Formulas for the δ-shell-plus-Coulomb potential for all partial waves

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Closed forms are given for a large number of quantities occurring in the theory of potential scattering in an arbitrary partial-wave state l for the cases (i) the δ-shell potential $V(r)=-\lambda R^{-2}\delta(r-R)$, and (ii) the δ-shell plus Coulomb potential. Furthermore, the trajectories of the poles of the total T operator in the complex k plane, with varying complex λ, are investigated in detail for zero, repulsive, and attractive Coulomb force, respectively. Expressions are given for the effective-range parameters, and the Coulomb-modified effective-range parameters, for *all* l, with application to the NN system, and the $N\alpha$ system, respectively. The connection between Coulomb-level shifts and effective-range parameters is considered. Improvements on the standard small-shift approximation, which is relatively poor, are suggested.

[NUCLEAR REACTIONS Charged-particle scattering. δ-shell plus Coulomb potential. Effective-range parameters for all l. Application to NN and $N\alpha$. Coulomb-level shifts.]

I. INTRODUCTION

The theory of nonrelativistic scattering by local and nonlocal potentials is well developed; see, for example, the books of Newton,[1] and de Alfaro and Regge.[2] In this paper we shall investigate the δ-shell potential, given by Eq. (2.7). It can be considered as a limit case of a *local* potential. It is also *separable* (rank one) in each partial wave l. Its simple form allows the explicit calculation of many quantities in scattering theory. This has made the δ-shell potential an increasingly popular educational tool, which is used in modern text books,[3] and in the construction of simple models; see, for example, Ref. 4. We are interested, in particular, in the δ-shell plus Coulomb potential. This interaction may be thought of as a model for the interaction between charged particles subject to short-range (strong, nuclear) interparticle forces. The Coulomb potential may be repulsive or attractive. The δ-shell potential is a short-range force characterized by two parameters: its range R, and its strength λ. For a Hermitian interaction λ must be real. We shall also consider the case of complex (nonreal) λ as a model for the non-Hermitian interactions occurring, for example, in exotic atoms.

We shall study, in particular, the relation between λ and the positions of the poles of the total T operator (i.e., of bound states, virtual states, resonances, etc.). We derive closed formulas for the effective-range parameters [Eqs. (6.7)–(6.10)] and the Coulomb-modified effective-range parameters [Eqs. (10.14)–(10.17)], for all l. For Coulomb attraction we discuss the phenomenon of reconstruction of spectra, and quantum defects. We also discuss the quality of the small-shift approximation for the energy-level shift caused by the short-range force.

In Secs. II and III, we establish most of our notation and conventions. In Secs. IV–VI, we give results for the pure δ-shell potential. Section VII recalls some formulas for the general case of a short-range plus Coulomb potential. Sections VIII–XI give the results for the δ-shell plus Coulomb case. Section XII concludes the paper with a discussion.

II. THE δ-SHELL POTENTIAL

A potential V is called local if in the representation $[\vec{r}]$

$$\langle \vec{r}' \mid V \mid \vec{r} \rangle = V(\vec{r})\delta(\vec{r}'-\vec{r}) , \tag{2.1}$$

where V is a real (for a Hermitian Hamiltonian) or complex (when the Hamiltonian is non-Hermitian)

function of $\vec{r}\equiv(x,y,z)$. Throughout we consider rotationally invariant potentials. This implies that $\langle\vec{r}'\,|\,V\,|\,\vec{r}\rangle$ is a function of r', r, and $\hat{r}'\cdot\hat{r}$ only, and for a local potential that $V(\vec{r})=V(r)$. In the latter case in the representation $[rl]$ we have

$$\langle r'\,|\,V_l\,|\,r\rangle=V(r)r^{-2}\delta(r'-r)\ . \tag{2.2}$$

In this paper we consider the short-range potential V_s which in each partial-wave state l is separable

$$V_{sl}=-\lambda\,|\,gl\rangle\langle gl\,|\ , \tag{2.3}$$

with form factor

$$|\,gl\rangle=|\,R\rangle,\ \ \langle r\,|\,gl\rangle=R^{-2}\delta(r-R)\ , \tag{2.4}$$

i.e., $|\,gl\rangle$ is a basis state vector of the representation $[rl]$. Here R is a real positive constant. In the momentum representation $[pl]$ we get

$$\langle p\,|\,gl\rangle=(2/\pi)^{1/2}(-i)^l j_l(pR)\ . \tag{2.5}$$

According to Eq. (2.3)

$$\begin{aligned}\langle r'\,|\,V_{sl}\,|\,r\rangle&=-\lambda\langle r'\,|\,gl\rangle\langle gl\,|\,r\rangle\\&=-\lambda R^{-2}\delta(r-R)\\&\quad\times r^{-2}\delta(r'-r)\ .\end{aligned} \tag{2.6}$$

Comparison with Eq. (2.2) shows that V_s may be considered as a limit case of a local potential. Henceforth we shall ignore this subtle distinction, and call V_s simply "local." We have

$$V_s(r)=-\lambda R^{-2}\delta(r-R)\ . \tag{2.7}$$

We shall refer to this potential as the δ-shell potential. It is both local and separable. Its range parameter is R, its strength parameter is λ. A positive (negative) value of λ corresponds to attraction (repulsion). Our units will be such that $2m=1=\hbar$. Here m is the reduced mass. The formulas in the following remain valid when λ and R are given different values in each partial wave, $(\lambda,R)\rightarrow(\lambda_l,R_l)$. In that case, strictly speaking, we no longer consider a local potential, but instead the lth partial-wave projection of a local potential. This will not give rise to confusion.

III. NOTATION AND CONVENTIONS

In this section we will further specify our notation. For the lth partial wave completeness and orthonormality relations read

$$\mathbb{1}_l=\int_0^\infty r^2dr\,|\,rl\rangle\langle rl\,|\ , \tag{3.1}$$

$$\mathbb{1}_l=\int_0^\infty p^2dp\,|\,pl\rangle\langle pl\,|\ , \tag{3.2}$$

$$\langle r'l\,|\,rl\rangle=r^{-2}\delta(r'-r)\ , \tag{3.3}$$

$$\langle p'l\,|\,pl\rangle=p^{-2}\delta(p'-p)\ . \tag{3.4}$$

The Green's operators $G_0(E)$ and $G_s(E)$ are connected with the free Hamiltonian H_0, and $H_s=H_0+V_s$, respectively. Throughout the energy variable is $E=(k+i\epsilon)^2$, $\epsilon\downarrow0$. In our notation we often suppress the dependence on E, and the l dependence. We have

$$G_0=(E-H_0)^{-1},\ \ G_s=(E-H_s)^{-1}\ , \tag{3.5}$$

$$G_s=G_0+G_0T_sG_0 \tag{3.6}$$

$$=G_0+G_0V_sG_s\ . \tag{3.7}$$

Here $T_s(E)$ denotes the T operator for the Hamiltonian H_s; it is related to V_s as follows,

$$T_s=V_s+V_sG_0T_s \tag{3.8}$$

$$=V_s+V_sG_sV_s\ . \tag{3.9}$$

Similar relations hold for the partial-wave projected operators: One merely attaches an extra subscript l to each operator.

"Outgoing" scattering states for H_0 and H_s are $|\,kl+\rangle_0$ and $|\,kl+\rangle_s$, respectively. They are connected by

$$|\,kl+\rangle_s=|\,kl+\rangle_0+G_{0l}T_{sl}\,|\,kl+\rangle_0\ . \tag{3.10}$$

The outgoing scattering states have δ-function normalization,

$${}_s\langle k'l+\,|\,kl+\rangle_s=k^{-2}\delta(k'-k)\ .$$

Note that $|\,kl+\rangle_0$ is the same state as $|\,kl\rangle$ or briefly, $|\,k\rangle$. The use of the former (subscripted) state makes the formal analogy with the charged case (Sec. VII) more transparent.

We shall consider the Jost Green's operators $\bar{G}_{0l}$ and $\bar{G}_{sl}$, the Jost states $|\,kl\uparrow\rangle_0$ and $|\,kl\uparrow\rangle_s$, and the related operator $\bar{T}_{sl}$.[5] They are connected through relations analogous to relations (3.6)–(3.9) (with subscripts l), and

$$|\,kl\uparrow\rangle_s=|\,kl\uparrow\rangle_0+\bar{G}_{0l}\bar{T}_{sl}\,|\,kl\uparrow\rangle_0 \tag{3.11}$$

$$=|\,kl\uparrow\rangle_0+\bar{G}_{sl}V_{sl}\,|\,kl\uparrow\rangle_0 \tag{3.12}$$

$$=|\,kl\uparrow\rangle_0+\bar{G}_{0l}V_{sl}\,|\,kl\uparrow\rangle_s\ . \tag{3.13}$$

We also recall that off-shell Jost states $|\,kql\uparrow\rangle$ can be defined, with q a momentum variable which in

general is taken off shell. Relations for $|kql\uparrow\rangle_s$ are obtained from Eqs. (3.11)–(3.13) by the simultaneous replacements

$$|kl\uparrow\rangle_s \rightarrow |kql\uparrow\rangle_s$$

and

$$|kl\uparrow\rangle_0 \rightarrow |ql\uparrow\rangle_0$$

throughout these formulas.

