Microscopic structure of an interacting boson model in terms of the Dyson boson mapping

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In an application of the generalized Dyson boson mapping to a shell model Hamiltonian acting in a single j shell, a clear distinction emerges between pair bosons and kinematically determined seniority bosons. As in the Otsuka-Arima-Iachello method it is found that the latter type of boson determines the structure of an interactive boson-model-like Hamiltonian for the single j-shell model. It is furthermore shown that the Dyson boson mapping formalism is equally well suited for investigating possible interactive boson-model-like structures in a multishell case, where dynamical considerations are expected to play a much more important role in determining the structure of physical bosons.

[NUCLEAR STRUCTURE Boson mapping, interacting boson model, shell model.]

I. INTRODUCTION

Although the success of the interacting boson model (IBM) is well established on a phenomenological level,[1-3] the same cannot as yet be said about microscopic support for the model. In fact, a better understanding of the conditions and assumptions required on a microscopic level in order for an IBM-like picture to emerge will at the same time indicate possible difficulties that the simple IBM model might run into[4,5] and if it is possible to remedy this by a straightforward extension of the model.

An SO(8) fermion model recently proposed by Ginocchio[6] seems to give some microscopic support to the IBM. In view of the special features of the model one should, however, exercise some caution in considering the results as giving direct support. The most unrealistic feature of this model is the complete decoupling of the S-D subspace from other shell model states.

Otsuka, Arima, and Iachello (OAI) (Ref. 7) considered a more realistic single j-shell (SJS) shell model Hamiltonian for which the S-D subspace is not decoupled from the rest of the shell model states. Their method of introducing bosons may be model dependent in the sense that it depends on the success of the seniority scheme in a single j shell. Even so, the OAI method[7] is invariably cited in support of a microscopically founded IBM.

In what follows we show that the Dyson boson mapping (DBM) (Refs. 8–11) constitutes a framework in which the goal of understanding the IBM microscopically can be pursued with what we believe to be minimal effort. At the same time the DBM formalism has at each stage of development physical and mathematical transparency and justification. The DBM method has recently been applied[12] to Ginocchio's SO(8) model.

While we aim here at illustrating the use and role of the DBM in obtaining a microscopically founded boson Hamiltonian for a realistic shell model fermion space, our analysis also shows that for the SJS case the OAI method is compatible with the DBM in the quest of obtaining the "physical" bosons. At this point it is already significant to note that the structure of the bosons emerging from the OAI analysis of the SJS model is at variance with the often held view of "physical" IBM bosons, the latter type being associated with nucleon pairs of angular momentum 0 and 2 (s and d bosons) while the former type can be viewed as seniority bosons.[7,13] This distinction emerges very clearly from the DBM analysis of the SJS model in which the seniority bosons are introduced in terms of pair bosons in a way analogous to the procedure proposed by Klein and Vallieres.[14] The fact that the structure of physical bosons can change drastically from the SJS case (where kinematics is the determining factor) to the multi-j-shell (MJS) case (where dynamics will play a much more important role) indicates that the OAI analysis of the SJS model cannot be considered as a general prescription for deriving the IBM Hamiltonian.

The DBM method has, on the other hand, the additional advantage that it can easily be extended from the SJS to the MJS case. Furthermore, the structure of the physical bosons in the MJS case will be determined by considerations similar to those used in the SJS case.

The present paper is organized as follows. In Sec. II the generalized DBM is briefly reviewed and then applied to the SJS model in Sec. III, first for the simple but illustrative case of a pairing interaction and then for a quadrupole-quadrupole interaction. Other approaches[15,16] to the present problem are briefly discussed and compared to the DMB in Sec. IV and in Sec. V a discussion of the results is presented together with some ideas about an extension to the MJS case.

II. THE GENERALIZED DYSON BOSON MAPPING

The DBM has been extensively discussed[8–11] and illustrated in analyses of analytically solvable fermion models such as the Sp(4) model[17] and Ginocchio's SO(8) model.[6] Here we only briefly review the basic structure of the formalism.

The idea behind any boson mapping is to set up an isomorphism between the fermion space and an ideal space, the latter comprised of bosons or bosons and ideal fermions,[18] usually depending on whether an even or odd fermion system is being considered. For the even case with only one type of fermion the generalized DBM is given by

$$a^1a^2 \to R^{12} = B^{12} - B^{13}B^{24}B_{34} , \tag{2.1}$$

$$a_2a_1 \to R_{12} = B_{12} , \tag{2.2}$$

$$a^1a_2 \to R_2{}^1 = B^{13}B_{23} , \tag{2.3}$$

where $n = (j_n m_n)$, while a^n and a_n denote fermion creation and annihilation operators in the original space. The operators R are the images in the ideal space of the corresponding bifermion operators and the operators B are boson creation and annihilation operators satisfying the boson algebra

$$\begin{aligned} &[B_{12}, B^{34}] = \delta_1{}^3\delta_2{}^4 - \delta_2{}^3\delta_1{}^4 , \\ &[B_{12}, B_{34}] = [B^{12}, B^{34}] = 0 , \\ &(B_{12})^\dagger = B^{12} . \end{aligned} \tag{2.4}$$

The vacuum state $|0)$ in the ideal space is defined by $B_{12}|0) = 0$. (We use the convention of an implied summation over repeated indices.)

