{\sqrt{2}} [a_j^\dagger a_j^\dagger]_\mu^{(2)} , \\
G_\mu^\dagger &= \frac{1}{\sqrt{2}} [a_j^\dagger a_j^\dagger]_\mu^{(4)} .
\end{aligned} \tag{14}$$

Otsuka[14] has used many degenerate orbitals in a calculation of first-order perturbations of the wave functions due to the g boson (the Feshbach method). Our results are qualitatively similar to his.

Our goal is to compare matrix elements of $V_{\pi\nu}$ between states inside the S-D space with matrix elements of $V_{\pi\nu}$ which couple the S-D model space and the S-D-G space. In the zeroth-order approximation of OAI, only the low-seniority states given in Eqs. (10) and (12) are necessary for the calculation of matrix elements of a two-body interaction. Hence the states we consider here need contain at most one pair of protons and one pair of neutrons with $L \neq 0$; the other pairs in our basis states are $L=0$ spectator pairs.

To split off these spectator pairs, we apply the quasispin reduction formalism of Racah to reduce a many-pair matrix element to a product of a two-pair matrix element and a reduction factor which correctly accounts for (most of) the Pauli effects due to the S pair spectators. In the case of states built from a single j shell, the quasispin reduction formulas are equivalent to seniority reduction formulas; for a detailed discussion of the quasispin or seniority reduction formalism, the reader is referred to Appendix 3 of Ref. 20. In Table I we show the proton- and neutron-number dependence of all nonzero matrix elements considered here. Having extracted the seniority reduction factors, we write an interaction matrix element as

$$\begin{aligned}
M_{kl} &= \langle \psi_k \mid V_{\pi\nu} \mid \psi_l \rangle = \langle \psi_k || V_{\pi\nu} || \psi_l \rangle \\
&= r(n_\pi, v_\pi, v'_\pi, j_\pi) r(n_\nu, v_\nu, v'_\nu, j_\nu) \\
&\quad \times \langle (v_\pi J_\pi; v_\nu J_\nu) J || V_{\pi\nu} || (v'_\pi J'_\pi; v'_\nu J'_\nu) J \rangle ,
\end{aligned} \tag{15}$$

where v_π and v'_π (v_ν and v'_ν) are the proton (neutron) seniorities in the states $|\psi_k\rangle$ and $|\psi_l\rangle$, respectively, and the factors $r(n,v,v',j)$ are those in Table I. In the second equality above we have used the Wigner-Eckart theorem[21]

$$\langle JM \mid O_q^{(K)} \mid J'M' \rangle = (J'KM'q \mid JM)\langle J||O^{(K)}||J'\rangle , \quad (16)$$

and the value of the Clebsch-Gordan coefficient $(J0M0 \mid JM)=1$. Owing to the separability of the proton-neutron interaction $V_{\pi\nu}$, the two-pair matrix elements in Eq. (16) can be rewritten as

$$\langle (v_\pi J_\pi ; v_\nu J_\nu)J || V_{\pi\nu} || (v'_\pi J'_\pi ; v'_\nu J'_\nu)J \rangle = (-)^{J+1} f \begin{Bmatrix} J_\pi & J'_\pi & 2 \\ J'_\nu & J_\nu & J \end{Bmatrix} \hat{J}_\pi \hat{J}_\nu \langle J_\pi || C_\pi^{(2)} || J'_\pi \rangle \langle J_\nu || C_\nu^{(2)} || J'_\nu \rangle$$

$$= (-)^{J+1} B \begin{Bmatrix} J_\pi & J'_\pi & 2 \\ J'_\nu & J_\nu & J \end{Bmatrix} \begin{Bmatrix} 2 & J_\pi & J'_\pi \\ j_\pi & j_\pi & j_\pi \end{Bmatrix} \begin{Bmatrix} 2 & J_\nu & J'_\nu \\ j_\nu & j_\nu & j_\nu \end{Bmatrix} \hat{J}_\pi \hat{J}'_\pi \hat{J}_\nu \hat{J}'_\nu , \quad (17)$$

where $\hat{J}=(2J+1)^{1/2}$ and $B=47.9$ MeV depends upon the interaction strength f as well as j_π and j_ν. The many-pair interaction matrix elements are calculated by combining Eqs. (15) and (17) with the r factors from Table I. The general form of the matrix element in Eq. (17) is pictured diagrammatically in Fig. 1(a), with the angular momentum labels from Eq. (17) shown. The proton (neutron) pair correlation is denoted by a circular vertex on the proton (neutron) lines. The wavy line connecting the proton and neutron vertices represents the separable proton-neutron quadrupole-quadrupole interaction characterized by $C_\pi^{(2)}$ and $C_\nu^{(2)}$. Spectator pair lines are not included in Fig. 1(a); Figs. 1(b) and (c) show two diagrams representing "bare" (unperturbed) matrix elements whose renormalization we will examine later.

