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Comparison of the mass quadrupole and hydrodynamic collective models

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Comparison is made between the mass quadrupole coordinate collective model and an appropriately extended version of the hydrodynamic collective model to ascertain whether or not the former is an exact microscopic formulation of the latter. We find that, while there are strong similarities, it is not.

[NUCLEAR STRUCTURE Bohr-Mottelson model, hydrodynamic model, mass quadrupole collective model; comparison, theory.]

I. INTRODUCTION

By any accounting, the Bohr-Mottelson (Refs. 1–3) (BM) collective model together with its extensions and generalizations has had spectacular success in describing the collective motion of a large class of nuclei. It is therefore not surprising that considerable effort has been expended in attempting to derive the BM model, or something close to it, from a microscopic starting point. Inevitably, such derived models must be compared with the BM model to establish contact between the microscopic theory and the phenomenology. One of the more recent attempts is the mass quadrupole collective model (MQC) of Rosensteel, Ihrig, and Rowe.[4,5] In the first of these papers the authors refer to the MQC model as yielding corrections to the BM model and in the second they refer to it as a microscopic formulation of the BM model.

The coordinates used in the two models are not precisely the same. The BM model coordinates are those of the quadrupole expansion of the nuclear surface, whereas the MQC model coordinates are those of the mass quadrupole tensor including its (monopole) trace. Therefore, in their usual formulations, the MQC model has an additional degree of freedom not possessed by the BM model. The BM model can be extended to include an analogous sixth degree of freedom which is usually restricted by a constant volume condition. A slight variation of the BM model in which a constant mean squared radius is imposed in place of constant volume affords the most direct comparision with the MQC model, since this new constraint then fixes the MQC monopole coordinate at a constant value. Furthermore, in this special case, it is very easy to derive a relationship between the five remaining coordinates of each model.

To compare the two models term by term it is necessary to express both in the same set of coordinates. Any differences between the models are, of course, independent of the choice of coordinates used for this comparison. Furthermore, one can compare the two models in any convenient special case or limit. Should they prove to be not identical in the special case they are clearly not identical in general. We have chosen the case of constant mean squared radius for simplicity of comparison.

The MQC model is philosophically appealing because of its microscopic connection and algebraic nature. Much is made of the fact that the MQC model is not restricted to small amplitude vibrations as is the case with the usual formulation of the BM model. The BM model kinetic energy can, however, be fairly easily extended to allow for larger amplitude oscillations. If the MQC model were an exact formulation of the BM model for arbitrary size oscillations, then an expansion of the MQC model kinetic energy in terms of BM model coordinates should agree with the BM model kinetic energy in all orders. It is the purpose of this paper to make that comparison. We chose to do so by formulating both as classical models and we find that they are identical only through terms of order three in the BM coordinates. In the fourth order terms the models are quite different.

In Sec. II we sketch the formulation of the BM model to higher order in the BM coordinates and their time derivatives. We also introduce the constant mean squared radius condition which is somewhat different from the constant volume constraint normally imposed. We show, however, that al-

though neither condition implies the other, through fourth order they yield the same kinetic energy expression. In Sec. III we derive the kinetic energy for the classical MQC model and, for completeness, present the quantum mechanical version as well. Our quantized expression differs from that of Refs. 4 and 5 only in that we explicitly remove the center of mass in our definition of the mass quadrupole tensor. In Sec. IV we establish the connection between the coordinates of the MQC and BM models in order to write the MQC model kinetic energy in terms of BM coordinates. We then compare the kinetic energy expressions of the two models and present our conclusions.

II. KINETIC ENERGY IN THE BOHR-MOTTELSON MODEL

A. Coordinates and model conditions

We begin by reviewing the six-coordinate version of the BM model. Throughout, we shall use the conventions of Rose[6] for Euler angles, rotation matrices, spherical harmonics, Clebsch-Gordan coefficients, spherical tensors, etc. In a coordinate frame fixed in the laboratory (lab) and centered at the nuclear center of mass (c.m.), the radius to the nuclear surface at polar angles θ and ϕ is written as

$$r=R(\theta,\phi) =R_0\left[1+\alpha_{00}Y_{00}^*+\sum_\mu \alpha_\mu Y_{2\mu}^*(\theta,\phi)\right] . \tag{1}$$

This defines six laboratory expansion coefficients α_{00}, α_μ which satisfy

$$\alpha_{00}^*=\alpha_{00};\quad \alpha_\mu^*=(-1)^\mu\alpha_{-\mu} , \tag{2}$$

because of the reality of R. These expansion coefficients become the generalized coordinates once certain further assumptions are made.

In the BM model, the nucleus is assumed at each instant to fill a surface specified by Eq. (1). In addition, one assumes (i) constant total mass, M; (ii) uniform mass density $\rho_0=m/V$ throughout the nuclear volume V; and (iii) irrotational flow of the nuclear fluid. One can pass to a five-dimensional version of the BM model by imposing one additional constraint such as (iv) constant volume

$$V=\tfrac{4}{3}\pi R_0{}^3 , \tag{3a}$$

or (v) constant mean squared radius

$$\frac{1}{V}\int\int\int r^2dV=\tfrac{3}{5}R_0{}^2 . \tag{3b}$$

Either of the conditions (3a) or (3b) may be used to eliminate α_{00} in favor of the α_μ. The constant volume (CV) condition has normally been the one chosen, but the constant mean squared radius (CMSR) condition permits a more ready comparison with the MQC model. It should be noted that for CMSR, both ρ_0 and V will be time dependent whereas for CV, both are constants. The model assumptions, together with a boundary condition connecting the fluid flow at the surface to the motion of the surface envelope, are sufficient to completely specify the kinetic energy, $T_{\rm BM}$, as a function of the α_μ and the $\dot\alpha_\mu=d\alpha_\mu/dt$ whether one chooses to impose the CV or the CMSR condition. We shall consider both versions of the BM model and indeed we shall show that through fourth order the kinetic energy is the same.