Since physical bosons have good angular momentum and are associated with collectivity of some as yet unspecified kind, the bosons introduced above are transcribed by introducing coupled and collective boson operators defined by

$$B^{(j_1j_2)JM} = \sum_{m_1m_2} \langle j_1m_1j_2m_2 \,|\, JM \rangle B^{j_1m_1j_2m_2} ;$$
$$B_{(j_1j_2)JM} = (B^{(j_1j_2)JM})^\dagger , \tag{2.5}$$

$$C^{JM\sigma} = \tfrac{1}{2} \sum \chi^{JM\sigma}_{j_1j_2} B^{(j_1j_2)JM} ;$$
$$C_{JM\sigma} = (C^{JM\sigma})^\dagger , \tag{2.6}$$

where $\{\chi^{JM\sigma}_{j_1j_2}\}$ is a complete set of two-particle wave functions. The normalization

$$\sum_{j_1j_2} \chi^{JM\sigma}_{j_1j_2} \chi^{j_1j_2}_{JM\sigma'} = 2\delta^{\sigma}_{\sigma'} \tag{2.7}$$

ensures that the collective boson commutator

$$[C_{JM\sigma}, C^{J'M'\sigma'}] = \delta_J^{J'}\delta_M^{M'}\delta_\sigma^{\sigma'}$$

holds. (See Appendix A and Ref. 11 for further details.) We would like to stress again the economy of the DBM which is free from the convergence problems facing other boson mappings. The nonunitarity of the DBM leads to a non-Hermitian Hamiltonian. Yet the Hamiltonian matrix represented in a physical basis is Hermitian. (See also Ref. 10.) Representation of the Hamiltonian in a boson basis is preferable (and more in line with the procedure of the IBM), but may involve the problem of overcompleteness as discussed later.

III. THE SINGLE j SHELL

A. General images of bifermion operators

For the single j shell there is just one ("collective") two-particle wave function for each J and the Dyson images of the angular momentum coupled bifermion operators

$$A^{JM} = \frac{1}{\sqrt{2}} [a^j \times a^j]^{JM} , \tag{3.1}$$

$$\begin{aligned} &\tilde{A}_{JM} = -\frac{1}{\sqrt{2}} [\tilde{a}_j \times \tilde{a}_j]_{JM} ; \\ &\tilde{A}_{JM} = (-)^{J+M} A_{J-M} , \end{aligned} \tag{3.2}$$

$$U^J_M = [a^j \times \tilde{a}_j]^J_M , \tag{3.3}$$

are readily found to be

$$(A^{JM})_D=B^{JM}-2\sum(\hat{J}_1\hat{J}_2\hat{J}_3\hat{L})^{1/2}\begin{Bmatrix} j & J_1 & j\\ j & J_2 & j\\ J_3 & L & J\end{Bmatrix}[[B^{J_1}\times B^{J_2}]^L\times\tilde{B}_{J_3}]^{JM}\ , \tag{3.4}$$

$$(A_{JM})_D=B_{JM}\ , \tag{3.5}$$

$$(U^J_M)_D=[B^{J_1}\times\tilde{B}_{J_2}]^J_M\ , \tag{3.6}$$

where $\hat{J}=(2J+1)$ and $\tilde{a}_{jm}=(-1)^{j-m}a_{j-m}$.

From the structure of these operators it can now be explicitly shown that the physical boson states, namely those obtained by operating repeatedly with $(A^{JM})_D$ on the boson vacuum state $|0)$, contain all the information of the usual seniority scheme classification (see Appendix B), i.e., the coefficients of the components of

$$[\ldots[B^{J'_1}\times B^{J'_2}]^{J'_3}\times\ldots]^J|0)$$

in

$$[\ldots[(A^{J_1})_D\times(A^{J_2})_D]^{J_3}\times\ldots]^J|0)$$

are just the two-particle coefficients of fractional parentage (cfp's) or linear combinations of them. (See also Ref. 19.) This is of course no surprise, since the seniority scheme is just a way of ensuring antisymmetrization, which in turn is the very basis of the generalized DBM. [It will, on the other hand, be very hard to obtain these cfp's from any infinite boson mapping ("expansion").]

