Partially separable t matrix

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The off-shell t matrix is expressed as a sum of *one* nonseparable and *one* separable term so that it is useful for applications to more-than-two-body problems. All poles are involved in this one separable term. Both the nonseparable and the separable terms of the kernel $G_0 t$ are regular at the origin. The nonseparable term of this kernel vanishes at large distances, while the separable term behaves asymptotically as the spherical Hankel function. These properties make our expression free from defects inherent in the Jost- or the K-matrix expressions, and many applications are anticipated. As the application, a compact expression of the many-level formula is presented. Also the application is suggested to the breakup three-body problem based on the Faddeev equation. It is demonstrated that the breakup amplitude is expressed in a simple and physically interesting form and we can calculate it in coordinate space.

[NUCLEAR REACTIONS Off-shell t matrix with one nonseparable and one separable term; many-level formula; three-body breakup amplitude; the Faddeev equation in coordinate space.]

I. INTRODUCTION

The t matrix plays a central role in the scattering theory. We are interested in expressing the t matrix in a form that is useful to various applications.

In the early days of studying three-body problems,[1] the Yamaguchi separable potential[2] played an important role, since this potential reduces the Faddeev equation,[3] which is a two-variable integral equation, to a one variable integral equation. This has stimulated some efforts to find a more sophisticated separable potential or fully separable expansions of the (off-shell) t matrix.[4] However, since a local potential cannot in principle be expressed as a finite sum of separable terms,[5] such an approach would not be adequate to be applied to a most sophisticated level of calculations of a three nucleon system. Our contemporary computer is big enough that we can perform three-body calculations without expressing the t matrix as a complete sum of separable terms. In fact, we have solved the three-body bound states with a partially separable t matrix (PST).[6,7]

Of course, the use of the t matrix is not confined to a few-body problem. Let us state the general background and the requirements in finding a PST with desirable properties.

If we try a full expansion of the t matrix by a complete orthonormal set of functions, the numerical calculation will become more and more inaccurate with the increasing number of nodes. In practice, we are forced to truncate the expansion with no guarantee of convergence. Even when a seeming convergence is obtained, often it only means a very slow convergence. In such a case, the truncation is made before we come to a true convergent value. One may say that an expansion by a complete set of functions is mathematical physics of an earlier time and with the computer age one should try to explore other ways.

First of all, we require that (I) *the t matrix be expressed as a sum of one separable and one nonseparable term.* The restriction to only one separable term is due to a purely practical reason, but is very important in preventing the increase of dimensionality in application. The inclusion of one nonseparable term is inevitable for a local potential. Let G_0 be the two-body free Green's function and V the two-body potential. Then the two-body t matrix satisfies the Lippmann-Schwinger equation

$$t = V + VG_0 t \ . \tag{1}$$

Suppose that G_0t is expressed in terms of nonseparable functions A, B, and C and a function F as

$$G_0t=A+|B\rangle F\langle C| \ . \tag{2}$$

Substituting (2) into (1), we find that the t matrix is expressed as

$$t=V(A+1)+V|B\rangle F\langle C| \ . \tag{3}$$

This is the form satisfying requirement (I). There are several ways of expressing the t matrix as PST.[8] Well-known examples are the K- and the Jost-matrix expressions. However, a simpleminded iteration of the K-matrix expression as the Neumann series often diverges. Of course, the use of such a divergent series is not desirable in applications. We have to remove the element causing this divergence. Fortunately, this can be done, if we require that (II) *no dangerous pole be involved in A, B, and C.* By the dangerous pole, we mean those poles which lie on the real momentum axis. This requirement means that all dangerous poles are contained in the function F in a manner that the poles of F are not on the real momentum axis, but shifted to some points on the complex momentum plane.

When this expression of the t matrix is used in the Faddeev equation, the separable term serves as the zeroth order term and the nonseparable term is treated as the perturbation. Since the separable term is the main part of the t matrix, the convergence of the whole calculation is almost clear *ab initio*. In fact, this was the case for the bound state.[6,7]

The Jost expression of G_0t that is useful in a two-body problem is not at all useful in a three-body system. This is because both the separable and nonseparable terms of the Jost expression of G_0t suffer from the singularity at the origin. To avoid such a difficulty, we require that (III) *the functions A and B should be regular at the origin.* This requirement is very important because in applications we use the separable and the nonseparable terms separately.

Further, to facilitate the calculation of the breakup amplitude taking into account the exact asymptotic behavior of the wave function, we require that (IV) *the operator A should vanish at large distances and the function* $|B\rangle$ *should asymptotically behave as the spherical Hankel function.* Then, since we can reduce the overlap of the spherical Hankel function with a certain function to a tractable form as demonstrated in Sec. IV, we can avoid a difficulty of numerical calculations at large distances, when Eq. (2) is used in a scattering equation of a more-than-two-body system. This greatly facilitates an accurate calculation of the breakup process.

We anticipate various applications of the present expression of the t matrix. For example, if we use it in the reaction theory, we can express the many-level resonance formula in a simple form. Also, the present expression is extremely suitable for calculations of the breakup process in the coordinate space. The present paper is an extension of the previous preliminary studies,[8,9] and at the same time serves to pave the way for a more complete treatment of the three-body scattering problem.

In Sec. II, first we study the general way of constructing PST and then discuss the Jost- and K-matrix expressions. Then, we present the most useful expression of PST. Some applications are suggested at the end of the section. In Sec. III we demonstrate how PST of Sec. II is applied to the three-body breakup channel. In Sec. IV we summarize the results. In the Appendix, some equations are proved.

II. PARTIALLY SEPARABLE T MATRIX

A. General theory

The t matrix satisfying requirement (I) may be obtained in the following manner.[8] Suppose that the Green's function G_0 is divided into two parts

$$G_0=\bar{G}_0+|a\rangle\langle b| \ . \tag{4}$$

Here, $\bar{G}_0$ is a Green's function satisfying the same equation as G_0,

$$G_0^{-1}\bar{G}_0=1 \ , \tag{5}$$

but subject to boundary conditions which are different from G_0. The second term $|a\rangle\langle b|$ is a separable operator. The separation (4) is not unique. Examples will be given shortly.

We define the matrix L and the wave matrix $\bar{\Omega}$ by

$$L=V+V\bar{G}_0L=V\bar{\Omega} \ , \tag{6}$$

where

$$\bar{\Omega}=\frac{1}{1-\bar{G}_0V} \ . \tag{7}$$

If we subtract Eq. (6) from the Lippmann-Schwinger equation (1), we obtain

$$\begin{aligned} t&=L+(1-V\bar{G}_0)^{-1}V|a\rangle\langle b|t \ , \\ &=V\bar{\Omega}+V\bar{\Omega}|a\rangle\langle b|t \ , \\ &=V\bar{\Omega}+V\bar{\Omega}|a\rangle\frac{1}{1-\langle b|V\bar{\Omega}|a\rangle}\langle b|V\bar{\Omega} \ . \end{aligned} \tag{8}$$

If we use this result on the left hand side of Eq. (1), we get

$$G_0 t=\bar{\Omega}-1+\bar{\Omega}\,|\,a\,\rangle\frac{1}{1-\langle b\,|\,V\bar{\Omega}\,|\,a\,\rangle}\langle b\,|\,V\bar{\Omega}\ . \tag{9}$$

Equations (8) and (9) take the forms of Eqs. (3) and (2), respectively, satisfying requirement (I). It is then obvious that for requirements (I), (III), and (IV) to be satisfied at the same time, the Green's function $\bar{G}_0$ used in A, B, C, and F of Eqs. (2) and (3) should be different from G_0. Comparing Eq. (2) with Eq. (9) and making use of Eq. (7), we see that requirements (III) and (IV) are alternatively expressed as:

(III′) *The Green's function* $\bar{G}_0$ *and the function* $|\,a\,\rangle$ *should be regular at the origin.*

(IV'_a) *The Green's function* $\bar{G}_0$ *should vanish at large distances from the origin.*

(IV'_b) *The function* $|\,a\,\rangle$ *should behave asymptotically as the spherical Hankel function.*

Let us write known examples of $\bar{G}_0$. With the spherical Bessel (Neumann) function $j_l(kr)[n_l(kr)]$, we obtain the spherical Hankel function by

$$h_l^{(+)}(kr)=-n_l(kr)+ij_l(kr)\ . \tag{10}$$

With the step function

$$\theta(x)=\begin{cases}1 \text{ for } x>0\ ,\\ 0 \text{ for } x<0\ ,\end{cases} \tag{11}$$

the usual Green's function G_0 is written as

$$G_0(r,r')=-k[\,h_l^{(+)}(kr)j_l(kr')\theta(r-r')+j_l(kr)h_l^{(+)}(kr')\theta(r'-r)]\ . \tag{12}$$

Equation (12) can alternatively be expressed as

$$G_0(r,r')=g_l(r,r')-kh_l^{(+)}(kr)j_l(kr')\ , \tag{13}$$

where we have defined a Green's function $g_l(r,r')$ that corresponds to $\bar{G}_0$ in Eq. (4) by

$$g_l(r,r')=-k[\,n_l(kr)j_l(kr')-j_l(kr)n_l(kr')]\theta(r'-r)\ . \tag{14}$$

This Green's function is used in the Jost expression of the t matrix. Another expression of $G_0(r,r')$ in the form of Eq. (4) is to decompose it into the principal value Green's function $\text{P}G_0(r,r')$ and the imaginary part

$$G_0(r,r')=\text{P}G_0(r,r')-ikj_l(kr)j_l(kr')\ . \tag{15}$$

In this case, $\text{P}G_0(r,r')$ corresponds to $\bar{G}_0$ in Eq. (4). This Green's function is defined by

$$\text{P}G_0(r,r')=k[\,n_l(kr)j_l(kr')\theta(r-r')+j_l(kr)n_l(kr')\theta(r'-r)]\ . \tag{16}$$

This Green's function is used in the K-matrix expression. The Green's function g satisfies (IV'_a) but not (III′), whereas $\text{P}G_0$ satisfies (III′) but not (IV'_a). Therefore, we should introduce another Green's function, if we require (III′) and (IV′) to be satisfied at the same time. Before we present a new expression of the t matrix, let us briefly review the Jost- and the K-matrix expressions of the t matrix.

B. Jost expression

With the Green's function (14), we define the Jost matrix J and its wave matrix ω by

$$J=V+VgJ=V\omega\ , \tag{17}$$

and

$$\omega=\frac{1}{1-gV}\ . \tag{18}$$

In terms of J and ω, the t matrix and the kernel $G_0 t$ are expressed as

$$t=J-J\,|\,kh^{(+)}\rangle\frac{1}{1+k\langle j\,|\,J\,|\,h^{(+)}\rangle}\langle j\,|\,J\ , \tag{19}$$

and

$$G_0 t=(\omega-1)-\omega\,|\,kh^{(+)}\rangle\frac{1}{1+k\langle j\,|\,J\,|\,h^{(+)}\rangle}\langle j\,|\,J\ , \tag{20}$$

where $|\,j\,\rangle$ and $|\,h^{(+)}\rangle$ stand for the spherical Bessel and Hankel function, respectively.

It is known that the Jost function $1+k\langle j\,|\,J\,|\,h^{(+)}\rangle$ has zeros in the lower half of the complex k plane, while J does not involve any pole. Thus, expression (19) satisfies requirement (II). The Green's function g satisfies requirement (IV'_a). As a result, Eq. (20) satisfies requirement (IV'_b), and hence (IV). However, since g does not satisfy (III′), requirement (III) is not satisfied. This is a serious defect of the Jost expression of the two-body t matrix, preventing applications to more-than-two-body problems. Besides this difficulty, the matrix element $\langle j\,|\,J\,|\,h^{(+)}\rangle$ is not always defined for coupled partial waves.[10]

C. K-matrix expression

With the Green's function (16), we define the K matrix and its wave matrix $\bar{\omega}$ by

$$K=V+V\mathrm{P}G_0K=V\bar{\omega}\ , \tag{21}$$

and

$$\bar{\omega}=\frac{1}{1-\mathrm{P}G_0V}\ . \tag{22}$$

Then we obtain the expressions

$$t=K-iK\,|\,kj\rangle\frac{1}{1+ik\langle j\,|\,K\,|\,j\rangle}\langle j\,|\,K \tag{23}$$

and

$$G_0t=(\bar{\omega}-1)-i\bar{\omega}\,|\,kj\rangle\frac{1}{1+ik\langle j\,|\,K\,|\,j\rangle}\langle j\,|\,K\ . \tag{24}$$

Since $\mathrm{P}G_0$ satisfies requirement (III'), expression (24) satisfies requirement (III). Since $\mathrm{P}G_0$ does not satisfy requirement (IV_a'), requirement (IV) is not satisfied. However, this is not a serious problem. A serious problem of Eqs. (23) or (24) is that K can be infinity at some points on the real momentum axis. [See Eq. (39).] This defect prevents the application of Eq. (23) to a problem of a more complicated system.

D. A new expression

We define the Sturm-Liouville function ψ_n and the corresponding eigenvalue λ_n by[11]

$$\mathrm{P}G_0V\psi_n=\lambda_n\psi_n(n=1,\ldots,\infty)\ . \tag{25}$$

In the present paper, we make use of those Sturm-Liouville states whose eigenvalues become unity or very close to unity at some energy. When we write $\hat{\psi}_n$ $(n=1,\ldots,N)$, it means that the Sturm-Liouville functions belonging to this set, normalized as

$$\langle j\,|\,V\,|\,\hat{\psi}_n\rangle=-\lambda_n,\quad(n=1,\ldots,N)\ . \tag{26}$$

(Later, we use the Sturm-Liouville function also belonging to this set, but normalized in another way. In this case, we use the notation ψ_n.) Let us introduce the function $\hat{\psi}$ by

$$\hat{\psi}=\sum_{n=1}^{N}\alpha_n^{(N)}\hat{\psi}_n\ , \tag{27}$$

where $\alpha_n^{(N)}$ is given by

$$\alpha_n^{(N)}=\frac{\lambda_n^{N-1}}{\prod_{m\neq n}^{N}(\lambda_n-\lambda_m)},\quad \alpha_1^{(1)}=1\ . \tag{28}$$

Using this function, we define the Green's function $\hat{g}$ by

$$\hat{g}=\mathrm{P}G_0+\,|\,\hat{\psi}\rangle\langle j\,|\ . \tag{29}$$

If we define the wave matrix $\hat{\omega}$ by

$$\hat{\omega}=\frac{1}{1-\hat{g}V}\ , \tag{30}$$

we can express the matrix t and G_0t as

$$t=V\hat{\omega}-V\hat{\omega}\,|\,\hat{\psi}+ikj\rangle\frac{1}{\prod_{n=1}^{N}(1-\lambda_n)+ik\langle j\,|\,V\hat{\omega}\,|\,j\rangle}\langle j\,|\,V\hat{\omega}\ , \tag{31}$$

$$G_0t=\hat{\omega}-1-\hat{\omega}\,|\,\hat{\psi}+ikj\rangle\frac{1}{\prod_{n=1}^{N}(1-\lambda_n)+ik\langle j\,|\,V\hat{\omega}\,|\,j\rangle}\langle j\,|\,V\hat{\omega}\ . \tag{32}$$

Clearly, these expressions satisfy requirement (I). The case $N=1$ is important for a low energy nucleon-nucleon interaction. This case has already been treated in Ref. 8.