We first consider the elimination of α_{00}. The nuclear volume V, given by

$$V=\int\int\int r^2dr\,d\Omega=\tfrac{1}{3}\int\int R^3(\theta,\phi)d\Omega ;\quad d\Omega=\sin\theta\,d\theta\,d\phi , \tag{4}$$

can be straightforwardly evaluated as a function of (α_{00},α_μ) by using Eq. (1) and properties of the Y_{LM}. We find

$$V=\tfrac{4}{3}\pi R_0{}^3\left[1+\frac{3\alpha_{00}}{\sqrt{4\pi}}+\frac{3\alpha_{00}{}^2}{4\pi}+\frac{3\sqrt{5}}{4\pi}[\alpha\alpha]_0 +\left[\frac{\alpha_{00}}{\sqrt{4\pi}}\right]^3+\frac{3}{4\pi}\left[\frac{5}{4\pi}\right]^{1/2}\alpha_{00}[\alpha\alpha]_0 -\frac{5}{4\pi\sqrt{14\pi}}[[\alpha\alpha]_2\alpha]_0\right] . \tag{5}$$

The α_μ are properly labeled components of a second rank spherical tensor, and we have used the notation

$$[AB]_{JM}=\sum_{M_AM_B} C(J_AJ_BJ;M_AM_BM)A_{J_AM_A}B_{J_BM_B} \tag{6}$$

to specify higher rank tensors constructed from tensors A and B. Thus

$$[\alpha\alpha]_{J\nu}=\sum_\mu C(22J;\mu,\nu-\mu,\nu)\alpha_\mu\alpha_{\nu-\mu} \tag{7a}$$

and

$$[[\alpha\alpha]_{J'}\alpha]_{J\nu}=\sum_{\mu_1\mu_2} C(J'2J;\mu_1\mu_2\nu) \times[\alpha\alpha]_{J'\mu_1}\alpha_{\mu_2} . \tag{7b}$$

Note that Eq. (5) is exact and for the CV condition, Eq. (3a), becomes a cubic equation in α_{00}. This could be solved exactly to find α_{00} as a function of the α_μ. Such a solution would not be in convenient form for our purposes. It is therefore somewhat simpler and very much more useful to note that the expansion of α_{00} in terms of the α_μ requires that all the terms of various orders in the α_μ must be zeroth rank tensors. We shall call a term of order k if it contains k factors of α_μ and/or $\dot{\alpha}_\mu$, and we shall use $\mathcal{O}(k)$ to indicate terms of order k and higher. For example, $[\alpha\alpha]_{JM}$, $[\dot{\alpha}\dot{\alpha}]_{JM}$, and $[\alpha\dot{\alpha}]_{JM}$ are all of order 2. We may then assume a form for α_{00} and substitute this into Eq. (5). By equating the coefficients of the independent tensors we find

$$\alpha_{00}=-\left[\frac{5}{4\pi}\right]^{1/2}[\alpha\alpha]_0+\frac{5}{12\pi}\left[\frac{2}{7}\right]^{1/2}[[\alpha\alpha]_2\alpha]_0+\frac{5}{24\pi^2}\left[\frac{5}{14}\right]^{1/2}[\alpha\alpha]_0[[\alpha\alpha]_2\alpha]_0+\mathcal{O}(6)\ \text{CV} . \tag{8}$$

For the CMSR condition, one proceeds similarly by first evaluating

$$\int\int\int r^2dV=\tfrac{1}{5}\int\int R^5(\theta,\phi)d\Omega . \tag{9a}$$

We find

$$\begin{aligned}\int\int\int r^2dV=\frac{R_0{}^5}{5}\Bigg\{&4\pi+5\sqrt{4\pi}\alpha_{00}+\frac{10}{\sqrt{4\pi}}\alpha_{00}{}^2+10\alpha_{00}{}^3+\frac{5}{4\pi}\alpha_{00}{}^4+(4\pi)^{-3/2}\alpha_{00}{}^5\\&+[\alpha\alpha]_0\left[10\sqrt{5}+30\left[\frac{5}{4\pi}\right]^{1/2}\alpha_{00}+\frac{30\sqrt{5}}{4\pi}\alpha_{00}{}^2+10\sqrt{5}(4\pi)^{-3/2}\alpha_{00}{}^3\right]\\&-[[\alpha\alpha]_2\alpha]_0\left[25\left[\frac{2}{7\pi}\right]^{1/2}+\frac{25}{\pi}\left[\frac{2}{7}\right]^{1/2}\alpha_{00}+\frac{25}{4\pi}\left[\frac{2}{7\pi}\right]^{1/2}\alpha_{00}{}^2\right]\Bigg\}+\mathcal{O}(4) .\end{aligned} \tag{9b}$$

Equations (5) and (9b) are then used in the CMSR condition, Eq. (3b). We find

$$\alpha_{00}=-\frac{7}{2}\left[\frac{5}{4\pi}\right]^{1/2}[\alpha\alpha]_0+\frac{45}{4\pi}\left[\frac{1}{14}\right]^{1/2}[[\alpha\alpha]_2\alpha]_0+\mathcal{O}(4)\ \text{CMSR} . \tag{10}$$

B. Velocity and kinetic energy

Let $\vec{v}(\vec{r})$ denote the velocity of a fluid element at position $\vec{r}=(r,\theta,\phi)$ in the laboratory frame. The usual assumption is that $\vec{v}$ is irrotational. Thus $\vec{v}$ satisfies

$$\vec{\nabla}\times\vec{v}=0 , \tag{11a}$$

as well as the equation of continuity

$$\vec{\nabla}\cdot(\rho_0\vec{v})=-\frac{\partial}{\partial t}\rho_0 . \tag{11b}$$

Equations (11) allow one to write

$$\vec{v}=-\vec{\nabla}\chi , \tag{12a}$$

where $\chi(r,\theta,\phi)$ satisfies Poisson's equation

$$\nabla^2\chi\equiv q=\begin{cases}0 & \text{CV}\\ \dot{\rho}_0/\rho_0 & \text{CMSR}\end{cases} . \tag{12b}$$

One should note that q is independent of position within the droplet.

The general solution to Eq. (12b) which is regular at the origin is

$$\chi(r,\theta,\phi)=\frac{qr^2}{6}+\sum_{lm}A_{lm}r^lY^*_{lm}(\theta,\phi) , \tag{13a}$$

where

$$A^*_{lm}=(-1)^mA_{l,-m} . \tag{13b}$$

The A_{lm} are determined by the physical boundary condition that the nuclear fluid does not escape the surface given by Eq. (1). That is, the normal component of $\vec{v}$ evaluated at the surface must equal the normal component of the radial velocity of the surface envelope. Thus, if $\vec{N}$, $\vec{e}_r$, and $\vec{v}_s$ denote an outward normal vector to the surface, the unit radial vector, and the fluid velocity at the surface point $R(\theta,\phi)$, respectively, then

$$\vec{v}_s\cdot\vec{N}=\left[\frac{d}{dt}R(\theta,\phi)\right]\vec{e}_r\cdot\vec{N}\ . \tag{14}$$