The Jost states have the following properties:

$$\langle r|kl\uparrow\rangle_0=(2/\pi)^{1/2}i^l h_l^{(+)}(kr)\ , \tag{3.14}$$

$$\langle p|kl\uparrow\rangle_0=\frac{2}{\pi k}\frac{(p/k)^l}{p^2-(k+i\epsilon)^2}\ , \tag{3.15}$$

$$\langle r|kl\uparrow\rangle_s=(2/\pi)^{1/2}(kr)^{-1}f_{sl}(k,r)\ , \tag{3.16}$$

$$\langle r|kql\uparrow\rangle_s=(2/\pi)^{1/2}(qr)^{-1}f_{sl}(k,q,r)\ , \tag{3.17}$$

where $f_{sl}(k,r)$ is the Jost solution of the radial Schrödinger equation, and $f_{sl}(k,q,r)$ is the off-shell Jost solution of the inhomogeneous radial Schrödinger equation.[6] For the spherical Hankel function $h_l^{(+)}$ we use the convention of Messiah.[7]

The Jost function $f_{sl}(k)$ and the off-shell Jost function $f_{sl}(k,q)$ follow from the relations,

$$f_{sl}(k)=\lim_{r\rightarrow 0}\langle r|kl\uparrow\rangle_s/\langle r|kl\uparrow\rangle_0\ , \tag{3.18}$$

$$f_{sl}(k,q)=\lim_{r\rightarrow 0}\langle r|kql\uparrow\rangle_s/\langle r|ql\uparrow\rangle_0\ , \tag{3.19}$$

respectively.

It is important to note that $\langle p|kl\uparrow\rangle_s$ is *not* a solution of the Schrödinger equation in momentum representation. This is related to the (at $r=0$) singular behavior of the irregular solution $\langle r|kl\uparrow\rangle_s$ of the Schrödinger equation in $[rl]$. If V_s is *local* the following important relations hold[1]:

$$\langle r'|G_{sl}|r\rangle=-\tfrac{1}{2}\pi k(-)^l\langle r_>|kl\uparrow\rangle_s \times\langle r_<|kl+\rangle_s \tag{3.20}$$

(local V_s only; the label s may be replaced by 0 or c),

$$\langle r'|\bar{G}_{sl}|r\rangle=\Theta(r-r')\frac{i\pi k}{4} \times[\langle r'|kl\uparrow\rangle_{ss}\langle kl\uparrow|r\rangle-\text{c.c.}] \tag{3.21}$$

(local V_s only), where Θ is the unit-step function, $\Theta(x)=1$, $x>0$; $\Theta(x)=0$, $x<0$. In Eq. (3.20) $r_<=\min(r',r)$, $r_>=\max(r',r)$.

If V_s is *separable* [see Eq. (2.3)] then also T_s and $\bar{T}_s$ are separable, according to

$$T_{sl}=-|gl\rangle\tau_{sl}(k^2)\langle gl|\ , \tag{3.22}$$

$$\tau_{sl}^{-1}=\lambda_l^{-1}+\langle gl|G_{0l}|gl\rangle\ , \tag{3.23}$$

$$\bar{T}_{sl}=-|gl\rangle\bar{\tau}_{sl}(k^2)\langle gl|\ , \tag{3.24}$$

$$\bar{\tau}_{sl}^{-1}=\lambda_l^{-1}+\langle gl|\bar{G}_{0l}|gl\rangle\ . \tag{3.25}$$

IV. FORMULAS FOR THE δ-SHELL POTENTIAL

The δ-shell potential has been discussed in Sec. II. It is local [cf. Eq. (2.7)], and separable in each partial wave [cf. Eq. (2.6)]. The following quantities are easily calculated:

$$\langle gl|kl+\rangle_0=(2/\pi)^{1/2}i^l j_l(kR)\ , \tag{4.1}$$

$$\langle gl|kl\uparrow\rangle_0=(2/\pi)^{1/2}i^l h_l^{(+)}(kR)\ , \tag{4.2}$$

$$\langle p|G_{0l}|gl\rangle=-(2/\pi)^{1/2}(-i)^l j_l(pR)[p^2-(k+i\epsilon)^2]^{-1}\ , \tag{4.3}$$

$$\langle p|\bar{G}_{0l}|gl\rangle=(2/\pi)^{1/2}(-i)^l[(p/k)^l j_l(kR)-j_l(pR)]/(p^2-k^2)\ , \tag{4.4}$$

$$\langle r|G_{0l}|gl\rangle=-kj_l(kr_<)h_l^{(+)}(kr_>),\ r_<=\min(R,r),\ r_>=\max(R,r)\ , \tag{4.5}$$

$$\langle r|\bar{G}_{0l}|gl\rangle=k\Theta(R-r)[j_l(kr)n_l(kR)-j_l(kR)n_l(kr)]\ , \tag{4.6}$$

$$\langle gl|G_{0l}|gl\rangle=-kj_l(kR)h_l^{(+)}(kR)\ , \tag{4.7}$$

$$\langle gl|\bar{G}_{0l}|gl\rangle=0\ . \tag{4.8}$$

The spherical Hankel function $h_l^{(+)}$ is defined by Messiah.[7] The spherical Bessel function

$$j_l(z)=(2z/\pi)^{-1/2}J_{l+1/2}(z)$$

and the spherical Neumann function

$$n_l(z)=(2z/\pi)^{-1/2}N_{l+1/2}(z)$$

are taken according to the convention of Ref. 8. Note that Messiah[7] denotes by n_l the same function with a change in sign. The above formulas immediately give explicit expressions for T_{sl} by using Eqs. (3.22), (3.23), and (4.7), and for $\bar{T}_{sl}$ by using Eqs. (3.24), (3.25), and (4.8). Note that

$$\bar{T}_{sl}=V_{sl}\ . \tag{4.9}$$

Thus one finds the scattering states from Eq. (3.10):

$$\langle r\mid kl+\rangle_s=(2/\pi)^{1/2}i^l\left[j_l(kr)+\frac{kj_l(kr_<)h_l^{(+)}(kr_>)j_l(kR)}{\lambda_l^{-1}-kj_l(kR)h_l^{(+)}(kR)}\right], \tag{4.10}$$

$$\langle p\mid kl+\rangle_s=k^{-2}\delta(p-k)+\frac{2}{\pi}\frac{j_l(pR)j_l(kR)}{\lambda_l^{-1}-kj_l(kR)h_l^{(+)}(kR)}\frac{1}{p^2-k^2}\ . \tag{4.11}$$

The Jost states and the off-shell Jost states one finds from Eq. (3.11); see also the remarks after Eq. (3.13). We give explicitly the off-shell Jost states,

$$\langle r\mid kql\uparrow\rangle_s=(2/\pi)^{1/2}i^l[\,h_l^{(+)}(qr)-\lambda_l k\Theta(R-r)\,\{j_l(kr)n_l(kR)-j_l(kR)n_l(kr)\}h_l^{(+)}(qR)]\ , \tag{4.12}$$

$$\langle p\mid kql\uparrow\rangle_s=\frac{2}{\pi q}\left[\frac{(p/q)^l}{p^2-q^2}-\lambda_l q\frac{(p/k)^l j_l(kR)-j_l(pR)}{p^2-k^2}h_l^{(+)}(qR)\right]. \tag{4.13}$$

The Jost states in $[rl]$ and in $[pl]$ are obtained from Eqs. (4.12) and (4.13), by deleting the symbol q from the left-hand side (lhs), and replacing q by $k+i\epsilon$ on the right-hand side (rhs), respectively. The Jost functions are obtained from Eqs. (3.18) and (3.19). For example, for S waves,

$$f_{s0}(k)=1-\lambda_0 k^{-1}R^{-2}\exp(ikR)\sin(kR)\ , \tag{4.14}$$

$$f_{s0}(k,q)=1-\lambda_0 k^{-1}R^{-2}\exp(iqR)\sin(kR)\ . \tag{4.15}$$

The δ-shell potential is an example of a so-called cutoff potential. For this type of potential many interesting general results and properties have been derived; see, for instance, Refs. 1, 2, and 9.