In deriving Eqs. (31) and (32), we have used the following important relations

$$1+\langle j\,|\,V\hat{\omega}\,|\,\hat{\psi}\rangle=1+\langle j\,|\,V[1+\sum_{l=1}^{N-1}(\hat{g}V)^l]\,|\,\hat{\psi}\rangle=\prod_{n=1}^{N}(1-\lambda_n) \tag{33}$$

and

$$(\hat{g}V)^l\,|\,\psi_n\rangle=0,\ \text{for}\ l\geq N\ \text{and}\ n=1,\ldots,N\ . \tag{34}$$

Equations (33) and (34) are proved in the Appendix. In Eq. (34), the function ψ_n satisfies Eq. (25) and is normalized as

$$\int\psi_nV\psi_m\,d\tau=\delta_{nm}\quad(n,m=1,\ldots,\infty)\ , \tag{35}$$

which is written for simplicity as

$$\langle\psi_n\,|\,V\,|\,\psi_m\rangle=\delta_{nm}\ . \tag{36}$$

The difference of the functions $\hat{\psi}_n$ and ψ_n for $n=1,\ldots,N$ is only the normalization factors.

Since Eq. (34) is satisfied, the equation

$$(\hat{g}V)^l \mid \hat{\psi}\rangle = 0 \text{ for } l \geq N \tag{37}$$

is valid.

From the orthonormality (36), we obtain the closure relation

$$\sum_{n=1}^{\infty} V \mid \psi_n\rangle\langle\psi_n \mid = \sum_{n=1}^{\infty} \mid \psi_n\rangle\langle\psi_n \mid V = 1 \ . \tag{38}$$

Now, we shall show that Eq. (31) satisfies requirement (II). If $\hat{g}$ were $\mathrm{P}G_0$, Eq. (30) reduces to Eq. (22), which is written by Eq. (38) as

$$\begin{aligned}\bar{\omega} &= 1 + \mathrm{P}G_0 V \bar{\omega} \\ &= 1 + \mathrm{P}G_0 V \sum_{n=1}^{\infty} \mid \psi_n\rangle\langle\psi_n \mid V\bar{\omega} \\ &= 1 + \sum_{n=1}^{\infty} \lambda_n \mid \psi_n\rangle\langle\psi_n \mid V\bar{\omega} \\ &= 1 + \sum_{n=1}^{\infty} \frac{\lambda_n}{1-\lambda_n} \mid \psi_n\rangle\langle\psi_n \mid V \ .\end{aligned} \tag{39}$$

If λ_n becomes unity at $E = E_n$, i.e.,

$$\lambda_n(E_n) = 1 \quad (n = 1, \ldots, N) \ , \tag{40}$$

the matrix $\bar{\omega}$ is singular at these energies.

We can avoid this difficulty if we use $\hat{g}$ as the Green's function. The wave matrix $\hat{\omega}$ is expanded as

$$\begin{aligned}\hat{\omega} &= 1 + \sum_{l=1}^{\infty} (\hat{g}V)^l \\ &= 1 + \sum_{l=1}^{N-1} (\hat{g}V)^l \\ &\quad + \sum_{l=N}^{\infty} \sum_{n=N+1}^{\infty} (\hat{g}V)^l \mid \psi_n\rangle\langle\psi_n \mid V \\ &= 1 + \sum_{l=1}^{N-1} (\hat{g}V)^l \\ &\quad + \sum_{n=N+1}^{\infty} \frac{1}{1-\lambda_n} (\hat{g}V)^N \mid \psi_n\rangle\langle\psi_n \mid V \ .\end{aligned} \tag{41}$$

As a result, the wave matrix $\hat{\omega}$ is free from any singularity and requirement (II) is satisfied by Eq. (31). All poles at E_n $(n = 1, \ldots, N)$ on the real axis which cause difficulty in the K-matrix theory are now absorbed in the denominator of Eq. (31) or (32), shifted to some points on the complex momentum plane.

If we use the function $\hat{\psi}_n$ defined by Eq. (26) also to $n \geq N+1$, and designate the ratio of the normalization factors of $\hat{\psi}_n$ and ψ_n by N_n, i.e., $\hat{\psi}_n = N_n \psi_n$, we get

$$(\hat{g}V) \mid \psi_n\rangle = \lambda_n \left(\mid \psi_n\rangle - \frac{1}{N_n} \mid \hat{\psi}\rangle \right) . \tag{42}$$

By the repeated use of this equation, we obtain

$$\begin{aligned}K_n &\equiv (\hat{g}V)^N \mid \psi_n\rangle\langle\psi_n \mid V \\ &= \lambda_n^N \left[\mid \psi_n\rangle - \frac{1}{N_n} \sum_{l=0}^{N-1} \left(\frac{\hat{g}V}{\lambda_n} \right)^l \mid \hat{\psi}\rangle \right] \langle\psi_n \mid V \ .\end{aligned} \tag{43}$$

At this moment, we are unable to prove the absolute convergence of the infinite series in Eq. (41). But it is very likely since the norm of the operator K_n is $\mid \lambda_n^N \mid$ and

$$0 < \mid \lambda_{n+1} \mid < \mid \lambda_n \mid < 1 (n \geq N+1) \ .$$

By equations (25) and (27), the function $\hat{\psi}$ is regular at the origin, and so is the Green's function $\hat{g}$ by Eq. (29). Therefore, $\hat{g}$ satisfies requirement (III') and Eq. (32) requirement (III).

Finally, let us show that the Green's function $\hat{g}$ vanishes at large distances from the origin. Equation (25) behaves asymptotically as

$$-\frac{\cos(kr - \frac{1}{2}l\pi)}{r} \langle j \mid V \mid \hat{\psi}_n\rangle = \lambda_n \lim_{r\to\infty} \hat{\psi}_n \ . \tag{44}$$

Therefore, if we use Eq. (27), we see that the function $\hat{\psi}$ behaves as

$$\begin{aligned}\lim_{r\to\infty} \hat{\psi} &= -\frac{\cos(kr - \frac{1}{2}l\pi)}{r} \sum_{n=1}^{N} \alpha_n^{(N)} \frac{\langle j \mid V \mid \hat{\psi}_n\rangle}{\lambda_n} \\ &= \frac{\cos(kr - \frac{1}{2}l\pi)}{r} \sum_{n=1}^{N} \alpha_n^{(N)} \ ,\end{aligned} \tag{45}$$

where we have used Eq. (26). In the Appendix, we show that

$$\sum_{n=1}^{N} \alpha_n^{(N)} = 1 \ . \tag{46}$$

Thus the function $\hat{\psi}$ behaves asymptotically as

$$\lim_{r\to\infty} \hat{\psi} = \frac{\cos(kr - \frac{1}{2}l\pi)}{r} \ . \tag{47}$$

By virtue of this equation and Eq. (29), the Green's function $\hat{g}$ vanishes at large distances and requirements (IV'_a) and (IV'_b) are satisfied. As a result, requirement (IV) is satisfied by Eq. (32).

In conclusion, Eqs. (31) and (32) satisfy all four

requirements (I), (II), (III), and (IV), and do not have any shortcoming that prevents the Jost- or the K-matrix expressions from applying to complex problems.