To form $\vec{N}$ we introduce

$$g(r,\theta,\phi)\equiv r-R(\theta,\phi)$$

and its gradient

$$\vec{\nabla}g=\left[\vec{e}_r\frac{\partial}{\partial r}-\frac{i}{r}\vec{e}_r\times\vec{L}\right]g\ , \tag{15}$$

in which $\vec{L}=-i\vec{r}\times\vec{\nabla}$ is a purely angular operator. Since g is constant on the nuclear surface, its gradient at $r=R(\theta,\phi)$ is normal to the surface. A more convenient normal is

$$\vec{N}=\frac{R(\theta,\phi)}{R_0}(\vec{\nabla}g)_s=\frac{R}{R_0}[\vec{e}_r-\vec{\nabla}R(\theta,\phi)]=\frac{R}{R_0}\left[\vec{e}_r+i\frac{R_0}{R}\left[\vec{e}_r\times\vec{L}\sum_\mu\alpha_\mu Y^*_{2\mu}\right]\right]\ . \tag{16}$$

We use f_s to denote the value of a function $f(r,\theta,\phi)$ at the surface point (R,θ,ϕ). From the form of the gradient operation in Eq. (15), and from Eqs. (12a) and (13a),

$$\vec{v}_s=-\frac{q}{3}R\vec{e}_r-\sum_{lm}A_{lm}lR^{l-1}\vec{e}_rY^*_{lm}+i\sum_{lm}A_{lm}R^{l-1}\vec{e}_r\times\vec{L}Y^*_{lm}\ . \tag{17}$$

We then substitute Eqs. (16) and (17) into the boundary condition, Eq. (14), and use elementary properties of the angular momentum operator $\vec{L}$ and the spherical harmonics to find

$$-\frac{R}{R_0}\left\{\frac{q}{3}R+\sum_{lm}A_{lm}lR^{l-1}Y^*_{lm}+5R_0\sum_{lJM}R^{l-2}(2l+1)\left[\frac{6l(2l+1)}{4\pi(2J+1)}\right]^{1/2}\right.$$
$$\left.\times C(2lJ;000)W(ll22;1J)[\alpha A_l]_{JM}Y^*_{JM}\right\}=\dot{R}\frac{R}{R_0}\ . \tag{18}$$

We want to use Eq. (18) to find the A_{lm} to order 3 which will be sufficient to determine the kinetic energy to order 4. From the form of Eq. (18) and the fact that $C(2lJ;000)=0$ unless $l+J$ is even, the A_{lm} have the following forms:

(i) $A_{lm}=0$, l odd ;

(ii) $A_{lm}=a\dot{\alpha}_m\delta_{l,2}+b^l[\alpha\dot{\alpha}]_{lm}+\sum_J C^l_J[[\alpha\alpha]_J\dot{\alpha}]_{lm}+\mathcal{O}(4)$, l even ; (19)

where a, b^2, b^4, and the C^l_J are constants to be determined. To evaluate q for the CMSR condition, we use $q=\dot{\rho}_0/\rho_0=-\dot{V}/V$ which follows from $M=\rho_0V$. We also use Eq. (5) for V and Eq. (10) for α_{00}. We then have

$$q=\begin{cases}0 & \text{CV}\\ \dfrac{15\sqrt{5}}{4\pi}[\alpha\dot{\alpha}]_0-\dfrac{375}{8\pi}\left[\dfrac{1}{14\pi}\right]^{1/2}[[\alpha\alpha]_2\dot{\alpha}]_0+\mathcal{O}(4) & \text{CMSR}\end{cases}\ . \tag{20}$$

To determine the A_{lm}, we substitute Eq. (19) into Eq. (18) and use Eq. (1) for R. We also use Eq. (20) for q and either Eq. (8) or Eq. (10) for α_{00}. By equating coefficients of the Y^*_{lm} order by order, we find

$$A_{lm}=-\tfrac{1}{2}\dot{\alpha}_m\delta_{l,2}+\frac{1}{4}\left[\frac{5}{14\pi}\right]^{1/2}[\alpha\dot{\alpha}]_{2m}\delta_{l,2}+\frac{3}{4R_0{}^2}\left[\frac{5}{14\pi}\right]^{1/2}[\alpha\dot{\alpha}]_{4m}\delta_{l,4}+\sum_J C^l_J[[\alpha\alpha]_J\dot{\alpha}]_{lm}+\mathcal{O}(4)\ , \tag{21}$$

where

$$
\begin{aligned}
&C_0^2=-\frac{129}{560\pi}\sqrt{5} \quad C_2^2=-\frac{201}{1568\pi} \quad C_4^2=-\frac{39\sqrt{5}}{3920\pi} && \text{CV}\\
&C_0^2=-\frac{187\sqrt{5}}{280\pi} \quad C_2^2=-\frac{1181}{1568\pi} \quad C_4^2=\frac{-1509\sqrt{5}}{3920\pi} && \text{CMSR}\\
&C_0^4=0 \qquad C_2^4=\frac{615}{1568\pi R_0^2} \quad C_4^4=\frac{75}{1568\pi R_0^2}\left[\frac{5}{22}\right]^{1/2} && \left.\begin{array}{l}\\ \\ \text{either CV or CMSR}\end{array}\right\}\\
&C_0^6=0 \qquad C_2^6=0 \qquad C_4^6=-\frac{75}{4\pi R_0^4}\left[\frac{5}{2002}\right]^{1/2} &&
\end{aligned}
\tag{22}
$$

We are now in a position to evaluate the BM model kinetic energy through order 4. The starting point is the usual

$$
\begin{aligned}
T_{\rm BM}&=\tfrac{1}{2}\int\int\int \rho_0 v^2 dV\\
&=\frac{\rho_0}{2}\int\int\int \vec{\nabla}\chi\cdot\vec{\nabla}\chi\, dV\\
&=\frac{\rho_0}{2}\int\int \vec{ds}\cdot(\chi\vec{\nabla}\chi)_s-\frac{\rho_0}{2}\int\int\int \chi q\, dV\,,
\end{aligned}
$$

where the last step follows from the vector identity

$$
\begin{aligned}
\vec{\nabla}\cdot[\chi\vec{\nabla}\chi]&=\vec{\nabla}\chi\cdot\vec{\nabla}\chi+\chi\nabla^2\chi\\
&=\vec{\nabla}\chi\cdot\vec{\nabla}\chi+\chi q
\end{aligned}
$$

upon use of Poisson's equation. In the first term, $\vec{ds}=ds\vec{e}_N$ is the outward normal surface area element which may be written

$$
\vec{ds}=\frac{R^2 d\Omega}{\vec{e}_r\cdot\vec{e}_N}\vec{e}_N=R^2 d\Omega\,\vec{\nabla}g\,|_s=RR_0 d\Omega\,\vec{N}\,.
$$

