Meson wave functions in the meson-nucleon shell model

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When the meson wave functions in the meson-nucleon shell model are chosen by analogy with the nucleon static model, direct effects of external mesons on the ground-state energy are minimized, and the Hartree approximation is realized in a particularly simple way.

[NUCLEAR STRUCTURE Pion wave functions.]

I. INTRODUCTION

The nuclear shell model was developed as a means of treating systems of nucleons interacting via two-nucleon forces. In the seventies, it became fashionable to attempt to treat directly the meson fields that are believed to give rise to the nucleon-nucleon forces; mean-field theory was applied to the ground configuration of meson-nucleon systems and results resembling the results of a Hartree-Fock procedure were obtained for systems of nucleons and scalar or vector mesons[1–4]; sometimes Hartree procedures resulted.[5,6] The mean-field procedure is only effective for treating fields with nonvanishing expectation values. Since the pion field has no expectation value in even-even nuclei, it contributes to the mean-field calculations only a repulsive self-energy term.[2,4,7] On the other hand, the work of the Paris group[8] indicates that the exchange of two pions gives rise to an attractive intermediate-range contribution to the nucleon-nucleon potential. Mean-field theory applied to a system of pions and nucleons cannot generate the equivalent of this intermediate-range attraction.

A recent article[9] describes how the mean-field procedure can be extended so as to effectively include fields such as the pion field (and the gluon field) that have vanishing expectation values. The extension is a meson-nucleon shell model (MNSM) in which the nucleon field is expanded in the usual way in a set of orthonormal modes or single-particle wave functions and, in addition, the meson field is also expanded in an appropriate set of orthonormal modes. In the usual nuclear shell model (NSM), the model space is restricted by requiring that the nucleons occupy a limited set of single-particle orbitals. The ground-state energy in the NSM is a functional of the nucleon mode functions, and setting to zero the functional derivative of the ground-state energy functional with respect to each of the nucleon mode functions gives a set of equations to be solved self-consistently for the mode functions. When the model space is a single Slater determinant, the self-consistent equations are the Hartree-Fock equations. In Ref. 9 a similar variational procedure was used to derive self-consistent equations for the nucleon mode functions and the meson mode functions in the MNSM.

This paper presents an improved procedure for determining the mesonic mode functions. The new method has the advantage that the meson mode functions are given explicitly as functionals of the nucleon mode functions and so do not have to be determined self-consistently; only the nucleon orbitals are self-consistent equations. Also, the connection with the mean-field procedure is strengthened, as the meson fields there are likewise given explicitly as functionals of the nucleon orbitals. Finally, the Hartree approximation is particularly transparent when the preferred meson wave functions are used. The new method for determining the meson orbitals is an extension of one previously developed[10] for meson orbitals in a simpler static model.

II. STATIC MODEL FOR THE MESON-NUCLEAR SYSTEM

Consider a system of nucleons and mesons with Yukawa interaction of the nucleon and meson fields. A noncovariant formulation is used here. The meson field is assumed to be invariant under space rotations; in covariant terminology, it can be a scalar or pseudoscalar field or the zeroth component of a vector or pseudovector field. Let the nucleon field be $\Psi(\vec{x})$ with Fourier transform $\tilde{\Psi}(\vec{p})$ and the meson field be $\Phi_\alpha(\vec{x})$ with the associated annihilation operator $a_\alpha(\vec{k})$; the index α is the isospin or color or other nonrotational symmetry index of the meson field and will be understood to be sub-

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ject to the usual summation convention in the following. The Hamiltonian is

$$\begin{aligned} H &= T_F + T_B + H_I + H_I^\dagger , \\ T_F &= \int \tilde{\Psi}^\dagger(\vec{p}) t(p) \tilde{\Psi}(\vec{p}) d\vec{p} , \\ T_B &= \int \omega(k) a_\alpha^\dagger(\vec{k}) a_\alpha(\vec{k}) d\vec{k} , \\ H_I &= -\int a_\alpha^\dagger(\vec{k}) \hat{S}_\alpha(\vec{k}) d\vec{k} , \\ \hat{S}_\alpha(\vec{k}) &= Y(k) \int \tilde{\Psi}^\dagger(\vec{p}) W_\alpha \left[\vec{k}, \frac{\vec{p}+\vec{q}}{2} \right] \\ &\quad \times \tilde{\Psi}(\vec{q}) \delta(\vec{k}+\vec{p}-\vec{q}) d\vec{p} d\vec{q} , \end{aligned} \tag{1}$$