B. Pairing interaction

The pairing model has served as an illustration of the OAI method,[7] as well as an example of how and why a non-Hermitian boson Hamiltonian can correspond to a Hermitian fermion Hamiltonian.[13] Although the Dyson boson mapping was first considered within an SU(2) framework,[20,21] the realization was given in terms of only one type of boson (corresponding to an s boson). The most general realization,[11] however, contains other bosons too, as is easily found from Eq. (3.4), now written as

$$\sqrt{\Omega}(S_+)_D=S^\dagger\left[\Omega-\hat{N}_S-2\sum_{J=1}^{\Omega}\hat{N}_{2J}\right]+\sum_{\substack{J_1J_2\neq 0\\ J_3}}(\hat{J}_1\hat{J}_2)^{1/2}\begin{Bmatrix} J_1 & J_2 & J_3\\ j & j & j\end{Bmatrix}[B^{J_1}\times B^{J_2}]^{J_3}\cdot B_{J_3}\ , \tag{3.7}$$

$$\sqrt{\Omega}(S_-)_D=S\ , \tag{3.8}$$

$$(S_z)_D=\hat{N}_S+\sum_{J=1}^{\Omega}N_{2J}-\frac{\Omega}{2}\ , \tag{3.9}$$

where $s^\dagger=B^{J=0}$, $\hat{N}_J=\sum_{JM}B^{JM}B_{JM}$, and $2\Omega=2j+1$, while the fermion operators are $S_+=\sqrt{\Omega}A^{00}$, $S_-=\sqrt{\Omega}A_{00}$, and $S_z=\frac{1}{2}(\hat{N}_F-\Omega)$. (See Table I.) The most general Dyson image of the pairing Hamiltonian, $H=-GS_+S_-$, is therefore

$$H_D=-G\left[\hat{N}_S\left[\Omega-\hat{N}_S+1-2\sum_{J=1}^{\Omega}N_{2J}\right]-\frac{1}{\sqrt{2\Omega}}\sum_J(B^J\cdot B^J)SS\right.$$
$$\left.+\sum_{J_1,J_2J_3\neq 0}(\hat{J}_1\hat{J}_2)^{1/2}\begin{Bmatrix} J_1 & J_2 & J_3\\ j & j & j\end{Bmatrix}[[B^{J_1}\times B^{J_2}]^{J_3}\times\tilde{B}_{J_3}]^{00}S\right]\ , \tag{3.10}$$

TABLE I. Notation for operators in fermion and boson spaces.

IBM or physical bosons	$s,s^\dagger,d_\mu,d^\mu\equiv d^\dagger_\mu,g_\mu,g^\mu,\ldots$
DBM or pair bosons	$S,S^\dagger,D_\mu,D^\mu,G_\mu,G^\mu,\ldots$
Fermion pair operators	$S_+\equiv\sqrt{\Omega}A^{00}$, $S_-\equiv\sqrt{\Omega}A_{00}$, $S_Z\equiv\frac{1}{2}(\hat{N}_F-\Omega)$
Dyson images of pair operators	$\breve{S}_+\equiv(S_+)_D$, $\breve{S}_-\equiv(S_-)_D$, $\breve{S}_Z\equiv(S_Z)_D$

This Hamiltonian is clearly non-Hermitian when represented in a boson basis. Since the Hamiltonian matrix is, however, triangular, the exact eigenvalues can be read off as

$$E(N_S,v)=-GN_S(\Omega-N_S+1-v)\ . \tag{3.11}$$

Note that the seniority is identified as $v=\sum N_{2J}$. (In Refs. 7 and 13 only the number of d bosons are identified with the seniority. This is certainly artificial in the pairing model, where all states without s bosons are degenerate.)

It is, furthermore, clear that if we drop the last two terms in H_D the resulting Hamiltonian, now diagonal instead of triangular in the boson basis, will still have the same eigenvalues. For the present example it is therefore almost trivial that the finite boson Hamiltonian corresponding to the fermion one can be either Hermitian or non-Hermitian, if we are only interested in reproducing eigenvalues.

A correspondence between eigenstates in the fermion and boson spaces in the sense of Ref. 13, however, requires a non-Hermitian boson Hamiltonian, the point being that "simple correspondence"[13] would only result from nonorthogonal boson eigenstates which in turn could not be obtained from a Hermitian boson Hamiltonian.

This result is readily understood in terms of the DBM formalism since the step that establishes the "simple correspondence" is equivalent to representing the boson Hamiltonian in a boson basis, as opposed to representation in a physical basis; i.e., the basis obtained from a fermion basis by replacing each bifermion operator with its Dyson image. The latter representation would yield a Hermitian Hamiltonian matrix and an obvious way of identifying boson and fermion states, but the cumbersome process of constructing the physical basis would nullify any advantage of switching to the boson picture, as already discussed by Hahne.[10] In other words, one can use the pair boson basis to span the true physical states without going through the complication of constructing the image of the fermion basis. One should, however, bear in mind that the boson basis might introduce overcompleteness, as we shall shortly discuss.