V. BOUND STATES AND RESONANCES FOR THE δ-SHELL POTENTIAL

Positions of resonances and bound, virtual, decaying, and "creative" states are given by the corresponding pole positions of the T operator. Resonance poles lie just below the positive real k axis. They produce the familiar Breit-Wigner dependence of the lth partial-wave cross section. In the E plane the pole occurs at $E_0-\frac{1}{2}i\Gamma$ on the second Riemann sheet, where Γ is the width of the Breit-Wigner peak, which is connected to the lifetime Γ^{-1}. Bound-state poles lie at the positive imaginary k axis. Decaying (unstable) and "creative" bound states occur only for nonreal λ. They lie just left or right of the positive imaginary k axis, respectively. A lifetime and a width can be associated with these states, too. The established nomenclature for virtual or antibound states is not unique. We shall call states which are associated with poles near and on the negative imaginary k axis virtual and antibound, respectively. An antibound state can lie close to the physical half-plane (i.e., $\mathrm{Im}k>0$), for example, when the attraction of V_s is not sufficiently strong to support a bound state, for $l=0$.

In this section, k will denote the position of a pole of T. According to Eqs. (3.23) and (4.7) the connection between the pole position k and strength parameter λ_l is given by

$$\lambda_l^{-1}=kj_l(kR)h_l^{(+)}(kR)\ . \tag{5.1}$$

The map (5.1) is shown in Figs. 1 and 2 for $l=0$ and $l=1$, respectively. It is not restricted to real values of λ. In the physical half-plane for real λ there are no poles other than on the imaginary axis. In the lower half-plane, when λ is real and varying, the corresponding pole trajectories exhibit a charac-

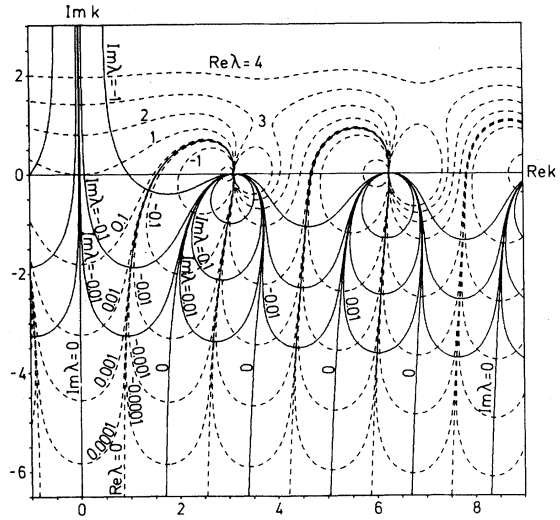

FIG. 1. Contour plot in the k plane, for the pure δ-shell potential and $l=0$. Scales are in units R^{-1}. Given the pole position k, one reads off the (complex) value of λ, in units R. Given λ, all corresponding complex pole positions can be found. Full lines correspond to contours where $\mathrm{Im}\lambda$ is kept fixed. Broken lines: $\mathrm{Re}\lambda$ is fixed. Since $\lambda(k)=\lambda^*(-k^*)$ the imaginary k axis is the axis of symmetry.

teristic behavior near the origin, i.e., for $|k|R \ll 1$. For $l=0$ the poles move along the imaginary axis ($\mathrm{Re}k=0$, antibound states). For $l>0$, when λ approaches from below that value where a bound state appears, the trajectories satisfy $\mathrm{Im}k \propto -(\mathrm{Re}k)^{2l}$; see also Refs. 10 and 11, and McVoy.[4] General characteristics, symmetries, saddle points, and the associated nonlinearity in the relation between λ and k are discussed in Refs. 4 and 12.

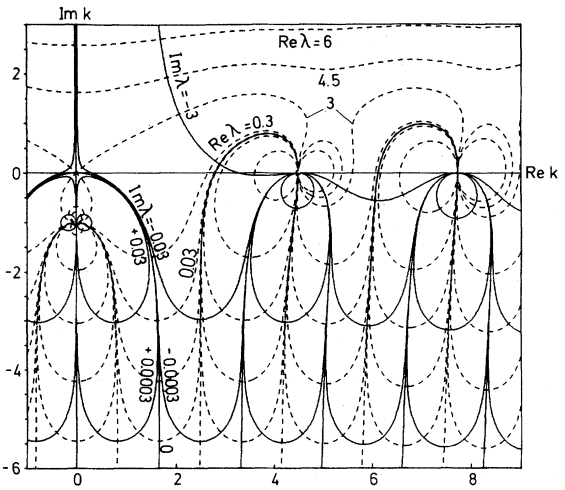

FIG. 2. Same as Fig. 1, for $l=1$.

When the strength parameter λ vanishes the poles must be far from the physical region (no scattering), and they are found to recede to equally-spaced limit points given by

$$kR=(2n+l+1)\pi/2-i\infty, \quad n\in\mathscr{I} ,$$

for $\lambda\uparrow 0$, and by

$$kR=(2n+l)\pi/2-i\infty, \quad n\in\mathscr{I} ,$$

for $\lambda\downarrow 0$. In the opposite extreme of infinite strength parameter only bound-state and virtual-state poles can move off to $\pm i\infty$. It should be noted that in each partial wave l the δ-shell potential can have at most one bound state. This is a direct consequence of the fact that it is a rank-*one* separable potential, cf. Eq. (2.3). Furthermore, when $|\lambda|\to\infty$, other poles reach limit points on the real axis, corresponding to the zeros of $j_l(kR)$. For $\lambda\to-\infty$ the barrier becomes completely reflecting; the resonances become infinitely narrow, cf. McVoy.[4] They can be thought of as positive-energy bound states for the impenetrable sphere.

VI. PHASE SHIFTS AND EFFECTIVE-RANGE PARAMETERS FOR THE δ-SHELL POTENTIAL

Phase shifts $\delta_{sl}(k)$ are connected to the on-shell matrix elements of the corresponding T matrix, according to

$$\langle k\,|\,T_{sl}[(k+i\epsilon)^2]\,|\,k\rangle = -(2/\pi)k^{-1}\exp(i\delta_{sl}(k))\sin\delta_{sl}(k) . \tag{6.1}$$

We observe that

$$\cot\delta_{sl}-i=-(2/\pi)k^{-1}/\langle k\,|\,T_{sl}\,|\,k\rangle . \tag{6.2}$$

The effective-range function $K_{sl}(k^2)$ is defined by

$$K_{sl}(k^2)\equiv k^{2l+1}\cot\delta_{sl}(k) . \tag{6.3}$$

It is known to be real meromorphic in k^2 in a large region containing the origin $k^2=0$, for a large class of potentials.[13] Its expansion coefficients in the expansion

$$K_{sl}(k^2)=-1/a_{sl}+\tfrac{1}{2}r_{sl}k^2-P_{sl}r_{sl}^3k^4+Q_{sl}r_{sl}^5k^6-\cdots \tag{6.4}$$

are related to the low-energy scattering parameters:

a_{sl} is the scattering length, r_{sl} is the effective range, and P_{sl} and Q_{sl} are so-called shape parameters.

For the δ-shell potential (2.4) the on-shell T matrix is given by

$$\langle k \mid T_{sl} \mid k \rangle = - \mid \langle k \mid gl \rangle \mid^2 / [\lambda_l^{-1} + \langle gl \mid G_{0l} \mid gl \rangle] , \tag{6.5}$$

and the effective-range function is even a real-meromorphic function of k^2 in the *entire* k^2 plane. [This follows from the fact that we have a *cutoff* potential; see the remark following Eq. (4.15).] We have

$$K_{sl}(k^2) = k^{2l+1}[(\lambda_l k)^{-1} + j_l(kR)n_l(kR)]/[j_l(kR)]^2 . \tag{6.6}$$

Using the well-known expansions of the spherical Bessel functions, we expand the rhs of (6.6) in powers of k^2. It is trivial but tedious to find the following elegant expressions:

$$-a_{sl}^{-1} = [(2l+1)!!]^2 R^{-2l-1} \left[\frac{R}{\lambda_l} - \frac{1}{2l+1} \right] , \tag{6.7}$$

$$\tfrac{1}{2} r_{sl} = [(2l+1)!!]^2 R^{-2l+1} \left[\frac{R}{\lambda_l} - \frac{1}{2l-1} \right] \frac{1}{2l+3} , \tag{6.8}$$

$$-P_{sl} r_{sl}^3 = [(2l+1)!!]^2 R^{-2l+3} \left[\frac{R}{\lambda_l} - \frac{1}{2l-3} \right] \frac{l+3}{(2l+3)^2(2l+5)} , \tag{6.9}$$

$$Q_{sl} r_{sl}^5 = [(2l+1)!!]^2 R^{-2l+5} \left[\frac{R}{\lambda_l} - \frac{1}{2l-5} \right] \frac{2l^2+15l+30}{3(2l+3)^3(2l+5)(2l+7)} . \tag{6.10}$$