TABLE I. Seniority reduction factors.

$$r(n,v,v-2,j) = (n - v/2 + 1)^{1/2} \times \left[\frac{2\Omega - 2n - v + 2}{2(\Omega - v + 1)} \right]^{1/2} ,$$

$$r(n,v,v,j) = \frac{\Omega - 2n}{\Omega - v} ,$$

and $\Omega = (2j+1)/2$.

n	$r(n,2,0,j=\frac{31}{2})$	$r(n,2,2,j=\frac{31}{2})$
1	1	1
2	1.366	0.857
3	1.612	0.714
4	1.789	0.571
5	1.915	0.429
6	2.000	0.286
7	2.049	0.143
8	2.066	0

D. Perturbation theory

In evaluating the coupling between the *S-D* space and *S-D-G* space, one could employ the Feshbach method[22] to correct the *S-D* wave function to first order in $V_{\pi\nu}$, use the perturbed wave functions to calculate matrix elements of the fermion Hamiltonian, and then map these corrected matrix elements onto matrix elements of the boson Hamiltonian to determine the renormalization of the boson parameters. This technique has been employed by Otsuka who found that the presence of $L=4$ pairs was not important except near the closed shell.[14]

A program of relating the renormalization of boson parameters to account for effects of states left out of the *s-d* boson space, to the renormalization of matrix elements in a paired-fermion space (coupling of *S-D* and *S-D-G* spaces) can also be accomplished by calculating the second-order corrections to paired-fermion matrix elements. These corrected matrix elements are mapped onto matrix elements of the boson Hamiltonian, thereby renormalizing the boson parameters. We use this method, taking $V_{\pi\nu}$ as the perturbing term. Figure 2(a) shows a diagrammatic representation of a typical second-order process, written generally as

$$\langle (J_\pi J_\nu)J \mid V_{\pi\nu}^{(2)} \mid (J'_\pi J'_\nu)J \rangle = \sum_{I_\pi I_\nu} \langle (J_\pi J_\nu)J \mid V_{\pi\nu} \mid (I_\pi I_\nu)J \rangle \times \frac{1}{\Delta E} \langle (I_\pi I_\nu)J \mid V_{\pi\nu} \mid (J'_\pi J'_\nu)J \rangle . \quad (18)$$

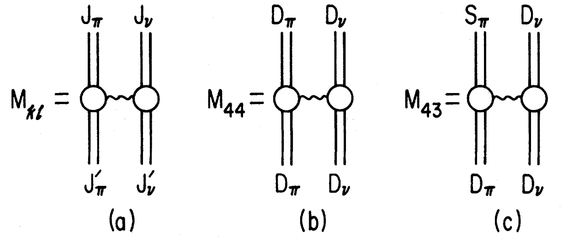

FIG. 1. (a) Diagrammatic representation of first-order matrix element M_{kl}. (b) First-order diagram for typical seniority-conserving matrix element M_{44}. (c) First-order diagram for typical seniority-changing matrix element M_{43}.

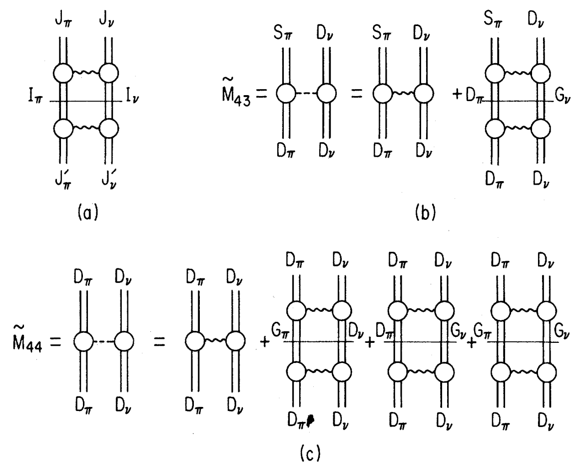

FIG. 2. (a) Diagrammatic representation of second-order process. (b) Sum of diagrams for renormalization of seniority-changing matrix element M_{43}. (c) Sum of diagrams for renormalization of seniority-conserving matrix element, M_{44}.

The energy denominator ΔE above will be discussed later. The matrix elements of $V_{\pi\nu}$ in Eq. (18) are found using the results of the previous subsection. The sum of a first-order diagram from Fig. 1 and the appropriate second-order diagrams of the type found in Fig. 2(a) is denoted $\widetilde{M}_{ij}$, with $i,j=1, 2, 3, 4$; Fig. 2(b) shows the sum of terms which corresponds to

$$\widetilde{M}_{43}=M_{43}+\langle (02)2\,|\,V_{\pi\nu}\,|\,(24)2\rangle \times \frac{1}{\Delta E}\langle (24)2\,|\,V_{\pi\nu}\,|\,(22)2\rangle \ . \qquad (19)$$

The energy denominator in Eq. (18) is written as $\Delta E=E_0-H_0$, where the unperturbed Hamiltonian is

$$H_0=\epsilon_{S_\pi}\hat{n}_{S_\pi}+\epsilon_{D_\pi}\hat{n}_{D_\pi}+\epsilon_{G_\pi}\hat{n}_{G_\pi} +\epsilon_{S_\nu}\hat{n}_{S_\nu}+\epsilon_{D_\nu}\hat{n}_{D_\nu}+\epsilon_{G_\nu}\hat{n}_{G_\nu} \ . \qquad (20)$$

The single-pair energies used in H_0 are $\epsilon_{S_\pi}=\epsilon_{S_\nu}=0$, $\epsilon_{D_\pi}=\epsilon_{D_\nu}=1.2$ MeV, $\epsilon_{G_\pi}=\epsilon_{G_\nu}=1.6$ MeV; the D and G pair energies are taken to be the energies of the first two excited states of ${}^{134}_{52}\mathrm{Te}_{82}$.[23] Proton and neutron energies are taken equal for simplicity. The effect of the choice of ϵ_G on the calculated renormalization of matrix elements and the influence of "dressing" the pair lines will be examined in the next subsection.