The potential V in Eqs. (31) and (32) cannot only be local but also nonlocal. It may also be a complex potential. Thus we can expect many applications in the theory of nuclear reactions. For instance, the many level resonance formula is very easily obtained. For the elastic scattering, the on-shell t matrix reads

$$\langle j \mid t \mid j \rangle = \frac{\langle j \mid V\hat{\omega} \mid j \rangle}{\prod_{n=1}^{N} (1-\lambda_n) + ik \langle j \mid V\hat{\omega} \mid j \rangle} . \tag{48}$$

Owing to Eq. (40), we can express $1-\lambda_n$ approximately as

$$1-\lambda_n(E) \simeq (E-E_n) \left[-\frac{\partial \lambda_n(E)}{\partial E} \right]_{E=E_n} . \tag{49}$$

If we make use of this expression in Eq. (48), we obtain the many level formula in a compact form.

In the remainder of the present paper, we will not go into detailed discussions of these subjects any further but confine ourselves to the treatment of the three-body breakup process.

III. APPLICATION TO THREE-BODY BREAKUP

About twenty years ago, the three-body breakup process aroused enthusiasm in constructing an equation for this process in a manner that the kernel is compact. Equations proposed at that time by various authors turned out to be the same and are now called the Faddeev equation.[12] Faddeev has proved that the kernel of the Faddeev equation may be extended over the entire Banach space to a completely continuous operator after five iterations (Sec. 7 of Ref. 3). In practice, the Faddeev equation was solved for the breakup process by employing a separable potential (in momentum space),[13] by using the Padé approximant for a local potential (in momentum space),[14] or numerically in the form of a partial differential equation with a local potential (in coordinate space).[15] The PST is proposed in the present paper for the purpose of treating the breakup process in the form of an ordinary differential equation with a local potential in coordinate space. The authors believe that in this form the application to nuclear reactions is done most easily.

We have already solved a bound state problem of the three nucleon system by using the PST.[6,7] For a bound state problem, the kernel of the Faddeev integral equation is confined in a finite region in coordinate space. As a result, the numerical calculations are performed without difficulty. However, at energies for which the breakup process is possible, the kernel of the Faddeev equation extends up to infinitely large distance in coordinate space. This makes numerical calculations in coordinate space very difficult unless some clever method is introduced. In the method of the partial differential equation,[15] the asymptotic behavior of the wave function is imposed as the boundary condition in the six dimensional space, thus limiting the region of numerical calculations. However, if there exists a final state interaction between a pair of particles as in the case of a singlet nucleon interaction, the pair is subject to this interaction without regard to the distance between the spectator and the center of mass of this pair. As a result, we have to perform numerical calculations up to an infinitely large distance for this coordinate. Obviously, this is impossible. Therefore, we anticipate that when the energy of the system becomes so low that the effect of the final state interaction becomes important, the method of solving the partial differential equation will find itself faced by this difficulty.

The PST proposed in the present paper is aimed at overcoming this difficulty. The Faddeev equation is constructed in a manner that the two-body interactions take place successively between different pairs of particles.[12] If we decompose the wave function satisfying the Faddeev equation into the spectrum of the spectator,[16] we need to solve a coupled set of ordinary differential equations for the relative coordinate of the pair. Thereby, if we adopt for $G_0 t$ the form given by Eq. (24), we need calculate only in a finite region, since our Green's function $\hat{g}$ vanishes at large distances. The contribution from the asymptotic region can be calculated by a method that is going to be demonstrated in this section. Thus in our method, numerical integrations are confined in a finite region for the coordinate of a pair of particles. The effect of the final state interaction can be calculated precisely, since the integration over the spectator coordinate is done analytically as demonstrated in Ref. 6 or 7. [See also Eq. (70) of this paper.]

In the Faddeev theory, the total wave function Ψ of the three-body system is expressed as a superposition of three components $\psi^{(i)} (i = 1, 2, \text{and } 3)$;

$$\Psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} , \tag{50}$$

where $\psi^{(1)}$ represents the wave function in which a

pair of particles 2 and 3 are interacting by the potential $V_1(\equiv V_{23})$ in the final state, while the particle 1 stands as a spectator.

Although Faddeev introduced this decomposition by a mathematical reason that the kernel of the integral equation be compact, the physical meaning of this decomposition is understood in the following manner. Suppose that the particle 1 propagates after the final state interactions $V_2(\equiv V_{31})$ or $V_3(\equiv V_{12})$. In this case, the propagation of particle 1 is described by $\psi^{(2)}$ or $\psi^{(3)}$, and these two wave functions should interfere. So the total wave function in this case should be the sum $\psi^{(2)}+\psi^{(3)}$. Now, it is easily understood by symmetry that the total wave function of the three-body system should be given by Eq. (50).

If we designate by $F^{(1)}$ the initial wave for the Faddeev component $\psi^{(1)}$, the Faddeev equation is represented as a set of coupled integral equations;

$$\psi^{(1)}=F^{(1)}+G_0 t_1(\psi^{(2)}+\psi^{(3)}) , \tag{51}$$

and two other equations obtained by the cyclic permutations of 1, 2, and 3. For simplicity, we shall use the permutation operator Q defined by

$$Q\psi^{(1)}=\psi^{(2)}+\psi^{(3)} . \tag{52}$$

To solve Eq. (51), we reduce the dimensionality of the variables. For this purpose, we introduce a complete set of functions $\{\phi_{1,p}\}$, where $\phi_{1,p}$ is the product of the normalized plane wave $u_l(p,y_1)$ of the spectator 1 with energy E_p, y_1 being the distance between the spectator 1 and the center of mass of the interacting pair 23, and the orthonormalized spin-isospin-angular wave function $|\alpha_1\rangle$ of the total system

$$\phi_{1,p}=u_l(p,y_1)|\alpha_1\rangle , \tag{53}$$

where

$$u_l(p,y_1)=\sqrt{2/\pi}pj_l(py_1) . \tag{54}$$

(The function $\phi_{1,p}$ is denoted by $f_{3,p}$ in Ref. 16, where the spectator was taken as the particle 3.) The spectator energy E_p and the energy of the interacting pair E_q are related by

$$E_q=E-E_p . \tag{55}$$

While E_p runs from 0 to ∞, E_q takes a positive value so far as $E_p<E$ and becomes negative for $E_p>E$. With respect to the breakup, the former corresponds to the open channel and the latter to the closed channel.

In Ref. 16, we demonstrated how Eq. (51) is decomposed into the elastic channel, the closed channel, and the breakup channel. To save space, we do not reproduce the demonstration here. In this paper, we are concerned only with the treatment of the breakup channel [(10a) of Ref. 16] and assume that the treatment of other channels is already known.[17] We also neglect the antisymmetrization.[18]

If we let $\phi_0^{(1)}$ include all contributions from the elastic and closed channels [(6a), (10b), and (10c) of Ref. 16], Eq. (51) can be written as

$$\psi^{(1)}=\phi_0^{(1)}+\sum_\alpha \int_0^E dE_p\,|\phi_{1,p}\rangle G_{0,q}t_{1,q}\langle\phi_{1,p}|Q\psi^{(1)} , \tag{56}$$

where $G_{0,q}$ is the two-body Green's function for the interacting pair 23,

$$G_{0,q}=(E_q-K_{23}+i\epsilon)^{-1} . \tag{57}$$

The subscript q on $t_{i,q}$ indicates that it is calculated at E_q. For simplicity, we write Eq. (56) as

$$|\psi\rangle=|\phi\rangle+\sum_\alpha \int' dp\,|u_l(p)\rangle|\alpha\rangle G_{0,q}t_a(q)\langle\alpha|\langle u_l(p)|Q\psi\rangle , \tag{58}$$

where we have omitted the particle index 1, and written $t_a(q)$ in place of t_q. The subscript a denotes the spin and angular momentum of the interacting pair. If we ignore the spin, the subscript a stands simply for the angular momentum L of the pair. From Eq. (56) or (58), we see that the breakup amplitude for the channel 1 with momentum p and spin-isospin α in the final state is given by

$$T_{\alpha,p}^{(1,B)}=-\langle j_L(qx)|t_a(q)\langle\alpha|\langle u_l(p)|Q\Psi\rangle . \tag{59}$$