VII. GENERAL FORMULAS FOR COULOMB PLUS SHORT-RANGE POTENTIAL

In this section we consider the potential

$$V = V_c + V_s , \tag{7.1}$$

where V_c is the Coulomb potential,

$$V_c(r) = -2s/r = 2k\gamma/r , \tag{7.2}$$

and V_s is a short-range potential. The strength of the Coulomb potential is related to the dimensionless Sommerfeld parameter $\gamma \equiv -s/k$. We define the Bohr radius $a_B = |s|^{-1}$ for both cases of Coulomb attraction and repulsion. We note that formulas (3.5)–(3.9) and (3.20) remain valid when the subscript s is replaced by c. In particular, the Coulomb Green's operator is defined by

$$G_c = (E - H_0 - V_c)^{-1} . \tag{7.3}$$

It is convenient to use the Gell-Mann–Goldberger[14] two-potential formalism. The total T operator can be written as

$$\begin{aligned} T &= T_c + T_{cs} \\ &= T_c + (1 + T_c G_0) t_{cs} (1 + G_0 T_c) . \end{aligned} \tag{7.4}$$

Here T_c is the Coulomb T operator, and $t_{cs}(E)$ is a short-range operator. It satisfies

$$t_{cs} = V_s + V_s G_c t_{cs} , \tag{7.5}$$

which is reminiscent of Eq. (3.8). The partial-wave analogs of Eqs. (7.1)–(7.5) have exactly the same form.

Outgoing scattering states for

$$H = H_0 + V = H_0 + V_c + V_s$$

will be denoted by $\mid kl + \rangle$. The following generalization of Eq. (3.10) to the charged case holds,

$$\mid kl + \rangle = \mid kl + \rangle_c + G_{cl} t_{csl} \mid kl + \rangle_c . \tag{7.6}$$

When V_s is separable [see Eq. (2.3)], then also t_{cs} is separable, according to

$$t_{csl} = - \mid gl \rangle \tau_{csl}(k^2) \langle gl \mid , \tag{7.7}$$

$$\tau_{csl}^{-1} = \lambda_l^{-1} + \langle gl \mid G_{cl} \mid gl \rangle . \tag{7.8}$$

Note again the formal analogy to the chargeless case. Sometimes it is convenient to consider Coulomb-modified form factors $g^c(E)$. They are associated with the states

$$\mid g^c l \rangle = (1 + T_{cl} G_{0l}) \mid gl \rangle . \tag{7.9}$$

VIII. FORMULAS FOR THE δ-SHELL PLUS COULOMB POTENTIAL

In this section some results of Sec. IV will be extended to the case of charged particles. In particular, we shall need

$$\langle gl \mid kl+\rangle_c = \langle R \mid kl+\rangle_c$$
$$= (2/\pi)^{1/2} e^{-1/2\pi\gamma} [\Gamma(l+1+i\gamma)/\Gamma(2l+2)](2ikR)^l e^{ikR}$$
$$\times {}_1F_1(l+1+i\gamma;2l+2;-2ikR) , \tag{8.1}$$

$$\langle gl \mid kl\uparrow\rangle_c = \langle R \mid kl\uparrow\rangle_c$$
$$= (2/\pi)^{1/2} e^{ikR+1/2\pi\gamma} (kR)^{-1} (-2ikR)^{l+1} U(l+1+i\gamma,2l+2,-2ikR) , \tag{8.2}$$

see for example, Refs. 8, 15, and 16. Furthermore, $\langle r \mid G_{cl} \mid gl\rangle$ and $\langle gl \mid G_{cl} \mid gl\rangle$ are needed. Inserting $\mid R\rangle$ for $\mid gl\rangle$ these formulas follow easily from Eq. (3.20) (with the label s replaced by c), and Eqs. (8.1) and (8.2). Thus a closed expression is obtained for t_{csl} and τ_{csl}; see Eqs. (7.7) and (7.8). In $[rl]$ the Coulomb-modified form factors (7.9) follow easily from

$$\langle r \mid g^c l\rangle = \langle r \mid gl\rangle - \frac{2k\gamma}{r}\frac{\pi}{2} k (-)^l \langle r_> \mid kl\uparrow\rangle_c \langle r_< \mid kl+\rangle_c . \tag{8.3}$$

In $[pl]$ the expressions for g^c take a formidable size. We only give a result for $l=0$, which is interesting, however, in view of its similarity to the result for the Yamaguchi form factor [Eq. (66) of Ref. 17]. For the δ-shell potential we find

$$\langle p \mid g^c l=0\rangle = \langle p \mid gl=0\rangle - (2/\pi)^{1/2}\frac{\gamma}{2pR}\left[\int_0^1 dt\, t^{i\gamma-1}\exp\left\{ikR\frac{1+ta}{1-ta}\right\} - \left[a\to\frac{1}{a}\right]\right] , \tag{8.4}$$

where $a=(p-k)/(p+k)$.

IX. BOUND STATES AND RESONANCES FOR THE δ-SHELL PLUS COULOMB POTENTIAL

In this section we extend the results of Sec. V to the charged case. The corresponding states of the total potential $V=V_c+V_s$ are given by the poles of the T operator T_c+T_{cs}. Pure Coulomb poles (of T_{cl}) occur at $k=is/n$, $n=l+1, l+2,\ldots$. They represent bound states for Coulomb attraction ($s>0$) and antibound states for Coulomb repulsion. If V_s is nonvanishing these pure Coulomb poles are canceled by corresponding poles in T_{cs}. For a separable potential (2.3) it follows that the poles of T are obtained by solving [see Eq. (7.8)]

$$\lambda_l^{-1} = -\langle gl \mid G_{cl} \mid gl\rangle . \tag{9.1}$$

The explicit form of the rhs of Eq. (9.1) follows from Eqs. (3.20), (8.1), and (8.2). We have solved by numerical means the map between pole position k and potential strength parameter λ for Coulomb attraction and repulsion, for $l=0,1,\ldots$, and for various values of $R/a_B=R\mid s\mid$, where a_B is the Bohr radius.

In Fig. 3 we show the result for Coulomb repulsion and $l=0$ (full lines). The ratio of the typical short-range parameter R and a_B has been chosen as 1/10 throughout this section. The results remain qualitatively the same for values of R/a_B ranging

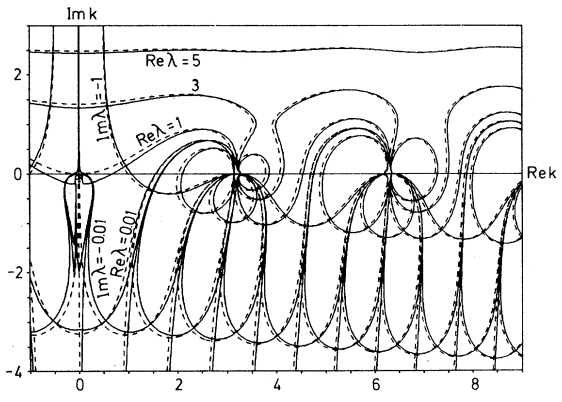

FIG. 3. Level contour plots for the δ-shell plus repulsive Coulomb potential (full lines), and for the pure δ-shell potential (broken lines), for $l=0$, and $R/a_B=0.1$. The levels of the contours (λ in units R) have been chosen the same in both cases (see indicated values). Scales are in units a_B^{-1}. Since $\lambda(k)=\lambda^*(-k^*)$ only the right-hand half-plane is shown.

from zero up to the order of 1. Also shown (broken lines) is the corresponding contour plot for the chargeless case. We observe that well outside the Coulombic region (i.e., well outside the region $|k| \lesssim a_B^{-1}$) and away from the Coulombic cut ($\mathrm{Im}k < 0$) the charged and chargeless cases are very similar. Inside the Coulombic region the maps are fully different. For the charged case this part of the map has been given on a larger scale in Fig. 4. The course of the trajectories here is determined almost exclusively by the Coulomb potential. In fact, the plot is almost identical to Fig. 2 of Ref. 18, where a related map (with a slightly different parametrization of the interaction strength parameter) is given. The trajectories in Figs. 3 and 4 are given with $\mathrm{Re}\lambda$ and $\mathrm{Im}\lambda$, respectively, kept fixed. A positive (negative) value of $\mathrm{Im}\lambda$ corresponds to an absorptive (creative) short-range interaction. In contrast to the map in Fig. 1, we observe in Fig. 3, and more clearly in Fig. 4, the possibility of a resonance very near the origin. Such a resonance is caused by the repulsive Coulomb barrier.