E. Particular cases

In this subsection we discuss the corrections through second-order of two particular matrix elements. The importance of the renormalization considered here will be made clear in Sec. III. In Fig. 3(a) we have plotted the ratio $\widetilde{M}_{44}/M_{44}$ for $J=2$ in $|\psi_4\rangle$ from Eq. (10) as a function of proton-pair number and neutron-pair number. The matrix element M_{44} is

$$M_{44}=\langle (S_\pi^{n_\pi-1}D_\pi;S_\nu^{n_\nu-1}D_\nu)J\,|\,V_{\pi\nu}\,|\,(S_\pi^{n_\pi-1}D_\pi;S_\nu^{n_\nu-1}D_\nu)J\rangle \ . \qquad (21)$$

The renormalized matrix element $\widetilde{M}_{44}$ consists of M_{44} and all possible corrections up to second order in $V_{\pi\nu}$, such that the intermediate states contain at least one G pair; $\widetilde{M}_{44}$ is represented diagrammatically in Fig. 2(c).

As Fig. 3(a) shows, the second-order correction to M_{44} is dramatic except in the case of a nearly half-filled proton shell, $n_\pi=7$. Even for this case, however, the neutron pair number dependence of $\widetilde{M}_{44}/M_{44}$ is still significant. The sizable renormalization of the "bare" matrix element M_{44} suggests that the effects of G pairs left out of the S-D paired-fermion space can be quite large, depending upon the number of pairs in the system. This result is consistent with the findings of McGrory[11] who has reported a diminished influence of the $J=4$ favored pair on low-lying, low-spin states as particles are added to a model nucleus.

In Fig. 3(b) we show the effect of the choice of ϵ_G on the calculated renormalization of M_{44}. Here we

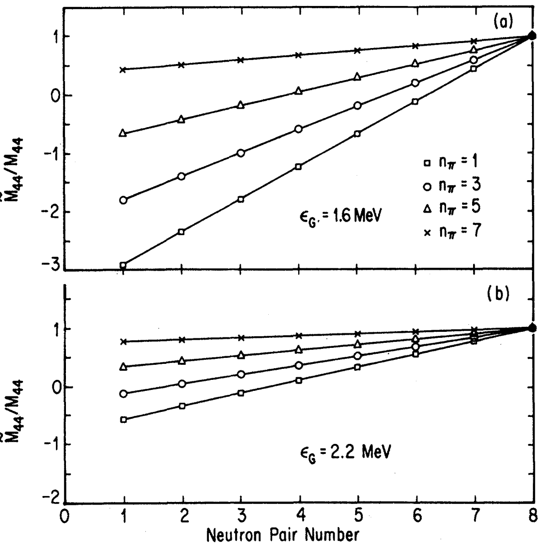

FIG. 3. (a) Ratio of renormalized matrix element $\tilde{M}_{44}$ to unperturbed matrix element M_{44} vs proton and neutron pair number, for single G-pair energy $\epsilon_G = 1.6$ MeV. (b) Same as (a), for $\epsilon_G = 2.2$ MeV.

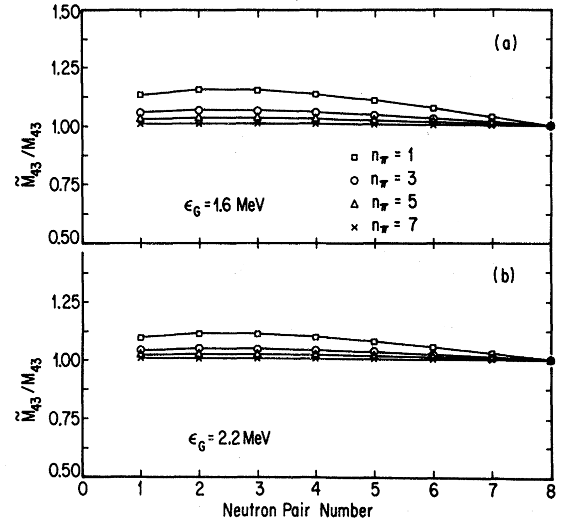

FIG. 4. (a) Ratio of renormalized matrix element $\tilde{M}_{43}$ to unperturbed matrix element M_{43} vs proton and neutron pair number, for $\epsilon_G = 1.6$ MeV. (b) Same as (a), for $\epsilon_G = 2.2$ MeV.

have used $\epsilon_G = 2.2$ MeV, following a suggestion of Wood[24] that such a value for the single g boson energy is appropriate for a fit to the $K^\pi = 3^+$ band of ^{168}Er. The energy of the two-quadrupole-phonon state would be $E(4^+) = 2\times\epsilon_D = 2.4$ MeV, so that the choice of $\epsilon_G \lesssim 2.4$ MeV for the *collective*, seniority $v = 2$ excitation seems plausible. Because our interest is in the general influence of the magnitude of ϵ_G on the perturbative corrections, we have not attempted to obtain a more realistic value of the single G pair energy. As Fig. 3(b) shows, the increase of ϵ_G from 1.6 MeV to $\epsilon_G = 2.2$ MeV makes the renormalization of M_{44} less dramatic, but the effect of excluding the G pair from the paired-fermion space is still apparent.