Let us express Eq. (31) in a simpler form of Eq. (3),

$$t_a(q)=V_a(A_a(q)+1) - V_a|B_a(q)\rangle F_a(q)\langle C_a(q)| , \tag{60}$$

where

$$A_a(q)=\hat{\omega}_a(q)-1=\hat{g}_a(q)V\hat{\omega}_a(q)\ , \tag{61a}$$

$$B_a(q)=\hat{\omega}_a(q)[\hat{\psi}_a(q,x)+iqj_L(qx)]\ , \tag{61b}$$

$$F_a(q)=\left[\prod_{n=1}^{N}(1-\lambda_n(q))+iq\langle j_L(qx)\,|\,V_a\hat{\omega}_a\,|\,j_L(qx)\rangle\right]^{-1}, \tag{61c}$$

and

$$C_a(q)=j_L(qx)V_a\hat{\omega}_a(q)\ . \tag{61d}$$

Further, we define a function $v_a(q)$ and an operator D by

$$v_a(q)=\langle C_a(q)\,|\,\langle\alpha\,|\,\langle u_l(p)\,|\,Q\psi\rangle\ , \tag{62}$$

and

$$D=\left[1-\sum_{\alpha}\int' dp\,Q\,|\,u_l(p)\rangle\,|\,\alpha\rangle A_a(q)\langle\alpha\,|\,\langle u_l(p)\,|\right]^{-1}. \tag{63}$$

If we use Eq. (58), it is readily seen that the function $v_a(q)$ satisfies a set of linear equations

$$v_a(q)+\sum_{\alpha'}\int' dp' M_{\alpha q,\alpha' q'}F_{a'}(q')v_{a'}(q')=\langle C_a(q)\,|\,\langle\alpha\,|\,\langle u_l(p)\,|\,DQ\,|\,\phi\rangle \tag{64}$$

with the matrix elements $M_{\alpha q,\alpha' q'}$ defined by

$$M_{\alpha q,\alpha' q'}=\langle C_a(q)\,|\,\langle\alpha\,|\,\langle u_l(p)\,|\,DQ\,|\,u_{l'}(p')\rangle\,|\,\alpha'\rangle\,|\,B_{a'}(q')\rangle\ . \tag{65}$$

If we use Eqs. (60) and (61) in Eq. (59), we can express the breakup amplitude as

$$T^{(1,B)}_{\alpha,p}=-F_a(q)v_a(q)\ . \tag{66}$$

This simple expression is very interesting. The factor $F_a(q)$ represents the pole of the final state interaction of the two-body subsystem involved in the three-body system.[19] For instance, for the singlet S state of the nuclear interaction, we may take $N=1$ and take the first Sturm-Liouville eigenstate as $\hat{\psi}$ of Eq. (25). For this state, the eigenvalue $\lambda_1(E)$ never becomes unity on the real k axis, but it takes values which are very close to one near zero energy.

Now, the problem of calculating the breakup amplitude is reduced to solve Eq. (64). The matrix elements $M_{\alpha q,\alpha' q'}$ in this equation are calculated by expanding the operator D of Eq. (63) into a power series in $A_a(q)$.

$$\begin{aligned}M_{\alpha q,\alpha' q'}=&\langle C_a(q)\,|\,\langle\alpha\,|\,\langle u_l(p)\,|\,Q\,|\,u_{l'}(p')\rangle\,|\,\alpha'\rangle\,|\,B_{a'}(q')\rangle\\&+\sum_{\alpha''}\int' dp''\langle C_a(q)\,|\,\langle\alpha\,|\,\langle u_l(p)\,|\,Q\,|\,u_{l''}(p'')\rangle\,|\,\alpha''\rangle A_{a''}(q'')\\&\times\langle\alpha''\,|\,\langle u_{l''}(p'')\,|\,Q\,|\,u_{l'}(p')\rangle\,|\,\alpha'\rangle\,|\,B_{a'}(q')\rangle+\cdots\ .\end{aligned} \tag{67}$$

A very important aspect of the present formulation is that the operator $A_a(q)$ vanishes at large distances due to Eq. (61a). This property is essential in the evaluation of matrix elements appearing on the right hand side of Eq. (67). Let us first take the first order term. The operator

$$\langle C_a(q)\,|\,\langle\alpha\,|\,\langle u_l(p)\,|\,Q\,|\,u_{l''}(p'')\rangle\,|\,\alpha''\rangle A_{a''}(q'')$$

in this term is obviously bounded and numerical calculation is feasible. Note that the function $C_a(q)$ is short ranged due to the potential V_a in Eq. (61d). The operator

$$A_{a''}(q'')\langle\alpha''\,|\,\langle u_{l''}(p'')\,|\,Q\,|\,u_{l'''}(p''')\rangle\,|\,\alpha'''\rangle A_{a'''}(q''')$$

needed in higher order terms is similar in nature. On the other hand, the function

$$A_{a''}(q'')\langle\alpha''\,|\,\langle u_{l''}(p'')\,|\,Q\,|\,u_{l'}(p'),\,|\,\alpha'\rangle\,|\,B_{a'}(q')\rangle$$

required in the first and higher order terms is similar in nature to the zeroth order term

$$\langle C_a(q) \mid \langle \alpha \mid \langle u_l(p) \mid Q \mid u_{l'}(p') \rangle \mid \alpha' \rangle \mid B_{a'}(q') \rangle ,$$

owing to the presence of the function $B_{a'}(q')$ that involves a contribution from large distance.

By Eq. (47) and due to the property of the operator $\hat{\omega}$ that it tends to 1 asymptotically, the functions $\mid B_a(q) \rangle$ and $\mid C_a(q) \rangle$ behave at large distances as $\mid h_L^{(+)}(qx) \rangle$ and $\mid V(x) j_L(qx) \rangle$, respectively. Since other terms of $\mid B_a(q) \rangle$ and $\mid C_a(q) \rangle$ vanish at large distances, it will be sufficient to study the behavior of

$$I_1 = \langle j_L(qx) V \mid \langle \alpha \mid \langle u_l(p) \mid Q \mid u_{l'}(p') \rangle \mid \alpha' \rangle \mid j_{L'}(q'x) \rangle \tag{68}$$

and

$$I_2 = \langle j_L(qx) V \mid \langle \alpha \mid \langle u_l(p) \mid Q \mid u_{l'}(p') \rangle \mid \alpha' \rangle \mid -n_{L'}(q'x) \rangle . \tag{69}$$

As described in a previous article,[6] the overlap integral of the spectator functions and the spin-angular functions

$$\langle \alpha \mid \langle u_l(p) \mid Q \mid u_{l'}(p') \rangle \mid \alpha' \rangle$$

is expressed together with the function $j_{L'}(q'x')$ as

$$\begin{aligned} \langle \alpha \mid \langle u_l(p) \mid Q \mid u_{l'}(p') \rangle \mid \alpha' \rangle \mid j_{L'}(q'x) \rangle &= pp' \int_{-1}^{1} du\, j_L(\lambda x) \Lambda_{pp'}(\alpha\alpha'; u) \int_0^{\infty} x'^2 dx'\, j_{L'}(\lambda_1 x') j_{L'}(q'x') \\ &= pp' \int_{-1}^{1} du\, j_L(\lambda x) \Lambda_{pp'}(\alpha\alpha'; u) \frac{\pi}{2} \frac{\delta(\lambda_1 - q')}{\lambda_1 q'} , \end{aligned} \tag{70}$$

where $u = \cos\theta_{\vec{p}\hat{\ }\vec{p}'}$ and

$$\vec{\lambda} = \vec{p}/2 + \vec{p}', \quad \vec{\lambda}_1 = -(\vec{p} + \vec{p}'/2) . \tag{71}$$