C. Quadrupole-quadrupole interaction

The Dyson image of the quadrupole-quadrupole interaction

$$H_{QQ}=-K(U^{(2)}\cdot U^{(2)}) \tag{3.12}$$

can, of course, be written by using Eq. (3.6), giving

$$(H_{QQ})_{D1}=-4K\sum_{J_1J_2J_3J_4}(\hat{J}_1\hat{J}_2\hat{J}_3\hat{J}_4)^{1/2}\begin{Bmatrix}J_1 & J_2 & 2\\ j & j & j\end{Bmatrix}\begin{Bmatrix}J_3 & J_4 & 2\\ j & j & j\end{Bmatrix}[B^{J_1}\times\widetilde{B}_{J_2}]^2[B^{J_3}\times\widetilde{B}_{J_4}]^2\ . \tag{3.13}$$

Another possibility is to rewrite H_{QQ} from its particle-hole multipole form into a particle-particle interaction, namely

$$H_{QQ}=-K\left[\frac{5}{2\Omega}\hat{N}_F-10\sum_J\begin{Bmatrix}j & j & J\\ j & j & 2\end{Bmatrix}A^J\cdot\widetilde{A}_J\right] \tag{3.14}$$

and then forming the Dyson image from Eqs. (3.4) and (3.5). We now get

$$(H_{QQ})_{D2}=-10K\left[\left[\frac{1}{2\Omega}-\begin{Bmatrix}j & j & J\\ j & j & 2\end{Bmatrix}\right]B^{JM}B_{JM}+2\sum\begin{Bmatrix}j & j & J\\ j & j & 2\end{Bmatrix}\begin{Bmatrix}j & j & J\\ j & j & K_3\\ K_1 & K_2 & L\end{Bmatrix}[B^{K_1}\times B^{K_2}]^L\cdot[\widetilde{B}_J\times\widetilde{B}_{K_3}]^L\right]\ . \tag{3.15}$$

Since the Dyson mapping is nonunitary, the Hamiltonians in Eqs. (3.13) and (3.15) are not identical. $(H_{QQ})_{D1}$ in Eq. (3.13) can be expressed as

$$(H_{QQ})_{D1}=-10K\left[\left[\frac{1}{2\Omega}-\begin{Bmatrix}j & j & J\\ j & j & 2\end{Bmatrix}\right]B^{JM}B_{JM}\right.$$
$$\left.+2\sum(\hat{J}_1\hat{J}_2\hat{J}_3\hat{J}_4)^{1/2}\begin{Bmatrix}J_1 & J_2 & 2\\ j & j & j\end{Bmatrix}\begin{Bmatrix}J_3 & J_4 & 2\\ j & j & j\end{Bmatrix}\begin{Bmatrix}J_1 & J_2 & 2\\ J_4 & J_3 & K\end{Bmatrix}[B^{J_1}\times B^{J_3}]^K\cdot[\widetilde{B}_{J_2}\times\widetilde{B}_{J_4}]^K\right]\ , \tag{3.16}$$

which is manifestly Hermitian, unlike $(H_{QQ})_{D2}$, which is non-Hermitian. These two Hamiltonians are nevertheless equivalent and have the same spectrum. It is only when we consider a truncation that the non-Hermitian Hamiltonian seems to present a superior choice as discussed in Refs. 11 and 22, although all aspects of this problem have not been clarified in general cases.

For the present example the non-Hermitian choice has some advantages (see Tables II and III) even if we do not truncate, as we proceed to show. Consider the case $j=\frac{7}{2}$. From the seniority scheme we know that there is only one four-particle 0^+ state and if we construct all the possible physical states, namely

$$[(A^J)_D\times(A^J)_D]^{00}\,|\,0);\quad J=0,2,4,6\ ,$$

we find that indeed only one state is (trivially) linearly independent. However, we have already pointed out that this type of construction becomes very cumbersome, especially for large fermion number N, and that the boson basis simplifies matrix element calculations considerably. The only problem is that the boson basis can be overcomplete—in the example above, the boson basis consists of four states, namely $(S^\dagger)^2\,|\,0)$, $(D^\dagger\cdot D^\dagger)\,|\,0)$, $(G^\dagger\cdot G^\dagger)\,|\,0)$, and $(I^\dagger\cdot I^\dagger)\,|\,0)$, and a diagonalization in this basis will yield three spurious solutions. As far as the recognition of these spurious solutions is concerned, no general prescription is known to us (see also Ref. 11), but in the present example the use of $(H_{QQ})_{D2}$ facilitates this recognition, since all spurious solutions have an eigenvalue that is either zero or complex. On the other hand, the Hermitian choice $(H_{QQ})_{D1}$ yields spurious eigenvalues, which are difficult to recognize (see Table III).

TABLE II. Non-Hermitian Hamiltonian matrix of the quadrupole-quadrupole interaction $-U^2\cdot U^2$ obtained by calculating matrix elements $\langle (B^{J_1})^2 J=0\,|\,(H_{QQ})_{D2}\,|\,(B^{J_2})^2 J=0\rangle$ for $j=\frac{7}{2}$. This matrix has one nonzero eigenvalue at -0.833 with the corresponding eigenstate

$$|\,\psi)=0.5\,|\,S^2)-0.373\,|\,D^2)-0.5\,|\,G^2)-0.601\,|\,I^2)\ ,$$

where the coefficients are just the numerical values of the cfp's $\langle j^2(J)j^2(J)0\{\,|\,j^4 v=0 J=0\rangle$ for $j=\frac{7}{2}$ and $J=0,2,4,6$.