Since $(\vec{\nabla}\chi)_s=-\vec{v}_s$ and $\vec{v}_s\cdot\vec{N}=\dot{R}R/R_0$, we have

$$
\vec{ds}\cdot(\chi\vec{\nabla}\chi)_s=-\chi_s R^2\dot{R}\, d\Omega\,.
$$

Also $q=\dot{\rho}_0/\rho_0$ is position independent, so for either CV or CMSR,

$$
T_{\rm BM}=-\frac{\rho_0}{2}\int\int d\Omega R^2\dot{R}\chi_s-\frac{\dot{\rho}_0}{2}\int\int\int \chi\, dV\,.
\tag{23}
$$

To evaluate $T_{\rm BM}$, we use $\rho_0=M/V$ and use Eq. (5) for V. We use Eq. (13a) for χ with the A_{lm} given by Eqs. (21) and (22). In Eq. (1) and in the corresponding expression for $\dot{R}$, one must use Eq. (8) for α_{00} for the CV condition or Eq. (10) for the CMSR condition. We then express $T_{\rm BM}/B'$ where

$$
B'=\frac{3MR_0^2}{8\pi\sqrt{5}}
\tag{24}
$$

is the mass parameter which appears later in the MQC model. The result is, remarkably, that for *either* the CV or CMSR condition,

$$
\begin{aligned}
\frac{T_{\rm BM}}{B'}=&\frac{5}{2}[\dot{\alpha}\dot{\alpha}]_0-\frac{45}{4}\left[\frac{5}{14\pi}\right]^{1/2}[[\dot{\alpha}\dot{\alpha}]_2\alpha]_0\\
&-\frac{127\sqrt{5}}{112\pi}[[\dot{\alpha}\dot{\alpha}]_0[\alpha\alpha]_0]_0\\
&-\frac{5515}{1568\pi}[[\dot{\alpha}\dot{\alpha}]_2[\alpha\alpha]_2]_0\\
&+\frac{393\sqrt{5}}{784\pi}[[\dot{\alpha}\dot{\alpha}]_4[\alpha\alpha]_4]_0+\mathcal{O}(5)\,.
\end{aligned}
\tag{25}
$$

Note that CV *does not* imply CMSR nor vice versa, but through 4th order they yield exactly the same kinetic energy. Presumably, the expressions for the kinetic energy must differ in higher order terms. We now turn to the MQC model.

III. KINETIC ENERGY IN THE MQC MODEL

A. Coordinates

In this section we shall derive a classical version of the MQC model. Our derivation is similar in spirit to several which have appeared previously,[7] but is closest in form to that of Dzyublik *et al.*[8] Unlike the BM model, the MQC model has a microscopic origin. Its six dimensionless collective coordinates $(\bar{\alpha}_\mu,\bar{p})$ are defined in terms of individual nucleon coordinates. Our starting point is the classical many-body kinetic energy

$$
T=\frac{m}{2}\sum_{n=1}^{A}\sum_{i=1}^{3}\dot{x}_{ni}^{\,2}
\tag{26a}
$$

in which x_{ni} is the ith Cartesian coordinate of the nth nucleon relative to some arbitrary laboratory frame origin; all nucleons are assumed to have the same mass m. Alternatively, we may write T in

terms of momenta p_{ni} conjugate to the x_{ni} as

$$T=\frac{1}{2m}\sum_{ni}p_{ni}^{2}\,. \tag{26b}$$

We are going to consider two successive canonical transformations of Eq. (26b). In the first of these we define

$$Q_l'\equiv\begin{cases}X_{\text{c.m.},i}=\dfrac{1}{A}\displaystyle\sum_n x_{ni}; & l=1,2,3\\ q_{ij}=\displaystyle\sum_n (x_{ni}-X_{\text{c.m.},i})(x_{nj}-X_{\text{c.m.},j}) \quad i\geq j; & l=4,5,\ldots,9\\ \omega_l=\text{unspecified functions of the } x_{ni}; & l=10,11,\ldots,3A\,,\end{cases} \tag{27a}$$

and the corresponding conjugate momenta

$$P_l'\equiv\begin{cases}P_{\text{c.m.},i}; & l=1,2,3\\ P_{ij}, i\geq j; & l=4,5,\ldots,9\\ P_{\omega,l}; & l=10,11,\ldots,3A\,.\end{cases} \tag{27b}$$

In Eq. (27a) the three $X_{\text{c.m.},i}$ are the Cartesian coordinates of the center of mass relative to the chosen laboratory frame. The six q_{ij} are the Cartesian components of the mass quadrupole moment relative to a set of axes parallel to the laboratory frame but centered at the center of mass. The remaining $3A$-9 coordinates need not be specified, since all terms in T involving the momenta $P_{\omega,l}$ are eventually discarded. Our definitions are parallel to those of Ref. 4, except that we have specifically removed the center of mass from the mass quadrupole moment.

Appropriate linear combinations of the q_{ij} may be found which transform under rotations like a monopole ($J=0$) and a quadrupole ($J=2$) tensor. These may be separately scaled as one likes. This introduces our second set of coordinates which are the MQC model coordinates $(\bar{\rho})^2$ and $\bar{\alpha}_\mu$, $\mu=\pm2,\pm1,0$. They are defined by

$$\bar{\rho}^2=\left[\frac{5}{3MR_0^{\,2}}\right]\sum_{n=1}^{A} mr_n^{\,2} \tag{28a}$$

and

$$\bar{\alpha}_\mu=\left[\frac{4\pi}{3MR_0^{\,2}}\right]\sum_{n=1}^{A} mr_n^{\,2}Y_{2\mu}(\theta_n,\phi_n) \tag{28b}$$

in which (r_n,θ_n,ϕ_n) are the spherical polar coordinates of the nth nucleon in the center-of-mass coordinate frame, and $M=Am$ is the total mass of the system. R_0 is a free scaling parameter with the units of length which we take equal to R_0 of Eq. (1). This choice is not mandatory but simplifies the comparison of the BM and MQC models. As we shall see later, the $\bar{\alpha}_\mu$ are related to, but not identically equal to, the α_μ of Eq. (1). $\bar{\rho}^2$ is a dimensionless measure of the nuclear mean-squared radius; for a uniform, spherical distribution of radius R_0, $\bar{\rho}^2=1$. The second canonical transformation which we shall effect will replace $[q_{ij};P_{ij}]$ by $[\bar{\alpha}_\mu,x;\bar{\pi}_\mu,P_x]$, where $\bar{\pi}_\mu$ is the momentum conjugate to $\bar{\alpha}_\mu$, and P_x is similarly conjugate to

$$x=\frac{\sqrt{5\pi}}{15}\bar{\rho}^2\,. \tag{29a}$$

It is convenient to use $[x,P_x]$ rather than $[\bar{\rho},P_{\bar{\rho}}]$ so that the second transformation is linear. The momentum conjugate to $\bar{\rho},P_{\bar{\rho}}$, will be seen to be related to P_x by

$$P_x=\frac{15}{2\sqrt{5\pi}\bar{\rho}}P_{\bar{\rho}}\,. \tag{29b}$$

We shall write final results for the MQC kinetic energy in terms of $[\bar{\alpha}_\mu,\bar{\rho};\bar{\pi}_\mu,P_{\bar{\rho}}]$.