where $t(p)$ and $\omega(k)$ are the energies of a free nucleon of momentum $\vec{p}$ and a free meson of momentum $\vec{k}$, respectively, and $Y(\vec{k})$ and $W_\alpha(\vec{k},\vec{K})$ are the form factors that characterize the particular Yukawa interaction; W_α represents the operator for the nucleon current that interacts with the field Φ_α, while Y is the factor that comes from the relation between $\Phi_\alpha(\vec{x})$ and $a_\alpha(\vec{k})$; representative forms for W and Y are given in Refs. 4 , 11, and 12.

First, rotational functions are used to expand $\hat{S}_\alpha$ and a_α

$$\begin{aligned} \hat{S}_\alpha(\vec{k}) &= \sum_{lm} \frac{Y_{l,m}(\hat{k})}{k} \hat{S}_{\alpha lm}(k) , \\ a_\alpha(\vec{k}) &= \sum_{lm} \frac{Y_{l,m}(\hat{k})}{k} a_{\alpha lm}(k) , \end{aligned} \tag{2}$$

so that H_I becomes

$$H_I = -\sum_{\alpha lm} \int_0^\infty a_{\alpha lm}^\dagger(k) \hat{S}_{\alpha lm}(k) dk . \tag{3}$$

Now let $\{f_{i\mu}\}$ be a complete set of orthonormal functions and expand Ψ

$$\tilde{\Psi}(\vec{p}) = \sum_{i=1}^{\infty} \sum_{\mu} B_{i\mu} \tilde{f}_{i\mu}(\vec{p}) , \tag{4}$$

where μ represents the isospin and angular momentum m values; i represents the angular momentum, isospin, and radial quantum numbers; and $B_{i\mu}$ is the annihilation operator for a nucleon in the state $i\mu$. Then $\hat{S}_{\alpha lm}(k)$ must take the form

$$\begin{aligned} \hat{S}_{\alpha lm}(k) &= \sum_{ijl} v_{ijl}(k) \rho_{\alpha m}^{ijl} , \\ \rho_{\alpha m}^{ijl} &= \{B_i^\dagger, B_j\}_{\alpha m}^{l} , \end{aligned} \tag{5}$$

where the curly braces in Eq. (5) are used to indicate coupling of the angular momentum and isospin parameters of the enclosed $B^\dagger$ and B operators to give total angular momentum lm and isospin index α and the same isospin representation as a_α. With these substitutions H_I takes the form

$$\begin{aligned} H_I &= -\sum_{ijl} \rho^{ijl} \cdot \int_0^\infty v_{ijl}(k) a_l^\dagger(k) dk \\ &\equiv -\sum_{ijl} \sum_{\alpha m} \rho_{\alpha m}^{ijl} \int_0^\infty v_{ijl}(k) a_{\alpha lm}^\dagger(k) dk . \end{aligned} \tag{6}$$

Now, as in the NSM, restrict the nucleon states to a limited number N of orbital quantum numbers, i, each with its full set of

$$(2J_i+1)(2T_i+1)$$

degenerate substates. Then the Hamiltonian is

$$\begin{aligned} H_N &= T_{NF} + T_B + H_{NI} + H_{NI}^\dagger , \\ T_{NF} &= \sum_{ij=1}^{N} \sum_{\mu} t_{ij} B_{i\mu}^\dagger B_{j\mu} \\ &= \sum_{ij=1}^{N} t'_{ij} \rho^{ij0} , \\ T_B &= \sum_{l} \int_0^\infty \omega(k) a_l^\dagger(k) \cdot a_l(k) dk , \\ H_{NI} &= -\sum_{ij=1}^{N} \sum_{l} \rho^{ijl} \cdot \int_0^\infty v_{ijl}(k) a_l^\dagger(k) dk . \end{aligned} \tag{7}$$

The operators ρ^{ijl} generate a finite algebra, so that for fixed nucleon orbitals $f_{i\mu}$ Eqs. (7) represent a static model for several mesons $a_l(k)$, each interacting with sources ρ^{ijl} through form factors $v_{ijl}(k)$.