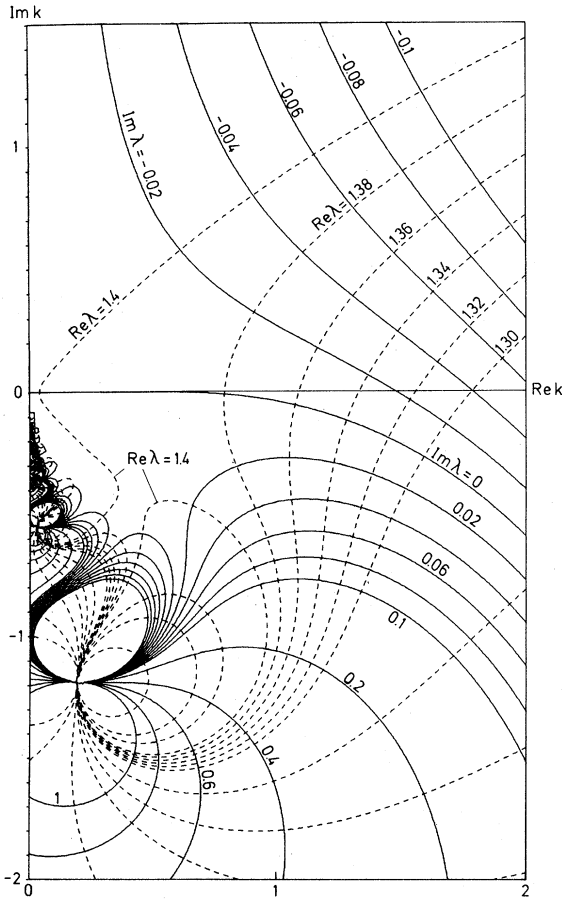

FIG. 4. Part of Fig. 3. Contour plot for the δ-shell plus repulsive Coulomb potential, for $l=0$, and $R/a_B=0.1$. Scales are in units a_B^{-1}, λ is given in units R.

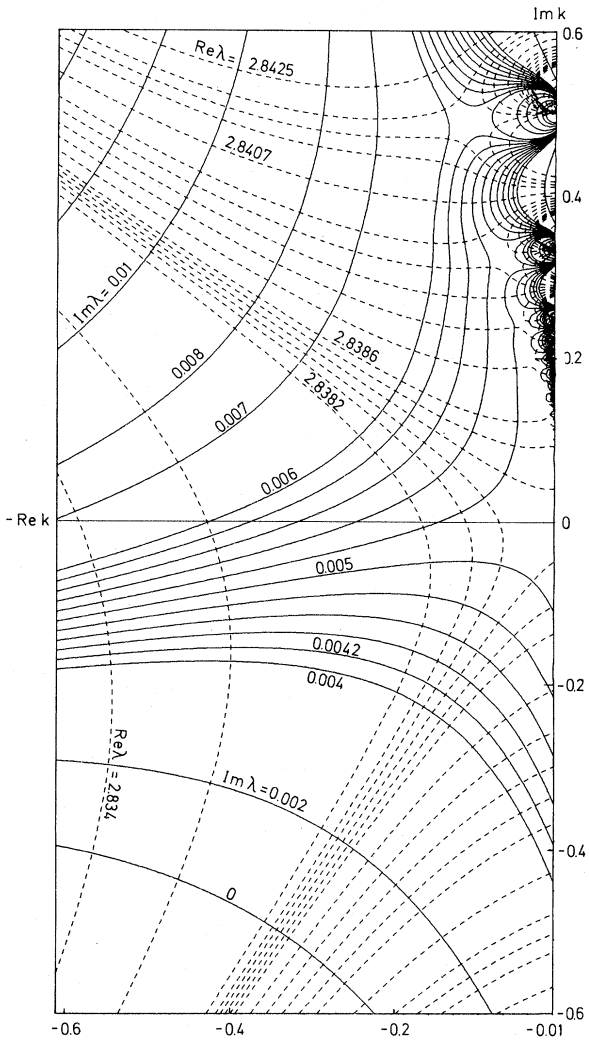

FIG. 5. Contour plot for the δ-shell plus Coulomb potential, for Coulomb attraction, for $l=1$, and $R/a_B=0.1$. Scales are in units a_B^{-1}; λ is given in units R.

In Fig. 5 we show a case of Coulomb attraction and $l=1$. Again $R/a_B=0.1$. Well outside the Coulombic region the picture is similar to the chargeless case; see Fig. 2. We have chosen to show the left half of the k plane for this case. In this part the decaying bound states are located, which can be associated with many physical systems, for example, exotic atoms. The shift of the Coulomb levels for nonvanishing V_s is clearly seen. Also the critical phenomena and the associated saddle points[19] are manifestly present. The origin ($k=0$) is a highly singular point. It is an accumulation point of

bound-state (or decaying bound-state) poles. Also, it is a branch point. Note that poles which cross the negative imaginary k axis with varying λ enter a different Riemann sheet.

X. COULOMB-MODIFIED PHASE SHIFTS AND COULOMB-MODIFIED EFFECTIVE-RANGE PARAMETERS

Coulomb-modified phase shifts $\delta_l^c(k)$ are connected to the physical on-shell T-matrix elements[16]

$$\langle kl\infty- \mid T_{csl}(k^2) \mid kl\infty+\rangle = -(2/\pi)k^{-1}\exp(2i\sigma_l)\exp(i\delta_l^c)\sin\delta_l^c \ , \tag{10.1}$$

so that

$$\cot\delta_l^c - i = -(2/\pi)k^{-1}\exp(2i\sigma_l)/\langle kl\infty- \mid T_{csl} \mid kl\infty+\rangle \ . \tag{10.2}$$

Here σ_l is the pure Coulomb phase shift,

$$\sigma_l = \arg\Gamma(l+1+i\gamma), \quad k>0 \ , \tag{10.3}$$

and $\sigma_l+\delta_l^c$ is the total phase shift. The Coulomb-modified effective-range function $K_{csl}(k^2)$ is defined by

$$K_{csl}(k^2) \equiv k^{2l+1}\begin{bmatrix} l+i\gamma \\ l \end{bmatrix}\begin{bmatrix} l-i\gamma \\ l \end{bmatrix}\left[2\gamma H(\gamma)+\frac{2\pi\gamma}{\exp(2\pi\gamma)-1}(\cot\delta_l^c - i)\right] \ . \tag{10.4}$$

It is known to be a real-meromorphic function of k^2 in a large domain of the k plane, for a large class of potentials.[20,21] Its expansion coefficients in the expansion

$$K_{csl}(k^2) = -1/a_{csl}+\tfrac{1}{2}r_{csl}k^2 - P_{csl}r_{csl}{}^3k^4+Q_{csl}r_{csl}{}^5k^6-\cdots \tag{10.5}$$

are related to the Coulomb-modified low-energy scattering parameters $a_{csl}, r_{csl}, P_{csl}, Q_{csl}, \ldots$. The function $H(\gamma)$ is related to the digamma function ψ,

$$H(\gamma) \equiv \psi(i\gamma)+(2i\gamma)^{-1}-\ln(-i\gamma\,\mathrm{sgn}(s)) \ . \tag{10.6}$$

For the δ-shell potential

$$\langle kl\infty- \mid T_{csl} \mid kl\infty+\rangle = -\exp(2i\sigma_l)\tau_{csl} \mid \langle R \mid kl+\rangle_c \mid ^2 \ . \tag{10.7}$$

The quantities occurring on the rhs of Eq. (10.7) have been given already in Eqs. (10.3), (7.8), and (3.20), and in (8.1) and (8.2). This yields

$$K_{csl}(k^2) = k^{2l+1}\begin{bmatrix} l+i\gamma \\ l \end{bmatrix}\begin{bmatrix} l-i\gamma \\ l \end{bmatrix} \times\left[2\gamma H(\gamma)+\frac{2\pi\gamma}{\exp(2\pi\gamma)-1}\left[\frac{2}{\pi k} \mid \langle R \mid kl+\rangle_c \mid ^{-2}\left\{\lambda_l{}^{-1}-\frac{\pi k}{2}(-)^l \langle R \mid kl\uparrow\rangle_c\langle R \mid kl+\rangle_c\right\}\right]\right] \ . \tag{10.8}$$

K_{csl} is a real-meromorphic function of k^2 in the entire k^2 plane, just like K_{sl}. The following expansions are known:

$$H(\gamma) \sim \ln(i\gamma)-\ln(-i\gamma\,\mathrm{sgn}(s))+\sum_{n=1}^{\infty} \mid B_{2n} \mid (2n)^{-1}\gamma^{-2n}, \quad 0< \mid arg(i\gamma) \mid <\pi \ , \tag{10.9}$$

$$\frac{(l!)^2}{\gamma^{2l}}\begin{bmatrix} l+i\gamma \\ l \end{bmatrix}\begin{bmatrix} l-i\gamma \\ l \end{bmatrix} = \sum_{n=0}^{l} c_n(l)\gamma^{-2n} \ , \tag{10.10}$$

$$\exp(ikr)_1F_1(l+1+i\gamma;2l+2;-2ikr) = \sum_{n=0}^{\infty} g_n\gamma^{-2n} \ , \tag{10.11}$$

$$\exp(ikr)\Gamma(i\gamma-l)U(l+1+i\gamma,2l+2,-2ikr)\sim\sum_{n=0}^{\infty}h_n\gamma^{-2n},\quad 0<|arg(i\gamma)|<\pi\ . \tag{10.12}$$