The $\bar{\alpha}_\mu$ and $\bar{\pi}_\mu^*$ are properly defined second rank spherical tensor components satisfying $\bar{\alpha}_\mu^*=(-1)^\mu\bar{\alpha}_{-\mu}$ and $\bar{\pi}_\mu^*=(-1)^\mu\bar{\pi}_{-\mu}$. From Eqs. (27) and (28) we may display the second canonical transformation in matrix form:

$$\begin{bmatrix}\bar{\alpha}_2\\ \bar{\alpha}_1\\ \bar{\alpha}_0\\ \bar{\alpha}_{-1}\\ \bar{\alpha}_{-2}\\ x\end{bmatrix}=\frac{\sqrt{5\pi}}{3AR_0^{\,2}}\begin{bmatrix}0 & 0 & 0 & -\sqrt{3/2} & i\sqrt{6} & \sqrt{3/2}\\ 0 & -i\sqrt{6} & -\sqrt{6} & 0 & 0 & 0\\ 2 & 0 & 0 & -1 & 0 & -1\\ 0 & -i\sqrt{6} & \sqrt{6} & 0 & 0 & 0\\ 0 & 0 & 0 & -\sqrt{3/2} & -i\sqrt{6} & \sqrt{3/2}\\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}q_{33}\\ q_{32}\\ q_{31}\\ q_{22}\\ q_{21}\\ q_{11}\end{bmatrix}\,. \tag{30}$$

B. Canonical transformations and model conditions

If we denote an old set of coordinates and momenta by $[q_l,p_l]$ and transform to a new set $[Q_l,P_l]$, where the transformation is generated by

$$F=\sum_l f_l(\vec{q})P_l \ ,$$

then

$$Q_l=\frac{\partial F}{\partial P_l}=f_l(\vec{q}) \ , \tag{31a}$$

$$p_l=\frac{\partial F}{\partial q_l}=\sum_k \frac{\partial f_k(q)}{\partial q_l}P_k \ . \tag{31b}$$

For a linear canonical transformation, we may regard the q_l, etc., as elements of column vectors. There the transformations will be of the form

$$\vec{Q}=A\vec{q} \tag{32}$$

and

$$\vec{P}=\tilde{A}^{-1}\vec{p} \ , \tag{33}$$

where A is the transformation matrix and $\tilde{A}^{-1}$ denotes the inverse of the transpose of A. In matrix notation an operator, $\mathcal{O}$, with the general form

$$\mathcal{O}=\frac{1}{2m}\tilde{\vec{p}}\mathcal{M}(\vec{q})\vec{p} \tag{34a}$$

then becomes

$$\mathcal{O}=\frac{1}{2m}\tilde{\vec{P}}A\mathcal{M}(A^{-1}\vec{Q})\tilde{A}\vec{P} \tag{34b}$$

under a linear canonical point transformation.

Our first canonical transformation is not linear and we employ Eqs. (31) directly to rewrite Eq. (26b) in terms of the $[Q_l';P_l']$ of Eqs. (27). We find

$$T=\frac{1}{2m}\sum_{n,i}\left[\sum_{j=1}^{3}\frac{\partial X_{\text{c.m.},j}}{\partial x_{ni}}P_{\text{c.m.},j}+\sum_{k\geq l}\frac{\partial q_{kl}}{\partial x_{ni}}P_{kl}+P_\omega \text{ terms}\right]^2$$

$$=T_{\text{c.m.}}+T_{\text{MQC}}+T_\omega \ , \tag{35a}$$

where

$$T_{\text{c.m.}}=\frac{1}{2M}\sum_j P_{\text{c.m.},j}^2 \ ,$$

$$T_{\text{MQC}}=\frac{1}{2m}\sum_{\substack{k\geq l\\ k'\geq l'}}[\delta_{kk'}q_{ll'}+\delta_{ll'}q_{kk'}+\delta_{lk'}q_{kl'}+\delta_{kl'}q_{lk'}]P_{kl}P_{k'l'} \ . \tag{35b}$$

T_ω in Eq. (35a) denotes all terms containing at least one factor of $P_{\omega,l}$, possibly mixed with P_{ij} terms. Thus we have obtained a clean separation of the center of mass and have defined the MQC model part of T. In T_{MQC}, if $k<k'$ then $q_{kk'}$ should be read as $q_{k'k}$, and similarly for the other terms. If we adopt the MQC model premise of truncating the motion to the nine-dimensional collective coordinate space (i.e., $P_{\omega,l}=0$) and view the remaining motion from the center of mass, then $T=T_{\text{MQC}}$.

Gulshani and Rowe[7] have shown that for linear irrotational flow the quantum conditions equivalent to setting all $P_{\omega,l}$ to zero (which effects the truncation) are possible only in the limit of small amplitude vibrations and rotations of a nearly spherical nucleus. For larger vibrations there is a strong coupling between the irrotational flow collective coordinates and the vorticity degrees of freedom. One presumes that if the hydrodynamic Bohr Hamiltonian were properly extended to include the remaining many-body degrees of freedom, a similar coupling to nonirrotational freedoms would be found. However, our work is confined to a comparison of the hydrodynamic and MQC models as presently formulated, and hence these other degrees of freedom do not enter.

The second canonical transformation is effected by applying Eqs. (30) and (34) to T_{MQC} and grouping the resulting terms as zero-coupled tensors. Then Eqs. (29) are used to transform $[x,P_x]$ to $[\bar{\rho},P_{\bar{\rho}}]$ with the final result

$$T_{\text{MQC}}=\frac{1}{2B'}\left\{-\left[\frac{35}{8\pi}\right]^{1/2}[\bar{\pi}^*\bar{\alpha}\bar{\pi}^*]_0+\bar{\rho}^2[\bar{\pi}^*\bar{\pi}^*]_0+\frac{5}{2\pi}[\bar{\alpha}\bar{\pi}^*]_0\frac{1}{\bar{\rho}}P_{\bar{\rho}}+\frac{\sqrt{5}}{8\pi}P_{\bar{\rho}}P_{\bar{\rho}}\right\} . \tag{36}$$

Here B' is as given in Eq. (24) with $M=Am$, and $[\bar{\pi}^*\bar{\alpha}\bar{\pi}^*]_0$ symbolizes $[[\bar{\pi}^*\bar{\alpha}]_2\bar{\pi}^*]_0$. One should remember that both Eqs. (35b) and (36) are classical expressions and are the classical analogs of results given in Ref. 4. One should also note that, as with Ref. 4, there is no restriction to small amplitude for the $\bar{\alpha}_\mu$ or $\bar{\rho}$.