A second example of the renormalization of a matrix element of $V_{\pi\nu}$ is shown in Fig. 2(b). In Fig. 4(a) we plot the ratio $\tilde{M}_{43}/M_{43}$ as a function of proton- and neutron-pair number. The renormalization of M_{43} is smaller in magnitude and of opposite sign to the correction to M_{44}, due to the presence of the S_π pair in the matrix element. Figure 4(b) is a plot of $\tilde{M}_{43}/M_{43}$ for $\epsilon_G = 2.2$ MeV.

Following a suggestion by Pittel,[25] we have also calculated the second-order corrections using a modified Brillouin-Wigner perturbation theory, in which $\Delta E = E_0 - H'$ and

$$H' = H_0 + V_{\pi\nu}^{\text{diag}} . \qquad (22)$$

The first term in H' is just the unperturbed Hamiltonian from Eq. (20). The addition of the diagonal part of $V_{\pi\nu}$ to the single-pair terms from H_0 serves to "dress" the proton and neutron pair lines in the intermediate state[26]; this changes the second-order diagram of Fig. 2(a) and would introduce some unlinked diagrams beyond the second-order terms considered here. The modification of the unperturbed energy of the intermediate state was done in an attempt to partially account for the fact that the energy of the $L = 4$, seniority $v = 2$, paired-fermion state should change as a function of proton and neutron pair number due to interactions with other valence pairs. We recognize the rather *ad hoc* nature of this modification to the energy denominator and realize that a microscopic approach to calculating the unperturbed single-pair energies as functions of valence pair number might be more suitable, but we proceed using the modified H' defined above in order to preserve the simplicity of our calculations for this test. In Fig. 5 we show the result of calculating $\tilde{M}_{44}/M_{44}$ with dressed pair lines in the intermediate state. When the number of proton and neutron pairs is not large (e.g., n_π or $n_\nu = 1$ or 3), these ratios are about three times larger than those shown in Fig. 3. This large ratio is due to a partial cancellation of the single pair part of ΔE by the diagonal part of $V_{\pi\nu}$. Given the *ad hoc* form of ΔE in Eq. (22) with its associated diagrammatical problems and the influence of this modification on important matrix elements, we prefer to use Rayleigh-Schrödinger perturbation theory [ΔE from Eq. (20)] for all of our calculations.

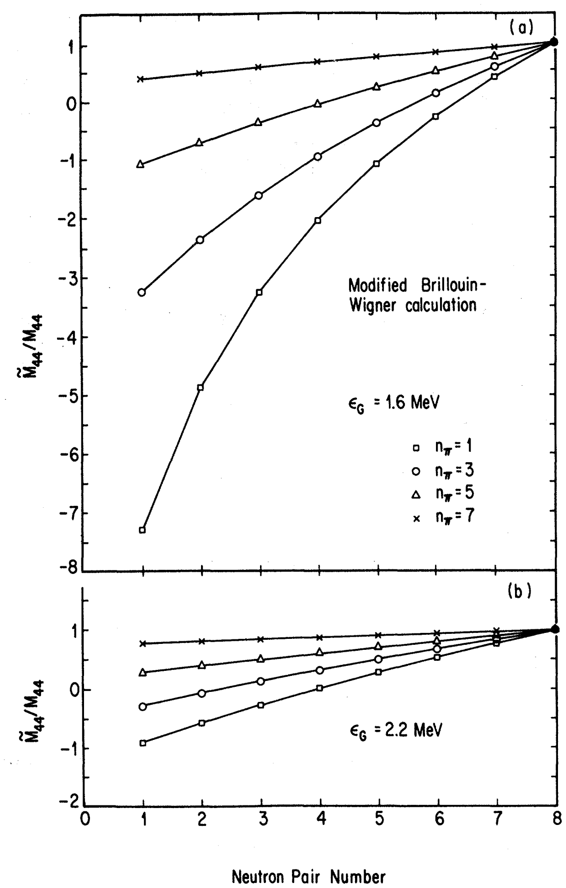

FIG. 5. (a) $\tilde{M}_{44}/M_{44}$ using modified Brillouin-Wigner perturbation theory, $\epsilon_G = 1.6$ MeV, to "dress" pair lines. (b) Same as (a), for $\epsilon_G = 2.2$ MeV.

III. RENORMALIZATION OF BOSON PARAMETERS

The OAI imaging technique has been applied in conjunction with second-order corrections to paired-fermion matrix elements of $V_{\pi\nu}$, in order to calculate the renormalization of the parameters of the boson Hamiltonian. In the OAI method, matrix elements of boson operators are constructed using matrix elements calculated in a paired-fermion space; renormalization of the fermion matrix elements to account for extra-model space effects results in the renormalization of the parameters of the boson operator.