The function $\Lambda_{pp'}(\alpha\alpha'; u)$ denotes the overlap of the spin-angular wave functions, explicitly given by Ref. 6, Eq. (31). The explicit form of this function is not important in the present discussion. After some calculations using Eq. (71), we express the δ function in Eq. (70) as

$$\delta(\lambda_1 - q') = \frac{2q'}{pp'} \delta\left[u - \frac{1}{pp'}(q'^2 - p^2 - p'^2/4) \right] . \tag{72}$$

Then, performing the u integration, we obtain

$$\langle \alpha \mid \langle u_l(p) \mid Q \mid u_{l'}(p') \rangle \mid \alpha' \rangle \mid j_{L'}(q'x) \rangle = \pi/q' j_L \left\{ \left[q'^2 + \tfrac{3}{4}(p'^2 - p^2) \right]^{1/2} x \right\} \Lambda_{pp'} \left[\alpha\alpha'; \frac{1}{pp'}(q'^2 - p^2 - p'^2/4) \right] . \tag{73}$$

Now it is evident that the integral I_1 can be numerically calculated.

Similarly, the integral I_2 is expressed as

$$\begin{aligned} \langle \alpha \mid \langle u_l(p) \mid Q \mid u_{l'}(p') \rangle \mid \alpha' \rangle \mid -n_{L'}(q'x) \rangle &= -pp' \int_{-1}^{1} du\, j_L(\lambda x) \Lambda_{pp'}(\alpha\alpha'; u) \int_0^{\infty} x'^2 dx'\, j_{L'}(\lambda_1 x') n_{L'}(q'x') \\ &= pp' \int_{-1}^{1} du\, j_L(\lambda x) \Lambda_{pp'}(\alpha\alpha'; u) q'^{-1} (\lambda_1/q')^{L'} \mathbf{P} \left[\frac{1}{\lambda_1^2 - q'^2} \right] , \end{aligned} \tag{74}$$

where we have made use of the formula given by Fuda[20]

$$\int_0^{\infty} x^2 j_l(px) h_l^{(+)}(kx) dx = \frac{k^{-1}(p/k)^l}{p^2 - (k^2 + i\epsilon)} . \tag{75}$$

The principal-value integral over u can be calculated without any difficulty if the pole of $(\lambda_1^2 - q'^2)^{-1}$ is between the upper and the lower boundary of the integral, where

$$-pp' < 2E - p^2 - \tfrac{7}{4}p'^2 < pp' \tag{76}$$

is satisfied. Here, we have used Eq. (55)

$$E = \tfrac{3}{4}p'^2 + \tfrac{1}{2}q'^2, \quad (m = \hbar = 1) . \tag{77}$$

On the other hand, if momentum p' satisfies the relation

$$2E-p^2-\tfrac{7}{4}p'^2=pp' \text{ or } -pp' , \tag{78}$$

the pole of $(\lambda_1^2-q'^2)^{-1}$ is on a boundary of the u integral. In this case the integral over u of Eq. (74) gives rise to a logarithmic divergence. However, the p' integral in Eq. (64) makes the contribution from this special point null. Therefore, we do not need to worry about this singularity. In conclusion, we can calculate Eq. (64) numerically and obtain the breakup amplitude by Eq. (66).

Finally, we mention that the discussions in Sec. III may amount to a proof for the existence of the solution of the Faddeev equation, although we do not claim any mathematical rigor. Because our kernel is confined in a finite region in coordinate space, we might be able to prove the existence of a unique solution of the Faddeev equation without going as far as five iterations. Without proof, it is almost clear that such a kernel, once iterated, should be compact in the three-body Banach space.

IV. SUMMARY

We have given an expression of the t matrix that is very useful in applications. It consists of one separable and one nonseparable term [Eq. (31)]. The nonseparable term does not involve any pole on the real momentum axis [Eqs. (29), (30), and (37)]. All poles are included in the separable term in a nondivergent manner [See

$$\prod_{n=1}^{N}(1-\lambda_n)$$

in Eq. (31)]. Both the separable and the nonseparable terms of the kernel G_0t [Eq. (32)] are regular at the origin. The Green's function $\hat{g}$ used in the t matrix vanishes at large distances from the origin. [See Eqs. (29) and (47).] As a result, the nonseparable term of the kernel [$\hat{\omega}-1$ of Eq. (32)] vanishes at large distances and the separable term behaves asymptotically as $h_L^{(+)}(kr)$. With these properties, the present expression of the t matrix is more useful in applications than the Jost- or the K-matrix expressions. The applications are suggested to the many level formula [Eq. (48)] and the three-body breakup problem (Sec. III). It is demonstrated that we can calculate Eq. (64) without any difficulty, and obtain the breakup amplitude expressed in a simple and physically interesting form [Eq. (66)].

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APPENDIX

In this Appendix, we prove that for the Sturm-Liouville function defined by Eq. (25), with Eqs. (26), (27), and (28), the formulas (33), (34), and (46) are valid.

First, we prove Eq. (46)

$$\sum_{n=1}^{N}\alpha_n^{(N)}=1 ,$$

where $\alpha_n^{(N)}$ is given by Eq. (28). Let us introduce the identity

$$f(\lambda)\equiv\frac{\lambda^N}{\prod_{m=1}^{N}(\lambda-\lambda_m)}$$

$$=1+\sum_{n=1}^{N}\frac{\lambda_n^N}{\lambda-\lambda_n}\prod_{m\neq n}^{N}\frac{1}{(\lambda_n-\lambda_m)} . \tag{A1}$$

If we put $\lambda=0$ in Eq. (A1), we immediately obtain Eq. (46). The proof of Eq. (A1) is done as follows. Let us define a constant f_n by

$$f(\lambda)=1+\frac{\lambda^N-\prod_{m=1}^{N}(\lambda-\lambda_m)}{\prod_{m=1}^{N}(\lambda-\lambda_m)}$$

$$=1+\sum_{n=1}^{N}\frac{f_n}{\lambda-\lambda_n} . \tag{A2}$$

The constant f_n is calculated by taking λ as a complex number.

$$f_n=\frac{1}{2\pi i}\oint_{|\lambda-\lambda_n|<\epsilon}\left[\frac{\lambda^N-\prod_{m=1}^{N}(\lambda-\lambda_m)}{\prod_{m=1}^{N}(\lambda-\lambda_m)}\right]d\lambda$$

$$=\frac{\lambda_n^N}{\prod_{m\neq n}^{N}(\lambda_n-\lambda_m)} . \tag{A3}$$

As a corollary, we get from (A1) an equation

$$\lambda^N=\prod_{m=1}^{N}(\lambda-\lambda_m)+\sum_{n=1}^{N}\frac{\lambda_n^N}{\prod_{m\neq n}^{N}(\lambda_n-\lambda_m)}\prod_{k\neq n}^{N}(\lambda-\lambda_k) . \tag{A4}$$

Next, we prove Eq. (34),

$$(\hat{g}V)^l \mid \psi_n \rangle = 0 ,$$

for $l \geq N$ and $n = 1, \ldots, N$. If we prove

$$(\hat{g}V)^N \mid \psi_n \rangle = 0 \tag{A5}$$

for $n = 1, \ldots, N$, then Eq. (34) is true. Since the functions $\hat{\psi}_n$ and ψ_n $(n = 1, \ldots, N)$ in the text differ only in the normalization factors, we prove

$$(\hat{g}V)^N \mid \hat{\psi}_n \rangle = 0 \tag{A6}$$

for $n = 1, \ldots, N$.