The coefficients B_{2n}, $c_n(l)$, g_n, and h_n are given in the Appendix. From these expressions we have obtained closed forms for the Coulomb-modified effective-range parameters. These forms are lengthy, and we give only the first two coefficients of the expansion on the rhs of Eq. (10.5). We shall use the parameter ν, where $|\nu|$ is the ratio of a typical short-range parameter ($2R$ in our case) and the Bohr radius $a_B=|s|^{-1}$,

$$\nu\equiv-2sR\equiv2k\gamma R\quad\text{(positive for Coulomb repulsion)}\ . \tag{10.13}$$

We find for Coulomb repulsion ($\nu=2R/a_B$):

$$-a_{csl}{}^{-1}=\frac{2s^{2l+1}}{(l!)^2(I_{2l+1})^2}\left\{-\frac{R}{\lambda_l}+2I_{2l+1}K_{2l+1}\right\}, \tag{10.14}$$

$$\tfrac{1}{2}r_{csl}=\frac{s^{2l-1}}{3(l!)^2(I_{2l+1})^2}\left\{-\nu\frac{R}{\lambda_l}\frac{lI_{2l+3}+\sqrt{\nu}I_{2l+2}}{I_{2l+1}}+2l(l+1)(2l+1)I_{2l+1}K_{2l+1}\right.$$
$$\left.-\tfrac{1}{2}(I_{2l+1})^2-l(l+1)+\tfrac{1}{2}\nu\right\}, \tag{10.15}$$

where I_n and K_n are shorthand notations for $I_n(2\sqrt{\nu})$ and $K_n(2\sqrt{\nu})$, respectively. For details about these Bessel functions of the third kind we refer to Ref. 22. For Coulomb attraction $-\nu$ is positive ($-\nu=2R/a_B$), and we find

$$-a_{csl}{}^{-1}=\frac{2s^{2l+1}}{(l!)^2(J_{2l+1})^2}\left\{\frac{R}{\lambda_l}+\pi J_{2l+1}N_{2l+1}\right\}, \tag{10.16}$$

$$\tfrac{1}{2}r_{csl}=\frac{s^{2l-1}}{3(l!)^2(J_{2l+1})^2}\left\{-\nu\frac{R}{\lambda_l}\frac{lJ_{2l+3}+\sqrt{-\nu}J_{2l+2}}{J_{2l+1}}+\pi l(l+1)(2l+1)J_{2l+1}N_{2l+1}\right.$$
$$\left.-\tfrac{1}{2}(J_{2l+1})^2+l(l+1)-\tfrac{1}{2}\nu\right\}, \tag{10.17}$$

where J_n and N_n stand for the Bessel function $J_n(2\sqrt{-\nu})$, and the Neumann function $N_n(2\sqrt{-\nu})$, respectively.[22] For vanishing Coulomb strength (i.e., in the limit that $s\uparrow0$ for repulsion, or in the limit $s\downarrow0$ for attraction) we have

$$\lim_{s\to0}a_{csl}=a_{sl},\quad\lim_{s\to0}r_{csl}=r_{sl}\ . \tag{10.18}$$

Here a_{sl} and r_{sl} are the low-energy scattering parameters given at the end of Sec. VI. For $l=0$ and Coulomb repulsion the following relationships exist between a_{csl} and a_{sl}, and between r_{csl} and r_{sl}, respectively:

$$-a_{cs0}{}^{-1}=-a_{s0}{}^{-1}\frac{\nu}{(I_1)^2}-R^{-1}\nu\frac{2I_1K_1-1}{(I_1)^2}\ , \tag{10.19}$$

$$\tfrac{1}{2}r_{cs0}=\tfrac{1}{2}r_{s0}\frac{2\sqrt{\nu}I_2}{(I_1)^3}+R\nu\frac{(I_1)^3-\nu I_1-2\nu\sqrt{\nu}I_2}{3\nu^2(I_1)^3}\ . \tag{10.20}$$

To lowest orders in ν we get

$$-a_{cs0}{}^{-1}=-a_{s0}{}^{-1}(1-\nu)-R^{-1}\nu(2C-1/2+\ln\nu)+O(\nu^2\ln\nu)\ , \tag{10.21}$$

$$\tfrac{1}{2}r_{cs0}=\tfrac{1}{2}r_{s0}(1-\tfrac{7}{6}\nu)+\tfrac{7}{36}R\nu+O(\nu^2)\ . \tag{10.22}$$

Here C is Euler's constant. The results (10.19)–(10.22) closely resemble the results obtained in Ref. 23 for the separable Yamaguchi potential. The corresponding results for Coulomb attraction are easily obtained. In particular, Eqs. (10.21) and (10.22) remain valid, provided we replace $\ln\nu$ by its real part, $\ln|\nu|$.

XI. RECONSTRUCTION OF SPECTRA, QUANTUM DEFECTS, AND THE SMALL-SHIFT APPROXIMATION

For Coulomb attraction ($s>0$) the position of the bound states is related to the scattering parameters.

We shall confine ourselves for the sake of illustration to $l=0$. Let us consider Eq. (10.4). Using Eq. (10.7) and

$$\psi(i\gamma)=\psi(-i\gamma)-(i\gamma)^{-1}+\pi i\coth\pi\gamma$$

it can be rewritten as

$$-\pi\frac{\cot\delta_{cs0}-i}{\exp(2\pi\gamma)-1}=\tfrac{1}{2}a_BK_{cs0}(k^2)+\psi(-i\gamma)+(-2i\gamma)^{-1}-\ln(-i\gamma)+\pi i\coth\pi\gamma\ . \tag{11.1}$$

Now K_{cs0} is a real-meromorphic function of k^2 at $k^2=0$, in a large domain containing $k^2=0$. At negative energies ($k^2<0$, $-i\gamma>0$) the sum of the first four terms on the rhs of Eq. (11.1) is a smoothly varying function of k^2. We define its value as

$$\pi\cot\pi\mu\equiv\tfrac{1}{2}a_BK_{cs0}(k^2)+\psi(-i\gamma)+(-2i\gamma)^{-1}-\ln(-i\gamma),\quad k^2<0\ , \tag{11.2}$$

so that

$$\frac{\cot\delta_{cs0}-i}{\exp(2\pi\gamma)-1}=-\cot\pi\mu-i\coth\pi\gamma=-\frac{\sinh\pi(\gamma+i\mu)}{\sinh\pi\gamma\sin\pi\mu},\quad k^2<0. \tag{11.3}$$

We introduce κ by putting $k=i\kappa$, where $-\kappa^2$ now designates the bound-state energy. Bound states occur when

$$\cot\delta_{cs0}-i=0\ ,$$

i.e., for

$$\sin[\pi((\kappa a_B)^{-1}+\mu)]=0\ ,$$

or

$$(\kappa a_B)^{-1}+\mu=n,\quad n=1,2,\ldots\ . \tag{11.4}$$

We stress that $i\kappa$ is no longer the independent momentum variable, but that it gives the bound-state position. For $V_s\to 0$ we have $K_{cs0}\to\infty$, and hence $\mu\to 0$: We retrieve the pure Coulomb bound states given by $\kappa a_B=n^{-1}$. [For this reason we have limited the values of n in Eq. (11.4) to 1,2,....] It is then natural to label κ and μ with n, such that for vanishing V_s

$$\lim_{V_s\to 0}\kappa_n(\mu_n)a_B=n^{-1},\quad n=1,2,\ldots\ , \tag{11.5}$$

since we expect μ to be a continuous function of the strength of V_s. Apparently μ_n are the quantum defects introduced by Seaton.[24,11] They can be obtained from the solution of

$$\pi\cot\pi\mu_n=\tfrac{1}{2}a_BK_{cs0}\left[\frac{-a_B^{-2}}{(n-\mu_n)^2}\right]+\psi(n-\mu_n)+\frac{1}{2(n-\mu_n)}-\ln(n-\mu_n)\ , \tag{11.6}$$

which connects μ_n and the effective-range parameters.