At this point it is time to generalize the notation so as to explictly include the parity of the field as well as the representation of the nonrotational symmetry group (isospin rotation group) to which it belongs. Instead of a_l or ρ^l, the operator will be written a_r or ρ^r, where r stands for (t,l,π), and l and π are the angular momentum and parity of the rotationally scalar field and t is its isospin or other symmetry group representation parameter. The 0 representation will be used for $(0,0,+)$; the 0 representation of any group is the one-dimensional one. In this notation, ρ^{ijr} is given by

$$\rho^{ijr} = \{B_i^\dagger, B_j\}^r , \tag{8}$$

and H_N becomes

$$\begin{aligned} H_N &= T_{NF} + T_B + H_{NI} + H_{NI}^\dagger , \\ T_{NF} &= \sum_{ij=1}^{N} t'_{ij} \rho^{ij0} , \\ T_B &= \sum_{r} \int_0^\infty \omega_r(k) a_r^\dagger(k) \cdot a_r(k) dk , \\ H_{NI} &= -\sum_{ij=1}^{N} \sum_{r} \rho^{ijr} \cdot \int_0^\infty v_{ijr}(k) a_r^\dagger(k) dk . \end{aligned} \tag{9}$$

III. MESON WAVE FUNCTIONS IN THE STATIC MODEL

Now it is the turn of the meson field to be expanded in terms of a set of orthonormal functions. In Ref. 10 it was shown that in the case of a single source form factor $v(k)$, the appropriate expansion functions are $v(k)/\omega(k)$ and an arbitrary set of functions orthogonal to it. The analogous set in the present case is obtained by first orthogonalizing the functions

$$v_{ijr}(k)/\omega_r(k)$$

for each r by a unitary transformation:

$$v_{r\nu}(k)=\sum_{ij=1}^{N} u^r_{\nu,ij}v_{ijr}(k) \,,$$
$$\int_0^\infty \frac{v^*_{r\nu}(k)v_{r\mu}(k)}{\omega_r{}^2(k)}dk=G_{r\nu}{}^2\delta_{\nu,\mu} \,. \tag{10}$$

For each r there are M_r orthonormal functions $\phi_{r\nu}(k)$, one for each nonzero $G_{r\nu}$

$$\phi_{r\nu}(k)=\frac{v_{r\nu}(k)}{G_{r\nu}\,\omega_r(k)} \,; \tag{11}$$

functions χ_{ri} orthogonal to these and relatively orthonormal can be chosen so as to form a complete set. Then the expansion of $a_r(k)$ is

$$a_r(k)=\sum_{\nu=1}^{M_r}\phi_{r\nu}(k)a_{r\nu}+\sum_i \chi_{ri}(k)a_{ri} \tag{12}$$

and H takes the form

$$H_N=H_S+T_E+H_{ES}+H^\dagger_{ES} \,. \tag{13}$$

The term H_S is the MNSM Hamiltonian

$$H_S=T_{SF}+T_{SB}+H_{SI}+H^\dagger_{SI} \,,$$
$$T_{SF}=\sum_\nu t_\nu \rho^{0\nu} \,,$$
$$T_{SB}=\sum_r\sum_{\nu\mu=1}^{M_r}\omega^r_{\nu\mu}a^\dagger_{r\nu}\cdot a_{r\mu} \,,$$
$$H_{SI}=-\sum_r\sum_{\nu\mu=1}^{M_r}\omega^r_{\nu\mu}G_{r\mu}\rho^{r\mu}\cdot a^\dagger_{r\nu} \,, \tag{14}$$
$$\omega^r_{\nu\mu}=\int_0^\infty \omega_r(k)\phi^*_{r\nu}(k)\phi_{r\mu}(k)dk$$
$$=\frac{1}{G_{r\nu}G_{r\mu}}\int_0^\infty \frac{v^*_{r\nu}(k)v_{r\mu}(k)}{\omega(k)}dk \,,$$