C. Quantum mechanical form of T_{MQC}

Our goal is to process Eq. (36) further and make a connection with Eq. (25), but for completeness and for comparison with Ref. 4 we digress briefly to present the quantum mechanical form of T_{MQC} implied by our classical form. In either Eq. (35b) or (36), the classical expression has the form

$$T_{\mathrm{MQC}}=\frac{1}{2m}\sum_{l,l'=1}^{6} G_{ll'}P_lP_{l'} \tag{37}$$

where G is a function of the coordinates only. To pass to the quantized form, we first rewrite Eq. (37) in terms of the time derivation of the generalized coordinates,

$$\dot{Q}_l=\frac{dQ_l}{dt}=\frac{\partial T_{\mathrm{MQC}}}{\partial P_l}=\frac{1}{m}\sum_{l'} G_{ll'}P_{l'} . \tag{38}$$

Then we have

$$P_{l'}=m\sum_{l} G^{-1}{}_{l'l}\dot{Q}_l \tag{39}$$

and

$$T_{\mathrm{MQC}}=\frac{m}{2}\sum_{ll'} G^{-1}{}_{ll'}\dot{Q}_l\dot{Q}_{l'} . \tag{40}$$

This is the form appropriate for writing the generalized Laplacian[9] so the kinetic energy operator becomes

$$T_{\mathrm{MQC}}=-\frac{\hbar^2}{2}\sum_{ll'}\frac{1}{\bar{G}}\frac{\partial}{\partial Q_l}\left[\bar{G}G_{ll'}\frac{\partial}{\partial Q_{l'}}\right] \tag{41}$$

in which $\bar{G}=[6/\det G]^{1/2}$.

When we apply this procedure to Eq. (35b) we find

$$T_{\mathrm{MQC}}=-\frac{\hbar^2}{2m}\sum_{\substack{k\geq l\\ k'\geq l'}}[\delta_{kk'}q_{ll'}+\delta_{ll'}q_{kk'}+\delta_{lk'}q_{kl'}+\delta_{kl'}q_{lk'}]\frac{\partial^2}{\partial q_{kl}\partial q_{k'l'}}-\frac{\hbar^2}{2m}(2A-2)\sum_{k}\frac{\partial}{\partial q_{kk}} . \tag{42}$$

One can achieve the same result by starting with the $3A$-dimensional Laplacian [from Eq. (26b)]

$$T=-\frac{\hbar^2}{2m}\sum_{ni}\frac{\partial^2}{\partial {x_{ni}}^2} ,$$

making the transformation of Eqs. (27) directly using the chain rule of differentiation, and then truncating to the nine-dimensional collective coordinate space by dropping all terms involving $\partial/\partial\omega_l$. In that case, in addition to T_{MQC}, one also finds the center of mass kinetic energy operator

$$T_{\mathrm{c.m.}}=-\frac{\hbar^2}{2M}\sum_{k}\frac{\partial^2}{\partial {X_{\mathrm{c.m.},k}}^2} .$$

Our expression (42) differs slightly from that in Ref. 4 due to our explicit removal of the center of mass from the quadrupole tensor.

The quantum mechanical expression appropriate to Eq. (36) is

$$T_{\mathrm{MQC}}=\frac{1}{2B'}\left[\bar{\rho}^2[\bar{\pi}^*\bar{\pi}^*]_0-\left[\frac{35}{8\pi}\right]^{1/2}[\bar{\pi}^*\bar{\alpha}\bar{\pi}^*]_0 -\frac{5i\hbar}{2\pi}[\bar{\alpha}\bar{\pi}^*]_0\frac{1}{\bar{\rho}}\frac{\partial}{\partial\bar{\rho}} -\frac{\hbar^2\sqrt{5}}{8\pi}\left[(3A-4)\frac{1}{\bar{\rho}}\frac{\partial}{\partial\bar{\rho}}+\frac{\partial^2}{\partial\bar{\rho}^2}\right]\right] \tag{43}$$

in which $\bar{\pi}^*_\mu$ is to be interpreted as $-i\hbar\partial/\partial\bar{\alpha}^*_\mu$. Again this is the analog of the operator given by Rosensteel and Ihrig[4] corrected for removal of center-of-mass motion. It can also be derived from Eq. (42) via the transformations of Eqs. (28). In making comparison with Ref. 4, their α_μ and π_μ should be identified with our $\bar{\alpha}^*_\mu$ and $\bar{\pi}^*_\mu$, respectively, since their counterpart of Eq. (1) involves $Y_{2\mu}$ rather than $Y^*_{2\mu}$. That is, their α_μ transforms under rotations like a $Y^*_{2\mu}$ rather than like a $Y_{2\mu}$ as does our $\bar{\alpha}_\mu$.

D. CMSR condition imposed

We now return to our main purpose, that of obtaining a CMSR version of T_{MQC} which can be compared with T_{BM}. To this end, we will rewrite Eq. (36) in terms of the $\bar{\alpha}_\mu$ and $\dot{\bar{\alpha}}_\mu$ assuming a constraint which is the discrete counterpart of Eq. (3b). As we shall see in Sec. IV this constraint implies $\dot{\bar{\rho}}=0$ and $\bar{\rho}=1$. Then, since

$$\dot{\bar{\rho}}=\frac{\partial T_{\mathrm{MQC}}}{\partial P_{\bar{\rho}}}=\frac{5}{2\pi}[\bar{\alpha}\bar{\pi}^*]_0\frac{1}{\bar{\rho}}+\frac{2\sqrt{5}}{8\pi}P_{\bar{\rho}} ,$$

we have

$$P_{\bar{\rho}}=-\frac{10}{\sqrt{5}}[\bar{\alpha}\bar{\pi}^*]_0 \quad (\bar{\rho}=1) \tag{44}$$

for this special case. Thus for CMSR,

$$T_{\mathrm{MQC}}=\frac{1}{2B'}\left[[\bar{\pi}^*\bar{\pi}^*]_0-\left[\frac{35}{8\pi}\right]^{1/2}[\bar{\pi}^*\bar{\alpha}\bar{\pi}^*]_0 - \frac{5\sqrt{5}}{2\pi}[\bar{\alpha}\bar{\pi}^*]_0[\bar{\alpha}\bar{\pi}^*]_0\right] . \tag{45}$$