By the definition of $\hat{g}$, Eq. (29) with Eqs. (26) and (27),

$$\hat{g}V \mid \hat{\psi}_n \rangle = (PG_0 + \mid \hat{\psi} \rangle \langle j \mid) V \mid \hat{\psi}_n \rangle = \sum_{m=1}^{N} C_{nm} \mid \hat{\psi}_m \rangle , \tag{A7}$$

where C_{nm} is given by

$$C_{nm} = \lambda_n \delta_{nm} - \lambda_n \alpha_m^{(N)} , \quad n, m = 1, \ldots, N . \tag{A8}$$

We form an $N \times N$ matrix $\underline{C}$ from the set $\{C_{nm}\}$. In general,

$$(\hat{g}V)^k \mid \hat{\psi}_n \rangle = \sum_{m=1}^{N} (\underline{C}^k)_{nm} \mid \hat{\psi}_m \rangle . \tag{A9}$$

Therefore, to prove Eq. (A6), we have to demonstrate that all matrix elements $(\underline{C}^N)_{nm}$ vanish.

When $N = 1$, $(\underline{C}^1)_{nm} = 0$, because $\alpha_1^{(1)} = 1$ from Eq. (28) or Eq. (46). For $N \geq 2$, we introduce a determinant

$$\phi^{(n)}(\lambda) = \mid \lambda \underline{1} - \underline{C} \mid ,$$

$$\phi^{(1)}(\lambda) = (\lambda - \lambda_1) + \lambda_1 \alpha_1 , \tag{A10}$$

or explicitly,

$$\phi^{(n)}(\lambda) = \begin{vmatrix} (\lambda-\lambda_1)+\lambda_1\alpha_1 & \lambda_1\alpha_2 & \cdots & \lambda_1\alpha_{n-1} & \lambda_1\alpha_n \\ \lambda_2\alpha_1 & (\lambda-\lambda_2)+\lambda_2\alpha_2 & & \lambda_2\alpha_{n-1} & \lambda_2\alpha_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_{n-1}\alpha_1 & \lambda_{n-1}\alpha_2 & & (\lambda-\lambda_{n-1})+\lambda_{n-1}\alpha_{n-1} & \lambda_{n-1}\alpha_n \\ \lambda_n\alpha_1 & \lambda_n\alpha_2 & \cdots & \lambda_n\alpha_{n-1} & (\lambda-\lambda_n)+\lambda_n\alpha_n \end{vmatrix} \tag{A11}$$

for $n = 2, 3, \ldots, N$. In Eq. (A11), we have suppressed the index (N) of $\alpha_i^{(N)}$ $(i = 1, \ldots, n)$, for simplicity. By the subtraction,

$$(\text{ the } n\text{th row}) - [\text{ the } (n-1)\text{-th row}] \times (\lambda_n / \lambda_{n-1}) ,$$

we rewrite Eq. (A11) as

$$\phi^{(n)}(\lambda) = \begin{vmatrix} (\lambda-\lambda_1)+\lambda_1\alpha_1 & \lambda_1\alpha_2 & \lambda_1\alpha_3 & \cdots & \lambda_1\alpha_{n-1} & \lambda_1\alpha_n \\ -(\lambda-\lambda_1)\dfrac{\lambda_2}{\lambda_1} & \lambda-\lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & -(\lambda-\lambda_2)\dfrac{\lambda_3}{\lambda_2} & \lambda-\lambda_3 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \lambda-\lambda_{n-1} & 0 \\ 0 & 0 & 0 & -(\lambda-\lambda_{n-1})\dfrac{\lambda_n}{\lambda_{n-1}} & \lambda-\lambda_n \end{vmatrix}$$

$$= (\lambda - \lambda_n)\phi^{(n-1)}(\lambda) + \alpha_n \lambda_n \prod_{m=1}^{n-1} (\lambda - \lambda_m)$$

$$= (\lambda - \lambda_n)(\lambda - \lambda_{n-1})\phi^{(n-2)}(\lambda) + \sum_{k=n-1}^{n} \alpha_k \lambda_k \prod_{m \neq k}^{n} (\lambda - \lambda_m)$$

$$= (\lambda - \lambda_n) \ldots (\lambda - \lambda_2)\phi^{(1)} + \sum_{k=2}^{n} \alpha_k \lambda_k \prod_{m \neq k}^{n} (\lambda - \lambda_m)$$

$$= \prod_{m=1}^{n} (\lambda - \lambda_m) + \sum_{k=1}^{n} \alpha_k \lambda_k \prod_{m \neq k}^{n} (\lambda - \lambda_m) . \tag{A12}$$

Therefore, if we use (A4), we obtain

$$\phi^{(N)}(\lambda)=\lambda^N . \tag{A13}$$

Now we recall the Cayley-Hamilton theorem that if

$$f(\lambda)\equiv|\lambda\underline{1}-\underline{C}|\ , \tag{A14}$$

then

$$f(\underline{C})=0 . \tag{A15}$$

As a direct consequence of this theorem, we obtain from Eq. (A13) that

$$\underline{C}^N=0 . \tag{A16}$$

This means that all matrix elements $(\underline{C}^N)_{nm}$ (n, $m=1,\ldots,N$) vanish and hence Eqs. (A6), (A5), and (34) are valid.

Before we prove Eq. (33), we demonstrate a corollary that if we define W_{nm} by

$$W_{nm}=\begin{cases}\displaystyle\prod_{k\neq n}^{N}(1-\lambda_k)+\sum_{k\neq n}^{N}\lambda_k\alpha_k\prod_{l\neq n,k}^{N}(1-\lambda_l), & \text{for } n=m\\ \displaystyle-\lambda_n\alpha_m\prod_{l\neq n,m}^{N}(1-\lambda_l), & \text{for } n\neq m\ ,\end{cases} \tag{A17}$$

then the matrix $\underline{W}$ is the inverse matrix of $(\underline{1}-\underline{C})$.

This is proved if we use Eq. (A8) for the matrix element of $(\underline{1}-\underline{C})$.

$$\begin{aligned}[(\underline{1}-\underline{C})W]_{nm}&=\sum_{k=1}^{N}[\delta_{nk}(1-\lambda_n)+\lambda_n\alpha_k]W_{km}\\&=(1-\lambda_n)W_{nm}+\lambda_n\sum_{k=1}^{N}\alpha_k W_{km}\\&=(1-\lambda_n)W_{nm}+\lambda_n\alpha_m\prod_{k\neq m}^{N}(1-\lambda_k) .\end{aligned} \tag{A18}$$

Here, we have used Eq. (A17) in the second term on the right hand side. If we again use Eq. (A17) in the first term, we get for the diagonal element

$$\begin{aligned}[(\underline{1}-\underline{C})W]_{nm}&=\sum_{k=1}^{N}(1-\lambda_k)\\&\quad+\sum_{k=1}^{N}\lambda_k\alpha_k\prod_{l\neq k}^{N}(1-\lambda_l)=1 ,\end{aligned} \tag{A19}$$

where we have used the relation obtained by putting $\lambda=1$ in Eq. (A4). If we use Eq. (A17) in Eq. (A18), we readily see that the nondiagonal matrix elements vanish. As a result, the matrix $\underline{W}$ is the inverse matrix of $(\underline{1}-\underline{C})$.