which operates in the subspace generated by the operators $\{B^\dagger_{i\mu},\, i=1,N\}$ and $a^\dagger_{r\nu}$ acting on the vacuum. In contrast to the NSM subspace, which is generated by a set of operators $B^\dagger_{i\mu}$ acting on the vacuum and is therefore finite, the MNSM subspace is infinite because the $a_{r\nu}$ are Boson operators. The part T_E of the Hamiltonian represents the kinetic energy of the external mesons,

$$T_E=\sum_r\sum_{ij}\omega^r_{ij}a^\dagger_{ri}\cdot a_{rj} \,,$$
$$\omega^r_{ij}=\int_0^\infty \omega_r(k)\chi^*_{ri}(k)\chi_{rj}(k)dk \,, \tag{15}$$

while H_{ES} describes the interaction of the external mesons with the MNSM subspace:

$$H_{ES}=\sum_r\sum_i a^\dagger_{ri}\cdot\hat{J}_{ri} \,,$$
$$\hat{J}_{ri}=\sum_\nu \eta^{r*}_{\nu i}(a_{r\nu}-G_{r\nu}\rho^{r\nu}) \,, \tag{16}$$
$$\eta^r_{\nu i}=\frac{1}{G_{r\nu}}\int_0^\infty v^*_{r\nu}(k)\chi_{ri}(k)dk \,.$$

The source current $\hat{J}$ in H_{ES} for the ri external meson acts within the MNSM subspace. The point of choosing the meson mode functions in the above special way is that $\hat{J}_{ri}$ has zero expectation value in any eigenstate of H_S, as follows from the commutation relation

$$[a_{r\nu},H_S]=\sum_\mu \omega^r_{\nu\mu}(a_{r\mu}-G_{r\mu}\rho^{r\mu}) \tag{17}$$

and the positive definite character of $\omega^r_{\nu\mu}$. Hence there are no second-order purely mesonic corrections to the energy of the ground state of H_S.

IV. REMARKS ON THE MESON-NUCLEON SHELL MODEL

The MNSM Hamltonian H_S of Eqs. (14) can evidently be written in the form

$$H_S=H_H+H_M \,,$$
$$H_M=\sum_r\sum_{\nu\mu=1}^{M_r}\omega^r_{\nu\mu}(a_{r\nu}-G_{r\nu}\rho^{r\nu})^\dagger\cdot(a_{r\mu}-G_{r\mu}\rho^{r\mu}) \,, \tag{18}$$
$$H_H=T_{SF}-\sum_r\sum_{\nu\mu=1}^{M_r}V^r_{\nu\mu}\rho^{r\nu\dagger}\cdot\rho^{r\mu}$$
$$=T_{SF}-\sum_r\sum_{ijkn}V^r_{ik,jn}\rho^{ijr\dagger}\cdot\rho^{knr} \,,$$

where the effective two fermion interaction V is given by

$$V^r_{ik,jn}=\int_0^\infty \frac{v^*_{ijr}(k)v_{knr}(k)}{\omega(k)}dk \ . \tag{19}$$

The function $v_{ijr}(k)$ is just the factor $Y_r(k)$, which is typically

$$(2\pi)^{-3/2}\omega^{-1/2}(k) \ ,$$

times the matrix element of the current operator W_α between the fermion states i and j; it follows that H_H is exactly the Hartree Hamiltonian for the fermion subspace with the fermion-fermion potential given by the direct one-meson-exchange potential for the meson feld under consideration. This result also holds for a scalar field that is the fourth component of a covariant vector or pseudovector field; the proof follows the lines given in Ref. 11. Like the mean-field treatment, the MNSM does not reproduce the Fock exchange terms. As have been noted previously,[1,2,4,9] these terms are largely contained in the free fermion self-energy terms that must be added to the ground-state energy of H_S to obtain the energy of the bound state of H_S relative to the energy of the same number of free fermions. Some of the Fock exchange is also in H_M. Note that H_M is positive semidefinite, so that the Hartree energy is a lower bound on the energy of the ground state of H_S. The Hartree Hamiltonian is usually only written for the ground state of a system of fermions; H_H of Eq. (18) has a whole spectrum of states and is actually an extension of the usual Hartree ground-state energy functional.