$J_1 \backslash J_2$	S^2	D^2	G^2	I^2
S^2	-1.8750	0.6522	-0.6250	-0.7512
D^2	1.3975	-0.4861	0.4658	0.5599
G^2	1.8750	-0.6522	0.6250	0.7512
I^2	2.2535	-0.7838	0.7512	0.9028

TABLE III. Hermitian Hamiltonian matrix of the quadrupole-quadrupole interaction $-U^2\cdot U^2$ obtained by calculating matrix elements $\langle (B^{J_1})^2 J=0\,|\,(H_{QQ})_{D1}\,|\,(B^{J_2})^2 J=0\rangle$ for $J=\frac{7}{2}$. Apart from the physical eigenvalue at -0.833, the above matrix has nonzero eigenvalues at 1.68, -2.69, and -4.67, showing that the use of an (overcomplete) boson basis for the Hermitian mapping of H_{QQ} introduces spurious states which cannot easily be identified.

$J_1 \backslash J_2$	S^2	D^2	G^2	I^2
S^2	-0.2500	-2.236	0	0
D^2	-2.236	-2.0377	-1.339	0
G^2	0	-1.339	0.8209	-0.5463
I^2	0	0	-0.5463	-0.3787

Before we again pick up the trail to the IBM which is formulated in terms of s and d bosons only, we would like to point out that this possible overcompleteness of a boson basis will emerge in any scheme attempting to arrive at an interacting boson model starting from microsopic considerations. It may well be present even in a phenomenological boson model. Incorporating g bosons might not generally be an improvement of the IBM, since this extension would for some cases introduce the spuriosities referred to above.

We now return to the problem of establishing a boson Hamiltonian which ideally would yield the exact shell model results for the low lying parts of the spectra. It seems that the DBM formalism addresses this problem on a more favorable level than, for example, the OAI method, in the sense that the boson states are determined by the dynamics of the boson Hamiltonian. In the OAI method, the construction of the Hamiltonian is determined by a specific choice of the basis states, which for the SJS is determined by the seniority scheme. Once a boson Hamiltonian is established it can be diagonalized in a boson basis and any physical quantity can then be calculated using the boson eigenstates.

A presupposed correspondence between boson and fermion states can, of course, assist in establishing a boson Hamiltonian (or other boson operator) as in the OAI method, but since we maintain that the microscopic structure of physical bosons (e.g., s and d in IBM) can vary throughout the Periodic Table, this approach seems to have limited applicability without *a priori* knowledge of this correspondence.

Let us illustrate these ideas schematically as in Fig. 1, where all the indicated operators are ideal boson operators. The fact that the phenomenological IBM is successful in many aspects suggests that the transformation $H_D \rightarrow H_{\rm IBM}$ exists generally such that the final truncation to s and d bosons is a good approximation. The ultimate aim would be to establish a similarity transformation to accomplish just this optimal decoupling of higher bosons. (See also the discussion by Klein and Vallieres.[14]) In view of the present lack of knowledge of such a transformation, the OAI method can be regarded as a "brute force" method of accomplishing the same aim. This method can be "translated" into the DBM framework as follows. We know that a truncation of the Dyson pair boson Hamiltonian to S and D bosons alone is a poor approximation for the SJS in view of the large mixing between boson states and the fact that this mixing persists even for states containing the maximum angular momentum boson. (See Appendix B.) Since, however, seniority is an almost good quantum number in the SJS, we can attempt to associate the physical s boson with the generalized seniority zero state and the d boson with a seniority two state, as in the OAI method.

As an example of introducing seniority bosons in the DBM we again turn to the pairing model of Sec. III B. We have already indicated that while the Dyson boson Hamiltonian (3.10) is non-Hermitian in a boson basis, it is Hermitian and diagonal in a physical basis $(S_+)_D^n\,|\,0)$. This observation is equivalent to the use of the simple truncated form

$$\sqrt{\Omega}(S_+)_{\rm sen}=S^+\left[\Omega-\hat{N}_S-2\sum_{J=1}^{\Omega}\hat{N}_{2J}\right] \qquad (3.17)$$

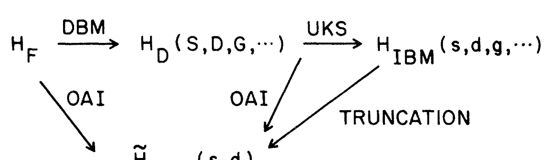

FIG. 1. Schematic illustration of the interrelation between different Hamiltonians. The method of Ref. 7 (OAI) constructs the boson Hamiltonian by equating matrix elements of the physical (seniority) boson space to those of the fermion space. The Dyson boson mapping (DBM) maps the fermion space onto the pair boson $(S,D,G,\ldots)$ space. The physical bosons $(s,d,g,\ldots)$ are then constructed from the dynamics (kinematics in the SJS case) of the Hamiltonian H_D. (For the SJS this amounts to obtaining $\hat{H}_{\rm IBM}$ from H_D by the OAI method.) The UKS mapping from H_D to $H_{\rm IBM}^{B}$ indicates an unknown similarity transformation.

instead of Eq. (3.7) to construct the boson Hamiltonian.