The IBM Hamiltonian is written as in Eq. (1),

$$H_B = H_{\pi\pi} + H_{\nu\nu} + H_{\pi\nu} \, . \tag{23}$$

Here, however, we allow $H_{\pi\pi(\nu\nu)}$ and $H_{\pi\nu}$ to assume the most general possible form for a scalar operator consisting of one- and two-boson interactions, which are constructed using boson creation and annihilation operators with $L = 0$ and $L = 2$. The like-boson interaction is ($\rho = \pi$ or ν)

$$\begin{aligned} H_{\rho\rho} &= \epsilon_{s_\rho}(s_\rho^\dagger s_\rho)^{(0)} + \epsilon_{d_\rho}(d_\rho^\dagger \tilde{d}_\rho)^{(0)} \\ &\quad + \text{two-body terms} \\ &= \epsilon_{s_\rho}\hat{n}_{s_\rho} + \epsilon_{d_\rho}\hat{n}_{d_\rho} \\ &\quad + \text{two-body terms} \, , \end{aligned} \tag{24}$$

where $\hat{n}_{s_\rho}$ and $\hat{n}_{d_\rho}$ are s and d boson number operators, respectively. The completely general form of the proton boson-neutron boson interaction is given by

$$\begin{aligned} H_{\pi\nu} = {} & c_1[(s_\pi^\dagger s_\pi)^{(0)}\cdot(s_\nu^\dagger s_\nu)^{(0)}]^{(0)} + c_2\sqrt{5}[(d_\pi^\dagger \tilde{d}_\pi)^{(0)}\cdot(s_\nu^\dagger s_\nu)^{(0)}]^{(0)} \\ & + c_3\sqrt{5}[(s_\pi^\dagger s_\pi)^{(0)}\cdot(d_\nu^\dagger \tilde{d}_\nu)^{(0)}]^{(0)} + c_4[(d_\pi^\dagger s_\pi)^{(2)}\cdot(s_\nu^\dagger \tilde{d}_\nu)^{(2)} + (s_\pi^\dagger \tilde{d}_\pi)^{(2)}(d_\nu^\dagger s_\nu)^{(2)}]^{(0)} \\ & + \frac{c_5}{\sqrt{5}}[(s_\pi^\dagger \tilde{d}_\pi)^{(2)}\cdot(s_\nu^\dagger \tilde{d}_\nu)^{(2)} + (d_\pi^\dagger s_\pi)^{(2)}\cdot(d_\nu^\dagger s_\nu)^{(2)}]^{(0)} \\ & + c_6[(d_\pi^\dagger \tilde{d}_\pi)^{(2)}\cdot(s_\nu^\dagger \tilde{d}_\nu)^{(2)} + (d_\pi^\dagger \tilde{d}_\pi)^{(2)}\cdot(d_\nu^\dagger s_\nu)^{(2)}]^{(0)} \\ & + c_7[(s_\pi^\dagger \tilde{d}_\pi)^{(2)}\cdot(d_\nu^\dagger \tilde{d}_\nu)^{(2)} + (d_\pi^\dagger s_\pi)^{(2)}\cdot(d_\nu^\dagger \tilde{d}_\nu)^{(2)}]^{(0)} \\ & + \sum_{L=0}^{4} \gamma_L(2L+1)(-)^L \sum_{L'=0}^{4} \begin{Bmatrix} 2 & 2 & L \\ 2 & 2 & L' \end{Bmatrix} [(d_\pi^\dagger \tilde{d}_\pi)^{(L')}\cdot(d_\nu^\dagger \tilde{d}_\nu)^{(L')}]^{(0)} \, . \end{aligned} \tag{25}$$

The first three terms of $H_{\pi\nu}$ can be rewritten to include the number operators

$$H_{\pi\nu}=c_1\hat{n}_{s_\pi}\hat{n}_{s_\nu}+c_2\hat{n}_{d_\pi}\hat{n}_{s_\nu}+c_3\hat{n}_{s_\pi}\hat{n}_{d_\nu}+\cdots . \tag{26}$$

The OAI method is used to relate matrix elements of $V_{\pi\nu}$ between paired-fermion states in the S-D space to matrix elements of $H_{\pi\nu}$ between corresponding boson states in the s-d space:

$$\langle\psi_i\,|\,V_{\pi\nu}\,|\,\psi_j\rangle\equiv(\phi_i\,|\,H_{\pi\nu}\,|\,\phi_j) , \tag{27}$$

where $|\psi_i\rangle$ corresponds to $|\phi_i)$: $|\psi_i\rangle\leftrightarrow|\phi_i)$. OAI show that, in their zeroth-order approximation, one need calculate only those matrix elements with proton and neutron seniorities $v_\pi=0,2$ and $v_\nu=0,2$ (in the S-D space) in order to parametrize a two-body number-conserving operator. Hence, we have calculated matrix elements of $H_{\pi\nu}$ between boson states containing, at most, one proton d boson and, at most, one neutron d boson; these matrix elements are equated to corresponding S-D space matrix elements of $V_{\pi\nu}$. There are 12 such matrix elements of $H_{\pi\nu}$ which are nonzero. We obtain a system of linear equations involving the 12 boson parameters of Eq. (25). This set of equations can be inverted to yield the boson parameters as functions of paired-fermion matrix elements of $V_{\pi\nu}$, viz.,