Now, the Hamilton's equation

$$\dot{\bar{\alpha}}_\mu=\frac{\partial T_{\mathrm{MQC}}}{\partial\bar{\pi}_\mu}=(-1)^\mu\frac{\partial T_{\mathrm{MQC}}}{\partial\bar{\pi}^*_{-\mu}}$$

yields

$$\dot{\bar{\alpha}}_\mu=\frac{1}{2B'}\left[\frac{2}{\sqrt{5}}\bar{\pi}^*_\mu-\left[\frac{7}{2\pi}\right]^{1/2}[\bar{\alpha}\bar{\pi}^*]_{2\mu} - \frac{5}{\pi}[[\bar{\alpha}\bar{\pi}^*]_0\bar{\alpha}]_{2\mu}\right] , \tag{46}$$

which is exact. Equation (46) represents a set of five coupled equations which could, in principle, be solved for the $\bar{\pi}^*_\mu$ in terms of the $\bar{\alpha}_\mu$ and $\dot{\bar{\alpha}}_\mu$. The nonlinear result would not be particularly convenient for our purposes. Rather we seek an expanded solution which, from the structure of Eq. (46), must have the form

$$\bar{\pi}^*_\mu=2B'\left[a_1\dot{\bar{\alpha}}_\mu+a_2[\bar{\alpha}\dot{\bar{\alpha}}]_{2\mu} + \sum_J a_{3J}[[\bar{\alpha}\bar{\alpha}]_J\dot{\bar{\alpha}}]_{2\mu}+\cdots\right] . \tag{47}$$

We then substitute Eq. (47) into Eq. (46) and equate coefficients of the independent tensors to find the a_k. Here, we require the result only to $\mathcal{O}(4)$ and we find

$$\bar{\pi}^*_\mu=2B'\left[\frac{\sqrt{5}}{2}\dot{\bar{\alpha}}_\mu+\frac{5}{4}\left[\frac{7}{2\pi}\right]^{1/2}[\bar{\alpha}\dot{\bar{\alpha}}]_{2\mu}+\frac{55}{16\pi}[[\bar{\alpha}\bar{\alpha}]_0\dot{\bar{\alpha}}]_{2\mu}+\frac{25\sqrt{5}}{32\pi}[[\bar{\alpha}\bar{\alpha}]_2\dot{\bar{\alpha}}]_{2\mu} + \frac{45}{8\pi}[[\bar{\alpha}\bar{\alpha}]_4\dot{\bar{\alpha}}]_{2\mu}+\mathcal{O}(4)\right] . \tag{48}$$

We then use Eq. (48) in Eq. (45) to write (CMSR)

$$T_{\mathrm{MQC}}=2B'\left[\frac{5}{4}[\dot{\bar{\alpha}}\dot{\bar{\alpha}}]_0+\frac{35}{8}\left[\frac{5}{14\pi}\right]^{1/2}[[\dot{\bar{\alpha}}\dot{\bar{\alpha}}]_2\bar{\alpha}]_0+\frac{55\sqrt{5}}{32\pi}[[\dot{\bar{\alpha}}\dot{\bar{\alpha}}]_0[\bar{\alpha}\bar{\alpha}]_0]_0+\frac{125}{64\pi}[[\dot{\bar{\alpha}}\dot{\bar{\alpha}}]_2[\bar{\alpha}\bar{\alpha}]_2]_0 + \frac{45\sqrt{5}}{16\pi}[[\dot{\bar{\alpha}}\dot{\bar{\alpha}}]_4[\bar{\alpha}\bar{\alpha}]_4]_0+\mathcal{O}(5)\right] . \tag{49}$$

To compare Eq. (49) with Eq. (25), we need only write the $\bar{\alpha}_\mu$ in terms of the α_μ for the CMSR condition. This we do in the next section.

IV. COMPARISON OF T_{MQC} WITH T_{BM} FOR CMSR

A natural connection between the $[\bar{\rho},\bar{\alpha}_\mu]$ coordinates of the MQC model and the BM model $[\alpha_{00},\alpha_\mu]$ is made by replacing the discrete sums of Eq. 28 with integrals. These defining equations for $\bar{\rho}$ and $\bar{\alpha}_\mu$, now in integral form, may then be evaluated using the BM model variables α_{00} and α_μ. Thus

$$\bar{\rho}^2=\left[\frac{5}{3MR_0{}^2}\right]\int\int\int\rho_0 r^2dV =\frac{5}{3R_0{}^2}\frac{1}{V}\int\int\int r^2dV \tag{50a}$$

and

$$\bar{\alpha}_\mu=\frac{4\pi}{3MR_0{}^2}\int\int\int\rho_0 r^2Y_{2\mu}(\theta,\phi)dV =\frac{4\pi}{3R_0{}^2}\frac{1}{V}\int\int\int r^2Y_{2\mu}(\theta,\phi)dV . \tag{50b}$$

The CMSR condition, Eq. (3b), together with Eq. (50a) yields immediately

$$\bar{\rho}^2=1\Rightarrow\dot{\bar{\rho}}=0\ \ (\mathrm{CMSR}) . \tag{51}$$

For CMSR, the volume of the BM nuclear droplet is found from Eqs. (5) and (10) to be

$$V=\tfrac{4}{3}\pi R_0{}^3\left[1-\frac{15\sqrt{5}}{8\pi}[\alpha\alpha]_0+\mathcal{O}(3)\right] . \tag{52}$$

Then, using Eqs. (1), (10), (50b), and (52) we have

$$\bar{\alpha}_\mu=\frac{4\pi}{3R_0{}^2}\frac{1}{V}\frac{R_0{}^5}{5}\int\int\left[1+\frac{\alpha_{00}}{\sqrt{4\pi}}+\sum_\nu\alpha_\nu Y^*_{2\nu}\right]^5 Y_{2\mu}d\Omega$$

$$=\alpha_\mu-2\left[\frac{5}{14\pi}\right]^{1/2}[\alpha\alpha]_{2\mu}-\frac{9\sqrt{5}}{8\pi}[[\alpha\alpha]_0\alpha]_{2\mu}+\frac{10}{14\pi}[[\alpha\alpha]_2\alpha]_{2\mu}+\frac{3\sqrt{5}}{7\pi}[[\alpha\alpha]_4\alpha]_{2\mu}+\mathcal{O}(4)\,. \tag{53}$$