The resolution of H_S given in Eqs. (18) is remarkable but not surprising. Its simplicity can be regarded as an additional justification of the choice of meson mode functions made in Eqs. (11) and (12).

The case of a single Slater determinant is the simplest special case of the MNSM; as was noted in Ref. 9, this case reduces to the usual mean-field approximation. For a single Slater determinant, all fermion operators ρ can be replaced by their respective expectation values and the expectation value of H_M is minimized in the mesonic state that satisfies

$$(a_{r\mu}-G_{r\mu}\rho^{r\mu})\,|\,\rangle=0 \tag{20}$$

for all $r\mu$; this condition can be satisfied because all the ρ's are c numbers in this case. The same procedure works if the fermion subspace is restricted to any single state vector for the fermions.

In other more complex cases it seems likely that the coherent meson-pair technique introduced in Ref. 13 will be useful in treating H_S of Eqs. (14) and (18). The MNSM has a coherent pair for each value of $r\mu$, so that a coherent pair state is

$$|\{n,y\}\rangle=\left[\prod_{r\mu} g_{d(r)+2n_{r\mu}}(y_{r\mu}a^\dagger_{r\mu}\cdot a^\dagger_{r\mu})\right]|\{n\}\rangle \ , \tag{21}$$

where the state $|\{n\}\rangle$ is a "basic" state that satisfies

$$\begin{aligned} &a_{r\mu}\cdot a_{r\mu}\,|\{n\}\rangle=0 \ ,\\ &a^\dagger_{r\mu}\cdot a_{r\mu}\,|\{n\}\rangle=n_{r\mu}\,|\{n\}\rangle \ ; \end{aligned} \tag{22}$$

the state $|\{n,y\}\rangle$ is the coherent pair state that satisfies

$$a_{r\mu}\cdot a_{r\mu}\,|\{n,y\}\rangle=y_{r\mu}\,|\{n,y\}\rangle \ . \tag{23}$$

The function $g_{d+2n}(x)$ is the coherent-pair function defined in Ref. 13 with d the degeneracy

$$d(r)=(2t+1)(2l+1)$$

of the meson operator a_r. For the standard p-wave pions of the nucleon static model d is 9. Matrix elements between coherent pair states are given in terms of basic-state matrix elements in Ref. 13. A general basic state has the structure

$$\{\,|\{n\}\rangle^{R_B},|\{i\}\rangle^{R_F}\}^R \ , \tag{24}$$

where now $|\{n\}\rangle^{R_B}$ is a purely mesonic basic state belonging to the combined representation R_B, where R stands for (T,J,π), and $|\{i\}\rangle^{R_F}$ is a purely fermionic state with occupied orbitals

$$\{i\}=i_1,i_2,\ldots,i_A$$

coupled to total $T_FJ_F\pi_F$. For any R, a choice of a set of basic states leads to a Hamiltonian matrix whose lowest eigenvalue is an approximate ground-state energy of H_S for the given R.

Techniques already exist for treating the external mesons. If Eq. (12) is written in the form

$$a_r(k)=\sum_{\nu=1}^{M_r}\phi_{r\nu}(k)a_{r\nu}+a_1(k) \ , \tag{25}$$

then the form of the expansion is as in Ref. 10, and the methods given there can be used to obtain binding energies and scattering amplitudes within the one-external-meson subspace.

V. SUMMARY

When the nucleus is treated as a system of nucleons and mesons with a Hamiltonian like that of Eq. (1), the usual expansion of the nucleon field in terms of a set of normalizable shell-model single-particle wave functions leads to a rather complicated "static model"of the type described by the Hamiltonians of Eqs. (7) and (9). In the expansion of the meson annihilation operator, a particular choice of the internal meson wave functions is advantageous: (a) It minimizes the direct effects of the "external" mesons on the ground-state energy of the system, and (b) it leads to a simple connection with the Hartree approximation.

The resulting MNSM Hamiltonian provides a framework for treating the effects of meson pairs by using methods previously applied to the static model of the nucleon[13]; external mesons can also be treated by using established techniques.[10]

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