$$c_1=M_{11}/n_\pi n_\nu , \tag{28a}$$

$$c_2=M_{22}/n_\nu-M_{11}(n_\pi-1)/(n_\pi n_\nu) , \tag{28b}$$

$$c_3=M_{33}/n_\pi-M_{11}(n_\nu-1)/(n_\pi n_\nu) , \tag{28c}$$

$$c_4=M_{23}/(n_\pi n_\nu)^{1/2} , \tag{28d}$$

$$c_5=M_{14}/(n_\pi n_\nu)^{1/2} , \tag{28e}$$

$$c_6=M_{24}/(n_\nu)^{1/2} , \tag{28f}$$

$$c_7=M_{34}/(n_\pi)^{1/2} , \tag{28g}$$

$$\gamma_L=M_{44}^L-M_{22}(n_\nu-1)/n_\pi+M_{33}(n_\pi-1)/n_\nu+M_{11}(n_\pi-1)(n_\nu-1)/n_\pi n_\nu , \tag{28h}$$

where

$$M_{44}^L=\langle(S_\pi^{n_\pi-1}D_\pi;S_\nu^{n_\nu-1}D_\nu)J=L\,|\,V_{\pi\nu}\,|\,(S_\pi^{n_\pi-1}D_\pi;S_\nu^{n_\nu-1}D_\nu)J=L\rangle,\ L=0-4 . \tag{28i}$$

It is clear from the above expressions that the effects which states left out of the S-D space have on the paired-fermion matrix elements will appear as renormalizations of the boson parameters of $H_{\pi\nu}$. With the exception of M_{11}, all of the paired-fermion matrix elements in Eqs. (28) have nonzero second-order corrections due to G pairs outside the S-D model space; hence, the parameters C_{2-7} and γ_{0-4} can be renormalized to take into account the exclusion of the g boson from the s-d model space.

We have calculated explicitly the renormalization of single d boson energies of the boson Hamiltonian. We rewrite H_B, singling out the relevant terms

$$H_B=\epsilon_{s_\pi}\hat{n}_{s_\pi}+\epsilon_{s_\nu}\hat{n}_{s_\nu}+\epsilon_{d_\pi}\hat{n}_{d_\pi}+\epsilon_{d_\nu}\hat{n}_{d_\nu}+c_1\hat{n}_{s_\pi}\hat{n}_{s_\nu}+c_2\hat{n}_{d_\pi}\hat{n}_{s_\nu}+c_3\hat{n}_{s_\pi}\hat{n}_{d_\nu}+\cdots . \tag{29}$$

Defining the number operators $\hat{n}_\pi=\hat{n}_{s_\pi}+\hat{n}_{d_\pi}$ and $\hat{n}_\nu=\hat{n}_{s_\nu}+\hat{n}_{d_\nu}$, we replace $\hat{n}_{s_\pi}$ and $\hat{n}_{s_\nu}$ in Eq. (29) to obtain

$$H_B=c_1\hat{n}_\pi\hat{n}_\nu+\epsilon_{s_\pi}\hat{n}_\pi+\epsilon_{s_\nu}\hat{n}_\nu+[\epsilon_{d_\pi}-\epsilon_{s_\pi}+c_2\hat{n}_\nu-c_1\hat{n}_\nu]\hat{n}_{d_\pi}+[\epsilon_{d_\nu}-\epsilon_{s_\nu}+c_3\hat{n}_\pi-c_1\hat{n}_\pi]\hat{n}_{d_\nu}+(c_1-c_2-c_3)\hat{n}_{d_\pi}\hat{n}_{d_\nu}+\cdots . \tag{30}$$

Ignoring the first three terms in Eq. (30) as unimportant for the calculation of *excited* states in nucleus with given n_π and n_ν, we write H_B as

$$H_B=\epsilon_\pi\hat{n}_{d_\pi}+\epsilon_\nu\hat{n}_{d_\nu}+(c_1-c_2-c_3)\hat{n}_{d_\pi}\hat{n}_{d_\nu}+\cdots . \tag{31}$$

where

$$\epsilon_\pi=\epsilon_\pi^0+(c_2-c_1)n_\nu=\epsilon_\pi^0+(M_{22}-M_{11}) , \tag{32a}$$

$$\epsilon_\nu=\epsilon_\nu^0+(c_3-c_1)n_\pi=\epsilon_\nu^0+(M_{33}-M_{11}) , \tag{32b}$$

and

$$\epsilon_\pi^0=\epsilon_{d_\pi}-\epsilon_{s_\pi},\ \epsilon_\nu^0=\epsilon_{d_\nu}-\epsilon_{s_\nu} .$$