This establishes the relationship between the MQC and BM quadrupole coordinates when the constant mean squared radius constraint is imposed. Finally, we substitute Eq. (53) and its time derivative into Eq. (49) to obtain the desired result

$$\frac{T_{\rm MQC}}{B'}=\frac{5}{2}[\dot\alpha\dot\alpha]_0-\frac{45}{4}\left[\frac{5}{14\pi}\right]^{1/2}[[\dot\alpha\dot\alpha]_2\alpha]_0-\frac{289\sqrt{5}}{112\pi}[[\dot\alpha\dot\alpha]_0[\alpha\alpha]_0]_0-\frac{8755}{1568\pi}[[\dot\alpha\dot\alpha]_2[\alpha\alpha]_2]_0$$

$$+\frac{366\sqrt{5}}{784\pi}[[\dot\alpha\dot\alpha]_4[\alpha\alpha]_4]_0+\mathcal{O}(5)\,. \tag{54}$$

Comparison of Eqs. (25) and (54) reveals that while $T_{\rm MQC}$ and $T_{\rm BM}$ are the same through 3rd order, they differ in higher order terms. In 4th order, the MQC model emphasizes the zero-zero or pairing coupling more than does the BM model. The really essential point is, however, that $T_{\rm MQC}$ and $T_{\rm BM}$ are simply not the same in 4th order. While it is true that $T_{\rm MQC}$ is derived without restriction to small oscillations, and that the $\bar{\alpha}_\mu$ are the components of the mass quadrupole tensor in the BM model, the MQC model is *not* an extension of the BM model to arbitrarily large amplitude vibrations. In that sense, $T_{\rm MQC}$ is not a "correction" to $T_{\rm BM}$ as was stated in Ref. 4, nor is the derivation of $T_{\rm MQC}$ a microscopic derivation of $T_{\rm BM}$ as stated in Ref. 5.

The question naturally arises as to whether the MQC model represents other than irrotational flow. It is easy to show that the MQC model implies curl-free flow in the sense of Gulshani and Rowe.[7] That is, the restriction to only collective (q_{ij}) motion implies $\vec{\nabla}_n\times\vec{v}_n=0$ for each of the A particles. For a classical fluid, one would define the fluid velocity "at a point," $\vec{v}(\vec{r})$, by averaging the velocities of the individual particles with an appropriate distribution function. This distribution function would reflect the nuclear forces that cause the fluid to form. $\vec{\nabla}\times\vec{v}$ would then be defined in terms of the limit of an integral of the $\vec{v}_n$ over a sphere.[10] It does not follow that $\vec{\nabla}\times\vec{v}$ will be a simple average of the $\vec{\nabla}_n\times\vec{v}_n$. Hence $\vec{\nabla}_n\times\vec{v}_n=0$ need not imply that the associated collective $\vec{v}(\vec{r})$ be curl free. Since we cannot uniquely determine $\vec{v}(\vec{r})=\langle\vec{v}_n\rangle$, we cannot assess whether such $\vec{v}$, if curl free, would satisfy the surface boundary condition. Both would be necessary to conclude that $\vec{v}=\vec{v}_{\rm BM}$. The essential feature here is that the $\vec{v}_n$ lead to collective coordinates and to a collective kinetic energy, $T_{\rm MQC}$, which can be meaningfully compared to $T_{\rm BM}$. This, of course, is what we have done.

The MQC and BM models are simply different models based upon different assumptions. It is therefore not surprising that differences should surface when the two models are subjected to close scrutiny. It is quite remarkable that the kinetic energy expressions are equivalent through terms of order 3. Based on Eq. (53) we had expected a difference in 3rd order. We have made our comparison only in the CMSR limit applied to both models. One could also compare them in the CV limit. By imposing no constraints of this sort and therefore retaining the α_{00} of the BM model, we could also compare the full six-dimensional versions of each model. However, if the model kinetic energies were equal, order by order, in the general case, they would be equal in any limit and vice versa. The CMSR limit is useful because, in the MQC model, the monopole $\bar{\rho}$ is not only decoupled from the quadrupole motion, but indeed $\bar{\rho}$ is constant.

The strong parallel between the two models in the form of the terms in the kinetic energy expressions is largely due to two things. Time reversal invariances and the tensorial nature of the $\bar{\alpha}_\mu$ and α_μ force the leading term to be $[\bar{\pi}^*\bar{\pi}^*]_0$ or $[\pi^*\pi^*]_0$, respectively, for $T_{\rm MQC}$ and $T_{\rm BM}$; also the next orders must be $[\bar{\pi}^*\bar{\alpha}\bar{\pi}^*]_0$ or $[\pi^*\alpha\pi^*]_0$. Secondly, our choice of scaling of $\bar{\alpha}_\mu$ so that $\bar{\alpha}_\mu=\alpha_\mu$ to lowest order makes the correspondence complete. Had we chosen to make the R_0 of the MQC model different from the R_0 of the BM model, then $T_{\rm MQC}$ in $[\bar{\rho},\bar{\alpha}_\mu]$ would appear quite different from $T_{\rm BM}$ expressed in $[\alpha_{00},\alpha_\mu]$. But then the relationship between $\bar{\alpha}_\mu$ and α_μ would be different and, in the end, it would remain that when expressed in BM coordinates,

$T_{MQC} = T_{BM}$ through 3rd order.

If one wants to deal with other than small vibrations within the Bohr-Mottelson model, it is necessary to extend T_{BM} to higher order as we have done in Eq. (25) for either CV or CMSR. One could proceed similarly for the unrestricted case by retaining α_{00} as an active degree of freedom. For consistency one should extend the potential energy $\mathscr{V}$ to higher order as well. We have not considered that here, but interestingly enough it is possible to compute the surface area term of $\mathscr{V}$ to arbitrary order.[11] Alternatively, one could use T_{MQC} either without restriction or by imposing one of CV or CMSR. In the latter cases it would be necessary to make a similar projection of the two-nucleon interactions onto the collective coordinate space. The means to do this have been introduced formally in Ref. 5, but the details are likely to present considerable difficulty. Even after projection most of the microscopic nature of the MQC model is retained through the algebraic group structure of the model. This structure would undoubtedly prove useful in calculations, but whether the extended BM model or the MQC model is the better phenomenological description of nature would have to be decided by comparison between experiment and the detailed predictions of each model. It is well known that the calculated value of B' in the BM model is two to three times larger than the value one would obtain by fitting the model to experiment. Since B' of the MQC model is the same as that for the BM model, the MQC model would experience a similar difficulty. However, by limiting the sums which define the mass quadrupole coordinates to those nucleons in the valence shells only, the MQC model could perhaps eliminate this difficulty.

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