Equation (11.6) contains the complete spectrum-reconstruction or Zel'dovich phenomenon: the sharp collective transition of the energy levels to the adjacent unperturbed values when the strength of the interaction is varied.[25,26] It is illustrated in Fig. 6, where $n-\mu_n$, $n=1,2,3,4$, is plotted as a function

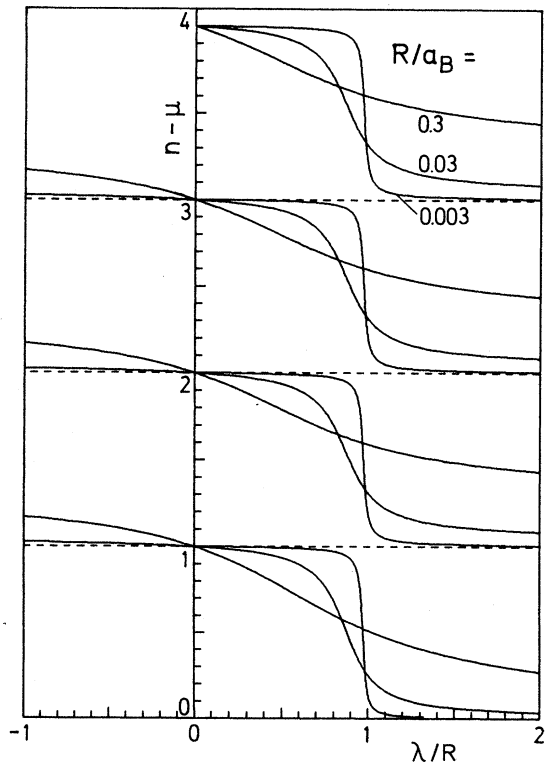

FIG. 6. Plot of $n-\mu_n$ as a function of the strength parameter λ/R, for the δ-shell plus Coulomb potential and S waves, for $n=1$, 2, 3, and 4, for the indicated values of R/a_B.

of λ/R, for the ratios $R/a_B=0.03$ (which is a realistic value for the proton-antiproton system), 0.3, and 0.003, respectively. The value of λ, where reconstruction occurs approximately, can be found by putting $\mu_n=\frac{1}{2}$ on the lhs of Eq. (11.6). For small ratios R/a_B it follows that this corresponds to a strength λ for which $|a_{cs0}| \gg a_B$. Equation (10.17) then gives the following approximate strength-parameter value,

$$\lambda_0/R \simeq 1+(2R/a_B)\ln(2R/a_B) . \tag{11.7}$$

The width $\Delta\lambda$ of the reconstruction region can be estimated similarly by setting $\mu_n=\frac{1}{4}$ and $\frac{3}{4}$, respectively, and taking

$$\Delta\lambda \equiv \lambda(\mu_n=\tfrac{3}{4})-\lambda(\mu_n=\tfrac{1}{4}) .$$

The estimate

$$\Delta\lambda/R \simeq 2\pi(2R/a_B)$$

is easily obtained. Considering Fig. 7, one observes that μ_n is almost independent of n for $n \geq 2$. For $\lambda \to \infty$, Eq. (11.6) shows that

$$1-\mu_n \simeq 2a_{cs0}(\lambda \to \infty)/a_B ,$$

for $n=2,3,\ldots$ Eq. (10.16) then shows that

$$1-\mu_n(\lambda \to \infty) \simeq \frac{2R}{a_B}\left[1+\frac{2R}{a_B}\ln\frac{2R}{a_B}+(C-1)\frac{2R}{a_B}\right] , \quad n=2,3,\ldots . \tag{11.8}$$

Note that for $n=1$ the estimate (11.8) does not hold; $n=1$ corresponds to the ground state, which for $\lambda \to \infty$ moves off to $k^2=-\infty$, and hence $1-\mu_1 \to 0$. The fact that

$$1-\mu_n(\lambda \to \infty)=\mu_{n-1}(\lambda \to -\infty) \simeq 2R/a_B , \quad n=2,3,\ldots ,$$

implying that the nth unperturbed level never reaches an adjacent unperturbed level, is intimately connected[27] to the rank-one character of the δ-shell potential.

An old and famous formula exists for the shift ΔE_{nl} (with respect to the pure Coulomb energy E_{nl}) caused by the short-range interaction V_s. It is the so-called small-shift approximation (SSA), derived by Deser *et al.*[28] for $l=0$, and reads

$$\frac{\Delta E_{n0}^{\mathrm{SSA}}}{E_{n0}} = -\frac{4}{n}\frac{a_s}{a_B} . \tag{11.9}$$

It is of great interest to test the quality of SSA in our case. It is trivial to compute the exact value of ΔE_{n0}. For clarity we shall name (for $n=1$) this value $\Delta E^{\mathrm{exact}}$. In Fig. 7 we have plotted the ratio $\Delta E^{\mathrm{SSA}}/\Delta E^{\mathrm{exact}}$, as calculated according to Eq. (11.9) for the ground state. One observes that the SSA is a rather poor approximation. In particular, *even in the limit of vanishing short-range potential its predicted energy shift is wrong, compared to the exact shift,* by some 6%. Indeed, one can easily show that generally

$$\lim_{\lambda \to 0}\frac{\Delta E^{\mathrm{SSA}}}{\Delta E^{\mathrm{exact}}} = \lim_{\lambda \to 0} -a_s K_{cs0}(-n^{-2}a_B{}^{-2}) .$$

In our explicit potential model the rhs of this equation reduces to

$$[\exp(-R/na_B)\,{}_1F_1(1-n;2;2R/na_B)]^{-2} .$$

A better approximation for ΔE_{n0} is obtained by using (cf. curve 2 of Fig. 7)

$$\frac{\Delta E_{n0}}{E_{n0}} \simeq -\frac{4}{n}\frac{a_{cs}}{a_B} , \tag{11.10}$$

see, for example, Deloff.[29] This formula gives a correct limiting behavior when $\lambda \to 0$. In Fig. 7 we show five other possible approximations. All five are based on Eq. (11.6), and use approximations for μ_n, which are then inserted into the expression

$$\frac{\Delta E_{n0}}{E_{n0}} = \frac{n^2}{(n-\mu_n)^2}-1 . \tag{11.11}$$

The four curves labeled 3 to 7 correspond to

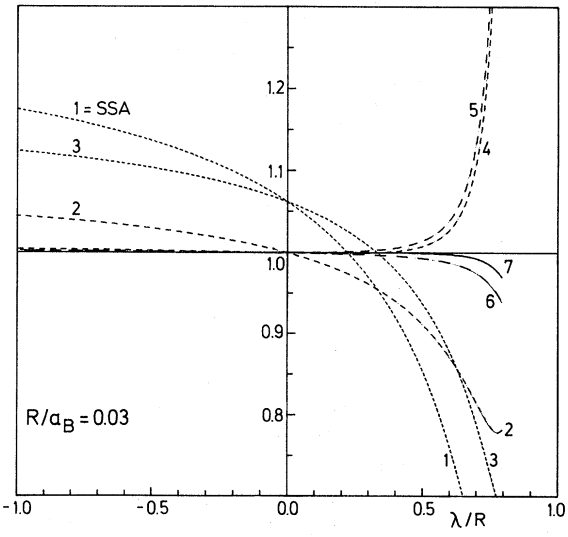

FIG. 7. Plot of $\Delta E^{\mathrm{approx}}/\Delta E^{\mathrm{exact}}$ vs λ/R, for $n=1$, $l=0$, $R/a_B=0.03$, for seven different approximations; see text. The curve labeled 1 is the standard SSA.

$$\mu_1 = -2a_s/a_B \ , \tag{11.12}$$

$$\mu_1 = -2a_{cs}/a_B \ , \tag{11.13}$$

$$\mu_1 = 1/(-\tfrac{1}{2}a_B/a_{cs} + \tfrac{1}{4}r_{cs}/a_B + \psi(1) + \tfrac{1}{2}) \ , \tag{11.14}$$

$$\mu_1 = \pi^{-1}\arctan(-2\pi a_{cs}/a_B) \ , \tag{11.15}$$

$$\mu_1 = \pi^{-1}\arctan[1/(-\tfrac{1}{2}a_B/a_{cs} + \tfrac{1}{4}r_{cs}/a_B + \psi(1) + \tfrac{1}{2})] \ , \tag{11.16}$$

respectively. Note that $n = 1$ has been taken. Also note that

$$|\psi(n) + \tfrac{1}{2}n^{-1} - \ln n| < 0.0773$$

for $n \geq 1$. Especially curve 7 shows an excellent agreement for the energy shift up to large values of λ.