The second terms in ϵ_π and ϵ_ν are

$$M_{22}-M_{11}\equiv\Delta\epsilon_\pi=\langle(S_\pi^{n_\pi-1}D_\pi;S_\nu^{n_\nu})J=2\,|\,V_{\pi\nu}\,|\,(S_\pi^{n_\pi-1}D_\pi;S_\nu^{n_\nu})J=2\rangle-\langle(S_\pi^{n_\pi};S_\nu^{n_\nu})J=0\,|\,V_{\pi\nu}\,|\,(S_\pi^{n_\pi};S_\nu^{n_\nu})J=0\rangle , \tag{33a}$$

$$M_{33}-M_{11}\equiv\Delta\epsilon_\nu=\langle (S_\pi^{n_\pi};S_\nu^{n_\nu-1}D_\nu)J=2\,|\,V_{\pi\nu}\,|\,(S_\pi^{n_\pi};S_\nu^{n_\nu-1}D_\nu)J=2\rangle$$
$$-\langle (S_\pi^{n_\pi};S_\nu^{n_\nu})J=0\,|\,V_{\pi\nu}\,|\,(S_\pi^{n_\pi};S_\nu^{n_\nu})J=0\rangle \ . \tag{33b}$$

Recall that M_{kl} (without a tilde) is an unperturbed matrix element.

The renormalization of the single proton d boson and single neutron d boson energies, ϵ_π and ϵ_ν, respectively, occurs through the second-order corrections to M_{22} and M_{33} to include effects of G pairs. The renormalized values are

$$\tilde{\epsilon}_\pi=\epsilon_\pi^0+(\tilde{M}_{22}-\tilde{M}_{11})=\epsilon_\pi^0+(\tilde{M}_{22}-M_{11}) \ , \tag{34}$$

$$\tilde{\epsilon}_\nu=\epsilon_\nu^0+(\tilde{M}_{33}-\tilde{M}_{11})=\epsilon_\nu^0+(\tilde{M}_{33}-M_{11}) \ . \tag{35}$$

The last terms in Eqs. (34) and (35) are renamed

$$\tilde{M}_{22}-M_{11}=\Delta\tilde{\epsilon}_\pi \ , \tag{36}$$

$$\tilde{M}_{33}-M_{11}=\Delta\tilde{\epsilon}_\nu \ . \tag{37}$$

The corrections $\Delta\tilde{\epsilon}_\pi$ and $\Delta\tilde{\epsilon}_\nu$ to ϵ_π^0 and ϵ_ν^0 are plotted in Figs. 6(a) and 7(a) for $\epsilon_G=1.6$ MeV and Figs. 6(b) and 7(b) for $\epsilon_G=2.2$ MeV. The essentially quadratic trend in $\Delta\tilde{\epsilon}_\pi$ as a function of n_ν for fixed proton pair number n_π can be seen directly from the expression

$$\Delta\tilde{\epsilon}_\pi=\tilde{M}_{22}=-\left[\frac{\Omega_\pi-2n_\pi}{\Omega_\pi-2}\right]^2\left[n_\nu\left[\frac{\Omega_\nu-n_\nu}{\Omega_\nu-1}\right]\right]|\langle (D_\pi S_\nu)J=2\,|\,V_{\pi\nu}\,|\,(G_\pi D_\nu)J=2\rangle|^2/\epsilon_G \ . \tag{38}$$

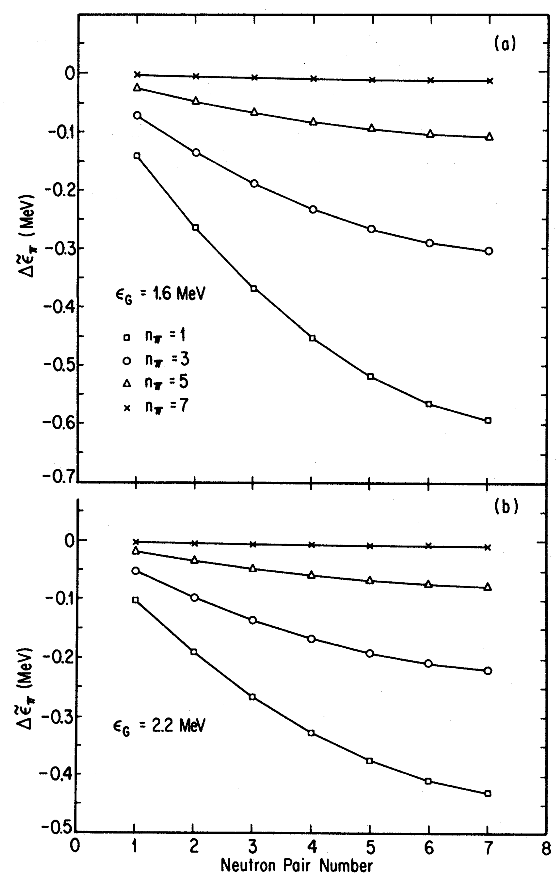

FIG. 6 (a) Renormalization of single d_π boson energy vs proton and neutron pair number, for $\epsilon_G=1.6$ MeV. (b) Same as (a), for $\epsilon_G=2.2$ MeV.