Now we prove Eq. (33),

$$1+\langle j\,|\,V\hat{\omega}\,|\,\hat{\psi}\rangle=\prod_{n=1}^{N}(1-\lambda_n) .$$

If we use Eq. (A9), the matrix element $\langle j\,|\,V\hat{\omega}\,|\,\hat{\psi}_n\rangle$ reads

$$\begin{aligned}\langle j\,|\,V\hat{\omega}\,|\,\hat{\psi}_n\rangle&=\langle j\,|\,V\frac{1}{1-\hat{g}V}\,|\,\hat{\psi}_n\rangle\\&=\langle j\,|\,V\sum_{m=1}^{N}(\underline{1}-\underline{C})_{nm}{}^{-1}\,|\,\hat{\psi}_m\rangle .\end{aligned} \tag{A20}$$

If we use Eq. (A17) for $(\underline{1}-\underline{C})_{nm}{}^{-1}$, and Eq. (26), we get

$$\langle j\,|\,V\hat{\omega}\,|\,\hat{\psi}_n\rangle=-\lambda_n\prod_{k\neq n}^{N}(1-\lambda_k) . \tag{A21}$$

As a result, we obtain

$$\begin{aligned}1+\langle j\,|\,V\hat{\omega}\,|\,\hat{\psi}\rangle&=1-\sum_{n=1}^{N}\alpha_n\lambda_n\prod_{k\neq n}^{N}(1-\lambda_k)\\&=\prod_{n=1}^{N}(1-\lambda_n) ,\end{aligned} \tag{A22}$$

where we have made use of the relation obtained from Eq. (A4) by putting $\lambda=1$.

[1]A. N. Mitra, Phys. Rev. 127, 1342 (1962); Nucl. Phys. 32, 529 (1962); R. D. Amado, Phys. Rev. 132, 485 (1963); R. Aaron, R. D. Amado, and Y. Y. Yam, *ibid.* 136, B650 (1964); 140, B1291 (1965); R. Aaron and R. D. Amado, *ibid.* 150, 857 (1966); C. Lovelace, *ibid.* 135, B1225 (1964); A. G. Sitenko and V. F. Kharchenko, Nucl. Phys. 49, 15 (1963).

[2]Y. Yamaguchi, Phys. Rev. 95, 1628 (1954); Y. Yamaguchi and Y. Yamaguchi, *ibid.* 95, 1635 (1954).

[3]L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in Quantum Scattering Theory,* translated from Russian, the Israel Program for Scientific Translations (Davey, New York, 1965).

[4]P. Noyes, Phys. Rev. Lett. 15, 538 (1965); K. L. Kowalski, *ibid.* 15, 798 (1965); M. Bolsterli and J. MacKenzie, Physics 2, 141 (1965); D. Y. Wong and G. Zambiotti, Phys. Rev. 154, 1540 (1967); V. F. Kharchenko and N. M. Petrov, Nucl. Phys. A137, 417 (1967); F. Tabakin, Phys. Rev. 174, 1208 (1968); M. G. Fuda, *ibid.* 160, 1064 (1968); T. A. Osborn, Nucl. Phys. A138, 305 (1969); J. S. Levinger, A. H. Lu, and R. Stagt, Phys. Rev. 179, 926 (1969); E. Harms, Phys. Rev. C 1, 1667 (1970); T. Sasakawa, Nucl. Phys. A160, 321 (1971); E. Harms and L. Laroze, *ibid.* A160, 449 (1971); V. F. Kharchenko, S. A. Strozhenko, and V. E. Kuzmichev, *ibid.* A188, 609 (1972); R. P. Bhatia, and J. F. Walker, *ibid.* A192, 658 (1972); A212, 214 (1973); D. J. Ernst, C. M. Shakin, and R. M. Thaler, Phys. Rev. C 8, 46 (1973); 9, 1780 (1974); S. C. Pieper, *ibid.* 9, 883 (1974); S. K. Adhikari, *ibid.* 10, 1623 (1974); S. Oryu, T. Ishihara, and S. Shioyama, Prog. Theor. Phys. 51, 1626 (1974); S. Oryu, *ibid.* 52, 550 (1974); S. K. Adhikari and I. H. Sloan, Nucl. Phys. A241, 429 (1975); A251, 297 (1975); Phys. Rev. C 11, 1133 (1975); K. H. Yang, W. J. Gerace, and J. F. Walker, Nucl. Phys. A240, 189 (1975); S. Oryu, M. Araki, and S. Satoh, Prog. Theor. Phys. Suppl. 61, 199 (1977).

[5]T. A. Osborn, J. Math. Phys. 14, 373 (1973).

[6]T. Sasakawa and T. Sawada, Phys. Rev. C 19, 2035 (1979).

[7]T. Sasakawa, H. Okuno, and T. Sawada, Phys. Rev. C 23, 905 (1981).

[8]T. Sasakawa and T. Sawada, Prog. Theor. Phys. Suppl. 61, 1 (1977).

[9]T. Sasakawa, J. Horáček, and T. Sawada, Science Report of Tohoku University Series 8 1, 119 (1980).

[10]T. Sasakawa and T. Sawada, Phys. Rev. C 11, 87 (1975).

[11]We do not use the Weinberg function [Phys. Rev. 130, 766 (1963)]. The reason is the following: If we use this set of functions in the treatment of the three-body problem in coordinate space, we need to calculate the matrix elements between the Weinberg functions for different energies. However, since the normalization factor of the Weinberg function is a complex number, the matrix element can sometimes be very large. Obviously, this is inconvenient in view of the stability of the numerical calculations. The Sturm-Liouville function defined by Eq. (25) was first introduced by R. Huby [Nucl. Phys. A138, 442 (1969)] and T. Sasakawa [Nucl. Phys. A160, 321 (1971)].

[12]In addition to Ref. 1 early articles include L. D. Faddeev, Zh. Eksp. Teor. Fiz. 39, 1459 (1960) [Sov. Phys.-JETP 12, 1014 (1960)]; L. Eyges, J. Math. Phys. 6, 1320 (1966); and E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. B2, 167 (1967).

[13]J. H. Hetherington and L. H. Schick, Phys. Rev. 137, B935 (1965); 139, B1164 (1965); 141, 1314 (1966); P. E. Shanley, *ibid.* 187, 1328 (1969); R. T. Cahill and I. H. Sloan, Nucl. Phys. A165, 161 (1971); Y. Avishai, Phys. Rev. D 3, 3232 (1971); 4, 400 (1971); W. Ebenhöh, Nucl. Phys. A191, 97 (1972); P. Dolleshall, *ibid.* A201, 264 (1973); A220, 491 (1974); J. Bruinsma and R. van Wageningen, *ibid.* A282, 1 (1977).

[14]J. A. Tjon, Phys. Rev. D 1, 2109 (1970); W. M. Kloet and J. A. Tjon, Ann. Phys.(N.Y.) 79, 407 (1973); Nucl. Phys. A210, 380 (1973).

[15]C. Gignoux, A. Laverne, and S. P. Merkuriev, Phys. Rev. Lett. 33, 1350 (1974); J. J. Benayoun, J. Chauvin, C. Gignoux, and A. Laverne, *ibid.* 36, 1438 (1976); S. P. Merkuriev, C. Gignoux, and A. Laverne, Ann. Phys. (N.Y.) 99, 30 (1976).

[16]T. Sasakawa, Phys. Rev. C 17, 2015 (1978). See also Ref. 9.

[17]The closed channel can be treated by a technique for the bound state. The elastic channel in the Faddeev equation takes the form of the scattering from a nonlocal potential. Methods for treating a nonlocal potential were studied in T. Sasakawa and T. Sawada, Phys. Rev. C 18, 96 (1978); Prog. Theor. Phys. Suppl. 61, 42 (1977); and in Ref. 9.

[18]T. Sasakawa and T. Sawada, Phys. Rev. C 22, 320 (1980).

[19]Equation (66) is a rigorous expression for the phenomenological theory of the final state interaction by A. B. Migdal [Sov. Phys.-JETP 1, 2 (1955)] and K. M. Watson [Phys. Rev. 88, 1163 (1952)].

[20]M. G. Fuda, Phys. Rev. C 14, 37 (1976).