Another way of stating the equivalence is to note that the "truncated" operator (3.17), together with $(S_-)_D$ and $(S_z)_D$ in Eqs. (3.8) and (3.9), satisfies an SU(2) algebra, which is a sufficient requirement for obtaining the true eigenvalues. One should, however, realize that the $S^\dagger$ in Eq. (3.17) cannot be associated with a pair of nucleons as before—it is now rather a seniority boson $s^\dagger$.

When the interaction deviates from pure pairing, the transition from the pair boson Hamiltonian (obtained by the Dyson mapping) to the seniority boson Hamiltonian is less clearcut. Let us formulate the process by considering the "smallest" Lie algebra to which the bifermion operators appearing in the fermion Hamiltonian belong, assuming that this algebra contains an SU(2) subalgebra. The DBM establishes the boson images of the elements of the larger algebra in terms of pair bosons with angular momentum up to $2j-1$. In particular, the image of S_+ will contain a "seniority part," i.e., Eq. (3.17) and additional terms as shown in Eq. (3.7). In the pure pairing case we have discussed the fact that neglecting the seniority changing part of $(S_+)_D$ still produces a boson realization of the SU(2) algebra. Whenever SU(2) is a subalgebra of some larger algebra, this change from Dyson image to seniority image of S_+ will destroy the property of the remaining Dyson images to form a realization of the larger algebra. In order to retain this property these other Dyson images will have to be redefined and this will be determined by a similarity transformation θ such that

$$\theta^{-1}(S_+)_D\theta=(S_+)_{\rm sen}\ . \qquad (3.18)$$

[In Ref. 23 this similarity transformation was, in effect, constructed for an SO(8) fermion algebra.]

This brings us back to the OAI method,[7] which first establishes a correspondence between boson states and fermion states with definite seniority and then proceeds to determine a boson Hamiltonian which will retain matrix elements in the seniority scheme. The success of this approach is based on the fact that in the seniority scheme matrix elements between states of larger N can be related to those of smaller N through the number-dependent factors that appear in reduction formulas.[7] If we therefore fix the IBM Hamiltonian parameters for seniority 0, 2, and 4 states and include these number-dependent factors in $H_{\rm IBM}$, it will preserve matrix elements of interactions diagonal or near diagonal in the *seniority scheme* for at least up to seniority 8 states, as illustrated by the numerical results in

Ref. 7. We note that the number-dependent factors above appear in the DBM when transforming from pair bosons to seniority bosons. (See Ref. 23 and Sec. IV.)

Returning to the DBM we conclude that the OAI approach is equivalent to determining an IBM Hamiltonian through the requirement

$$(0|(S)_D^{N-N_d}(\hat{D})_D^{N_d}|H_D(S,D,\dots)|(S_+)_D^{N-N_d'}(\tilde{D}^+)_D^{N_d}|0)=(0|s^{N-N_d}d^{N_d}|\tilde{H}_{\mathrm{IBM}}(s,d)|(s^+)^{N-N_d'}(d^+)^{N_d'}|0),$$

where $(\hat{D})_D$ is the Dyson image of the $J=2$ projected seniority increasing operator (see Ref. 7). The reason behind this equivalence to the OAI method can be found in the fact that the physical states in the matrix elements on the lhs are equivalent to states with definite seniority (see Appendix B).

IV. COMPARISON WITH OTHER APPROACHES

Apart from the OAI approach, microscopic considerations that could possibly support the IBM have also been investigated in Refs. 15 and 16 and we briefly indicate here their relation to the DBM approach.

We first comment on the nuclear field theory (NFT) approach by Broglia *et al.*[16] where some empirical diagram rules "still lacking a 'first principle' derivation" are introduced. One of these empirical rules involves the *ad hoc* introduction of the N-dependent factors that typically arise in a transformation from pair type to seniority type bosons. It seems, therefore, that the first principle involved concerns this important distinction between bosons and the possibility of transforming from one type to the other. The necessity to introduce the empirical rules in the NFT approach is nothing but a manifestation of the fact that NFT is formulated in terms of pair bosons. Restricting the NFT calculation to pair bosons with $J=0$ and $J=2$ amounts to truncation in the DBM to S- and D-pair bosons, which we have already shown to be a bad approximation.