XII. DISCUSSION

The δ-shell potential is convenient to use. It has the unique property that it is both local, and separable in each partial wave. Its local character makes it easily visualized in the coordinate representation. Its separable character is a big computational advantage: The frequent use of separable forces in n-body calculations is primarily based on this fact. Its simple form has enabled us to derive many closed analytical expressions for the quantities commonly encountered in scattering theory of charged and uncharged particles, *for all l.* In particular, we have derived formulas for the low-energy effective-range parameters [Eqs. (6.7)–(6.10)], and the Coulomb-modified effective-range parameters [Eqs. (10.14)–(10.17)], *for all l.*

As an illustration, let us consider proton-proton scattering in the $^1S_0$ partial wave. The first three of the following four experimental values,[30]

$$a_{cs0} = -7.822 \pm 0.004 \text{ fm} \ ,$$

$$r_{cs0} = 2.830 \pm 0.017 \text{ fm} \ ,$$

$$a_B = 57.60 \text{ fm} \ ,$$

$$P_{cs0} = 0.051 \pm 0.014 \ ,$$

give range- and strength-parameter values

$$R = 2.126 \pm 0.013 \text{ fm} \ ,$$

$$\lambda_0 = 1.880 \pm 0.010 \text{ fm} \ .$$

These give in turn the value

$$P_{cs0} = -0.0377 \pm 0.0001$$

and the values

$$a_{s0} = -16.25 \pm 0.08 \text{ fm} \ ,$$

$$r_{s0} = 3.02 \pm 0.02 \text{ fm} \ ,$$

$$P_{s0} = -0.0187 \pm 0.0003 \ .$$

The experimental $^1S_0$ values for neutron-neutron scattering are[31]

$$a_{s0} = -18.6 \pm 0.5 \text{ fm} \ ,$$

$$r_{s0} = 2.83 \pm 0.11 \text{ fm} \ .$$

Within the limitations of this simple model and the assumption of point charges this gives support to a high degree of charge symmetry of the strong nuclear forces. A recent more realistic potential model[32] gives an even closer agreement.

A second illustration is provided by the $N-\alpha$ system. In the $P_{3/2}$ wave we have from proton-α data[33]

$$a_{cs1} = -44.83 \pm 0.51 \text{ fm}^3 \ ,$$

$$r_{cs1} = -0.365 \pm 0.013 \text{ fm}^{-1} \ ,$$

$$a_B = 18.00 \text{ fm} \ ,$$

$$P_{cs1} r_{cs1}^{\ 3} = -0.60 \pm 0.04 \text{ fm} \ .$$

The first three numbers determine R and λ_1,

$$R = 2.81 \pm 0.05 \text{ fm} \ ,$$

$$\lambda_1 = 7.49 \pm 0.07 \text{ fm} \ ,$$

giving in turn the value

$$P_{cs1} r_{cs1}^{\ 3} = -0.53 \pm 0.03 \text{ fm}$$

(which is remarkably close to the experimental value -0.60 ± 0.04 fm), and

$$a_{s1} = -59 \pm 3 \text{ fm}^3 \ ,$$

$$r_{s1} = -0.80 \pm 0.02 \text{ fm}^{-1} \ ,$$

$$P_{s1} r_{s1}^{\ 3} = -0.795 \pm 0.015 \text{ fm} \ ,$$

showing remarkable agreement with experimental neutron-α data[34]

$$a_{s1} = -63.057 \pm 0.003 \text{ fm}^3 ,$$

$$r_{s1} = -0.8805 \pm 0.0012 \text{ fm}^{-1} ,$$

$$P_{s1} r_{s1}^3 = -0.717 \pm 0.016 \text{ fm} .$$

This supports charge symmetry within the same limitations as in the NN case.

Our formulas (10.19)–(10.22) serve as an explicit and useful potential-model check for the general approximate relations between effective-range parameters and Coulomb-modified effective-range parameters, which have been recently presented[35] for $l > 0$. To our knowledge the only other explicit potential model for which closed expressions have been published for Coulomb-modified effective-range parameters for all l is that of Ref. 36.

In Secs. II and IV, and in Secs. VII and VIII, we have given many explicit formulas for various states and various matrix elements, in the uncharged and the charged case, respectively. Furthermore, we note that explicit formulas can be obtained for the scattering states $|kl+\rangle$, the Jost states $|kl\uparrow\rangle$, the off-shell Jost states $|kql\uparrow\rangle$, the so-called regular states $|kl\textcircled{R}\rangle$, and the corresponding radial solutions, and Jost states, for the interaction $V_c + V_s$, since the corresponding quantities for V_c are known.[5,16,37] The formulas for all l remain relatively simple. The only other interaction we know for which so many explicit formulas can be derived for all l is the separable potential with simple rational form factors; see, for example, Refs. 36 and 38. The latter interaction is essentially nonlocal.

The δ-shell potential is special in some further respects. Though $\bar{T}_l$ is *real* by construction (for real V_l) the δ-shell interaction is unique in the respect that the equality of $\bar{T}_l$ and V_l holds; see Eq. (4.9). In general, $\langle p' | \bar{G}_{0l} | p \rangle$ and $\langle p' | \bar{T}_l | p \rangle$ are *not* symmetric, i.e.,

$$\langle p' | \bar{G}_{0l} | p \rangle \neq \langle p | \bar{G}_{0l} | p' \rangle$$

and

$$\langle p' | \bar{T}_l | p \rangle \neq \langle p | \bar{T}_l | p' \rangle .$$

However, for the δ-shell potential the equality

$$\langle p' | \bar{T}_l | p \rangle = \langle p | \bar{T}_l | p' \rangle$$

does hold. Quite generally $\bar{G}_{0l} \neq \bar{G}_{0l}^{\dagger}$. We recall that $\bar{G}^{\dagger}(k^2)$ is the Green's function for "the regular solution." (See Ref. 5.)

A potential consisting of one δ function with a range parameter R may be too simple to describe realistically the interaction between physical particles. One then may use a superposition of δ-shell potentials with different ranges $R_i, i = 1, \ldots, N$, and strength parameters λ_i which can be chosen of different sign. This is a rank-N potential, which admits at most N bound states. The extension of the formalism to the rank-N case is straightforward.

In Secs. V and IX we have explored the relationship between pole positions k and the strength parameter λ, with the Coulomb potential taken zero, repulsive, and attractive, respectively, for some representative values of l. The results nicely illustrate the general theory in Ref. 12. They agree with general results obtained for the Coulomb plus any short-range potential.[10,18,19,39] They are similar to other explicit models.[40] In Ref. 10 the course of the trajectories near the origin in the fourth quadrant of the k plane is analyzed in detail. Our results confirm this analysis.

In our explicit potential model we have studied reconstruction of spectra (see Ref. 26), quantum defects, and the small-shift approximation in Sec. XI. Related work has been carried out by Popov and collaborators[41] for different potential models. We stress that the standard SSA, given by Eq. (11.9), is a very poor approximation, even for small values of the ratio of the typical short-range parameter and a_B. A better approximation is to replace a_s by a_{cs}. This is convenient because often a_{cs} can be determined directly from experiment. Much better approximations for the energy shift result from using Eq. (11.11), and Eq. (11.15) when a_{cs} is known, or Eq. (11.16) when both a_{cs} and r_{cs} are known, cf. Fig. 7. Even for complex-valued scattering lengths Eqs. (11.11)–(11.16) can be used.

APPENDIX

In this appendix we specify the expansion coefficients occurring in Eqs. (10.9)–(10.12). The B_{2n} are the Bernoulli numbers.[8,22] Note that in Ref. 15 a different convention for the Bernoulli numbers is used. The expansion coefficients $c_n(l)$, g_n, and h_n are obtained from Ref. 15. The $c_n(l)$, $n = 0, \ldots, l$, are given by

$$c_0(l) = 1 , \; c_1(l) = l(l+1)(2l+1)/6 ,$$

$$c_2(l) = (2l-1)(l-1)l(l+1)(2l+1)(5l+6)/360 ,$$

$$c_3(l) = (2l-3)(2l-1)(l-2)(l-1)l(l+1)(2l+1)(35l^2+91l+60)/45360, \ldots , \; c_l(l) = (l!)^2 .$$

The g_n and h_n follow from Eqs. (6.12), (6.13), and (8.28) of Ref. 15, respectively. The connection between our notation [see Eqs. (8.1) and (8.2) and Eqs. (10.11) and (10.12)] and the notation of Ref. 15 [see Eqs. (2.5) and (2.6) of Ref. 15] is given by the correspondence $l \leftrightarrow L$, $\gamma \leftrightarrow \eta$, $kr \leftrightarrow \rho$, and

$$\langle r \mid kl+ \rangle_c \leftrightarrow i^L e^{i\sigma_L} (2/\pi)^{1/2} \rho^{-1} F_L(\rho) ,$$

$$\langle r \mid kl\uparrow \rangle_c \leftrightarrow i^L e^{-i\sigma_L} (2/\pi)^{1/2} \rho^{-1} H_L(\rho) .$$

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