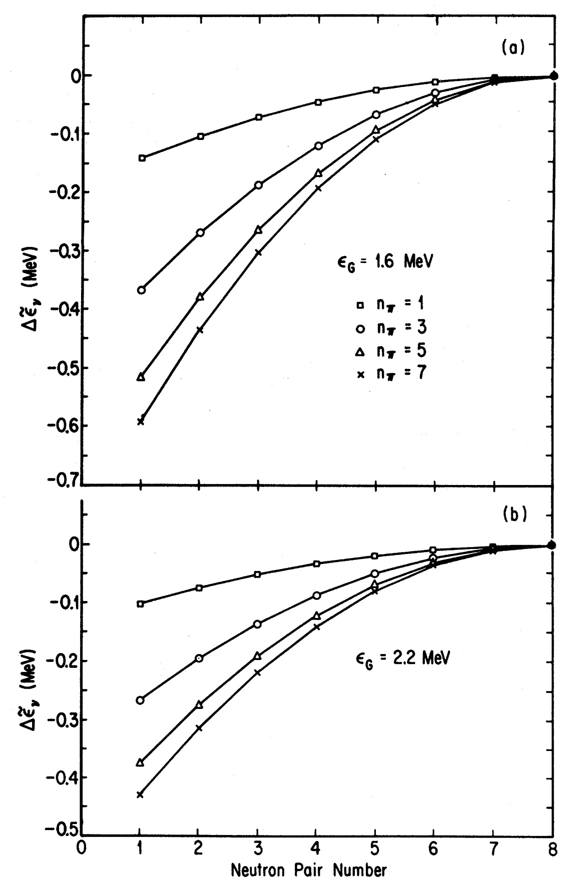

FIG. 7. (a) Renormalization of single d_ν boson energy vs proton and neutron pair number, for $\epsilon=1.6$ MeV. (b) Same as (a), for $\epsilon_G=2.2$ MeV.

(For the quadrupole-quadrupole interaction $V_{\pi\nu}$ used here, $M_{11}=M_{22}=0$.) As more neutron pairs are added, the attractive second-order proton-neutron interaction pushes $\Delta\tilde{\epsilon}_\pi$ toward more negative values. In contrast, the second-order correction to ϵ_ν^0 is

$$\Delta\tilde{\epsilon}_\nu=\tilde{M}_{33}=-\left[n_\pi\left[\frac{\Omega-n_\pi}{\Omega_\pi-1}\right]\right]\left[\frac{\Omega_\nu-2n_\nu}{\Omega_\nu-2}\right]^2 |\langle(S_\pi D_\nu)J=2\,|\,V_{\pi\nu}\,|\,(D_\pi G_\nu)J=2\rangle|^2/\epsilon_G \;. \tag{39}$$

For fixed proton pair number, the factor $(\Omega_\nu-2n_\nu)^2$ causes $\Delta\tilde{\epsilon}_\nu$ to vanish identically when the neutron shell is half filled. Physically, this means that the strength of the proton-neutron interaction between the states

$$|(S_\pi^{n_\pi};S_\nu^{n_\nu-1}D_\nu)J=2\rangle$$

and

$$|(S_\pi^{n_\pi-1}D_\pi;S_\nu^{n_\nu-1}G_\nu)J=2\rangle$$

is "diluted" as more neutron pairs are added to the system.

The renormalization of other terms in the proton-neutron interaction $H_{\pi\nu}$ of Eq. (25) could be examined in a similar manner; other workers are currently undertaking a systematic study of such extra-model space effects, utilizing a generalized seniority basis for many nondegenerate proton and neutron orbitals.[18]

IV. CONCLUSIONS

In this paper we have employed the OAI imaging method to map second-order corrections of paired-fermion matrix elements onto corresponding boson matrix elements. This perturbative calculation of the effect of excluding the g boson from the IBM model space led to an explicit renormalization of the parameters of the boson Hamiltonian. Particular examples of the renormalization were discussed in detail. We found that the correction terms depended strongly upon the numbers of proton and neutron valence pairs in the nucleus. An example of this dependence is the ratio of the renormalized seniority-conserving matrix element $\tilde{M}_{44}$ to the unperturbed value M_{44}. For the case of three proton pairs, $\tilde{M}_{44}$ varies from -450% of M_{44} for one neutron pair, to -30% of M_{44} for seven neutron pairs; $\tilde{M}_{44}$ vanishes identically for eight pairs, the half-filled shell. The observed dependence on neutron pair number is due solely to Pauli effects (spectator pairs).

The decreasing importance of the $L=4$ degree of freedom in the collective states has also been observed by Otsuka[14] and by McGrory.[13] These authors conclude that the truncation of the boson model space to include only states containing s and d bosons is valid for low-lying, low-spin collective states. Our calculations confirm their findings, provided that there are several proton and neutron valence pairs in the nucleus. If n_π and n_ν are small, the coupling of the S-D states to states outside this model space is so large that one must question whether it is reasonable to use any perturbative approach to renormalize the boson Hamiltonian.

ACKNOWLEDGMENTS

We would like to thank H. A. Weidenmüller, S. Pittel, and J. L. Wood for valuable discussions relating to this work. This material is based upon work supported in part by the National Science Foundation Grant No. PHY-81-00141.

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