The approach by Suzuki *et al.*[15] is much more successful since their starting point is equivalent to taking the seniority image $(S_+)_{\mathrm{sen}}$ in Eq. (3.17) as the boson image for S_+. As we have already shown this would require a consistent similarity transformation on all the Dyson images of operators A^{JM} and A_{JM} with $J\neq 0$. There is, however, another way out, namely the one taken by Suzuki *et al.*[15] Instead of being required to introduce all the possible pair bosons in the Dyson images of A^{JM} and A_{JM} with $J\neq 0$, one could prefer an ideal space description in terms of only one boson, namely the seniority s boson, together with any number of ideal fermions.[18] In terms of a mapping procedure this would require a mapping of the closed algebra formed by the fermion operators S_+, S_-, S_z, a^{jm}, and a_{jm}.

The expressions given by Suzuki *et al.*[15] can be regarded as an accomplishment of this mapping within a Holstein-Primakoff framework (see also Ref. 24). Note that in Ref. 15 operators corresponding to the ideal fermions satisfy a modified fermion algebra. As shown in Ref. 11, however, the mapping can be constructed by adopting the usual fermion algebra for the ideal fermion operators and absorbing the effect of the modified algebra in the mapping. This is also the technical difference between the methods adopted in Refs. 18 and 24. The Dyson mapping for the single fermion operators is then obtained in terms of a seniority boson and ideal fermions

$$a^{jm}\to A^{jm}=\alpha^{jm}\hat{\rho}(\Omega-\hat{N})+\frac{1}{\sqrt{\Omega}}s^{\dagger}\tilde{c}_{jm}\,, \tag{4.1}$$

$$\tilde{a}_{jm}\to\tilde{A}_{jm}=\tilde{c}_{jm}+\sqrt{\Omega}s\alpha^{jm}\hat{\rho}\,, \tag{4.2}$$

$$\alpha^{jm}=c^{jm}-(c^j\cdot c^j)\hat{\rho}\,\tilde{c}_{jm}\,, \tag{4.3}$$

$$\hat{\rho}=\left[\Omega-\sum c^{jm}c_{jm}\right]^{-1}=(\Omega-\hat{n})^{-1}\,, \tag{4.4}$$

$$\hat{N}=\hat{N}_S+\hat{n}\,. \tag{4.5}$$

The ideal fermion operators c satisfy the usual fermion algebra and commute with the boson operators s and $s^{\dagger}$.

It is instructive that the mapping (4.1) and (4.2) already contains the typical N-dependent factors associated with a seniority mapping. By using this mapping together with the seniority mapping (3.16), (3.8), and (3.9) for S_+, S_-, and S_0, it is now possible to obtain an exact boson Hamiltonian in terms of seniority s bosons and ideal fermions where the seniority is now $\nu=\hat{n}$. In order to obtain an IBM-like Hamiltonian the ideal fermions have to be eliminated in favor of seniority d bosons and this can be accomplished by again equating matrix elements in the manner of OAI, as shown in Ref. 15. The ideas expressed above can be schematically summarized as in Fig. 2.

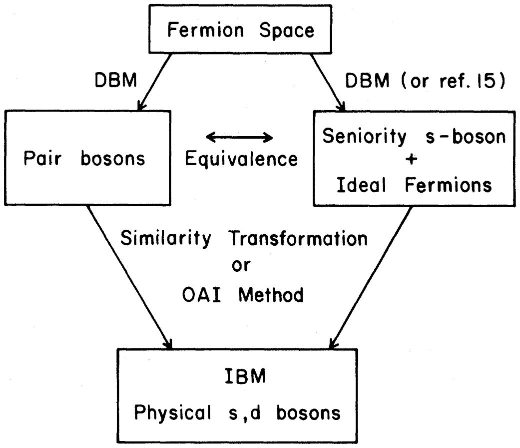

FIG. 2. Schematic illustration of different possible mapping routes from fermion space to the IBM.

V. CONCLUSIONS

Returning to the schematic display in Fig. 1, the following conclusions can now be drawn. (i) By introducing the intermediate step of the DBM before applying the OAI method to obtain H_{IBM}, it is very clear that the physical s and d bosons are not associated with fermion pairs of angular momentum 0 and 2 (as would be the case if we could just truncate H_D), but rather with seniority 0 and 2 pairs.[7,13] (ii) This does not rule out the possibility that in the multishell case (where seniority becomes less important) the Tamn-Dancoff approximation (TDA) [or random-phase approximation (RPA)] pair bosons might already constitute the physical bosons. Only in such a situation could the IBM be considered as making fundamental progress in the sense that it gives a unified, simple picture of low-lying nuclear spectra by relying on the particulars of dynamical nucleon correlations. It would be difficult to apply the kinematically oriented OAI method[7] in such a situation, while the DBM will remain a useful microscopic tool in the derivation of the boson Hamiltonian. (See Appendix A and Ref. 25, where only the lowest TDA bosons were used.) (iii) Whatever may be the case, the DBM presents an elegant way of introducing shell model bosons (it can even be useful in normal shell model type calculations where no truncation is desired—viz., the calculation of cfp's in Appendix B). (iv) Ultimately there might be different microscopic considerations supporting an IBM in different nuclear mass regions. In other words, the step that transforms

$$H_D(S,D,G,\dots)\rightarrow H_{\mathrm{IBM}}(s,d)$$

might be possible for different reasons under different circumstances. In the SJS investigated here, the step above was guaranteed by the seniority scheme, its reduction formulas, and the fact that interactions such as the pairing, quadrupole-quadrupole, or delta interactions are (almost) diagonal in this scheme.

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APPENDIX A: MULTISHELL DYSON MAPPING

Let us consider the effective Hamiltonian

$$H_F=\sum_{jm}\epsilon_j a^{jm}a_{jm}+\tfrac{1}{2}\sum V_J(j_1j_2j_3j_4)([a^{j_1}\times a^{j_2}]^J[\tilde{a}_{j_4}\times\tilde{a}_{j_3}]^J) \tag{A1}$$

of the multishell fermion system. The corresponding Dyson image Hamiltonian in the boson space is

$$\begin{aligned}H_B=&\sum_{\substack{j_1j_2\\JM}}\epsilon_{j_1}B^{(j_1j_2)JM}B_{(j_1j_2)JM}+\tfrac{1}{2}\sum V_J(j_1j_2j_3j_4)B^{(j_1j_2)J}\cdot\tilde{B}_{(j_3j_4)J}\\&+\tfrac{1}{2}\sum V_J(j_1j_2j_3j_4)\sum(-)^{J_1-J_2+J'}(\hat{J}_1\hat{J}_2\hat{J}_3)^{1/2}\hat{J}\begin{Bmatrix}j_1 & j_1' & J_1\\ j_2 & j_2' & J_2\\ J & J_3 & J'\end{Bmatrix}\\&\qquad\times([B^{(j_1j_1')J_1}\times B^{(j_2j_2')J_2}]^{J'}\cdot[\tilde{B}_{(j_1'j_2')J_3}\times\tilde{B}_{(j_3j_4)J}]^{J'})\ .\end{aligned} \tag{A2}$$

The dynamical bosons are, however, determined by the dynamics of the effective interaction. If the interaction Hamiltonian (A1 or A2) possess a few degrees of collectivity, e.g., low-lying phonon states, one can transform the Hamiltonian (A2) into the collective coordinates which are given by Eqs. (2.5)–(2.7). Substituting Eq. (2.6) into Eq. (A2), one obtains the collective boson Hamiltonian in terms of the collective boson operator. If only the dynamical correlation between nucleons is important, the collective boson Hamiltonian can then be truncated to the collective bosons.

APPENDIX B: SENIORITY BOSON WAVE FUNCTIONS

The single j shell (SJS) Dyson mapping is given by Eqs. (3.4)–(3.6). In this appendix, we construct the seniority image boson wave function. The two fermion states with angular momentum JM, namely $|j^2J,M\rangle$, has the boson image $R^{JM}|0)=B^{JM}|0)$. B^{JM} is a shorthand notation for $B^{(jj)JM}$, the nucleon pair boson image. The four nucleon state with seniority quantum number v is given by

$$|j^4,vJM\rangle=\sum |j^2(J_1)j^2(J_2)JM\rangle[j^2(J_1)j^2(J_2)J\,|\}j^4vJ]\,, \tag{B1}$$

where the [| }] is the cfp. The corresponding image in the boson space is given by

$$(R^K\times R^L)^{JM}|0)=(B^K\times B^L)^{JM}|0)-2(-)^J\sum(\hat{J}_1\hat{J}_2\hat{K}\hat{L})^{1/2}\begin{Bmatrix} j & j & J_1\\ j & j & J_2\\ K & L & J\end{Bmatrix}(B^{J_1}\times B^{J_2})^{JM}|0)\,. \tag{B2}$$

The cfp of Eq. (B1) can be read off directly from Eq. (B2) (see Ref. 26). Similarly the boson state corresponding to a six fermion seniority state is given by

$$(R^{I'}\times(R^K\times R^L)^{J'})^{JM}|0)\,.$$

Since these operators are boson operators, the manipulation is somewhat easier. As a specific example, we consider the seniority zero state

$$\breve{S}_+\times(\breve{S}_+\times\breve{S}_+)|0)=\frac{2j-3}{2j+1}\left[\frac{6(2j-1)}{2j+1}\right]^{1/2}\left\{S^+[\breve{S}_+\breve{S}_+|0)]_N-\frac{4}{2(2j-1)(2j-3)}\times\sum_{J_3\neq 0}\sqrt{\hat{J}_3}\left[B^{J_3}\times[\breve{S}_+R^{J_3}|0)]_N\right]^0\right\}, \tag{B3}$$

where $[\breve{S}_+\breve{S}_+|0)]_N$ and $[\breve{S}_+R^{J_3}|0)]_N$ are the normalized seniority 0 and 2 state, respectively. Thus the coefficient in Eq. (B3) is just the cfp of the six fermion system.

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