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# Coupled adiabatic approximation in the three-body problem

J. L. Ballot and M. Fabre de la Ripelle
*Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France*

J. S. Levinger
*Department of Physics, Rensselaer Polytechnic Institute, Troy, New York 12181*
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In the framework of the hyperspherical formalism, we present a study of the coupled adiabatic approximation for the case of three nucleons interacting via central spin-dependent two-body potentials. We analyze the convergence of the ground state eigenvalues versus the grand orbital quantum number ($2K$) and compare the results to that of the coupled equations. We also compare with two simpler but less accurate approximations: the uncoupled adiabatic approximation and the extreme adiabatic approximation. The former provides an upper and the latter provides a lower bound to the ground state energy.

[NUCLEAR STRUCTURE Few body bound states, reduction of hyperspherical equations applied to triton, quantum few-body problem.]

## I. INTRODUCTION

In recent years the ground state of the three nucleon system has been analyzed by many authors using different methods which were stimulated by the feasibility of solving numerically, in various ways, the few particle equations. Among these approaches we may distinguish between the Faddeev integral equation methods[1–6] and the Schrödinger variational methods[7–17] which have yielded in the last decade successful calculations in the bound state problem of the $^3$He and $^3$H nuclei.

In this work we use an adiabatic approximation, proposed by one of us ten years ago,[18] to find the triton binding energy and wave function for central two-body potentials. The idea of the adiabatic approximation was introduced in the past by Born and Oppenheimer to solve the bound of electrons and nuclei in a diatomic molecule. That is, the electronic wave function and eigenvalue $U$ are calculated for frozen internuclear coordinate $R$. This eigenvalue $U(R)$ is then used to determine the vibrational and rotational level of the molecule. This extreme adiabatic approximation was later proved by Born[19,20] and the reader can find a discussion of these approximations by Kolos.[21]

Recently, this method was used by authors treating two-electron and three-electron problems with hyperspherical harmonic expansions.[22–26] The extreme adiabatic approximation (EAA) was used both for the ground state and for the continuum state. More recently, the extreme adiabatic approximation and the uncoupled adiabatic approximation (UAA) were used[27,28] to find numerical values for the triton binding energy for different choices of a central two body potential and the accuracy was judged by comparison with the corresponding value of the triton binding energy calculated as the solution of coupled differential equations (CE) (Refs. 10 and 12).

In Sec. II we present the mathematical development of the coupled adiabatic approximation and demonstrate the basic inequalities

$$E(\mathrm{EAA}) \leq E(\mathrm{CE}) \leq E(\mathrm{CAA}) \leq E(\mathrm{UAA}) ,$$

where the binding energies for fixed grand orbital quantum number ($2K$) are those in the EAA, the coupled differential equation (which gives the exact result for a given value of $K$), the coupled adiabatic approximation (CAA), and the UAA, respectively.

In Sec. III we apply our three forms of the adiabatic approximation to find the triton energy and wave function for five different two-body potentials. The accuracy of the approximation is found by comparison with the results of the coupled differential equations[12] for the same two-body poten-

 

tial.

In the last section we estimate accuracy with the coupled adiabatic approximation and also with the EAA and UAA. We discuss possible applications of these adiabatic approximations to several other problems.

## II. THE ADIABATIC APPROXIMATION

The Schrödinger equation for a wave function

$$\Psi(r,\Omega)=r^{-(D-1)/2}u(r,\Omega)$$

in hyperspherical coordinates is as follows:

$$\left\{-\frac{\hbar^2}{m}\left[\frac{d^2}{dr^2}+(L^2(\Omega)-(D-1)(D-3)/4)r^{-2}\right]+V(r,\Omega)-E\right\}u(r,\Omega)=0\ . \tag{2.1}$$

$D=3(A-1)$ is the space dimensionality for a system of $A$ particles after elimination of the center of mass $X$. $L^2(\Omega)$ is the grand orbital operator while $V(r,\Omega)$ is the potential.

The idea leading to the adiabatic approximation is the following. Instead of using the hyperspherical harmonic (h.h.) basis in order to solve the Schrödinger equation one chooses the basis $B_\lambda(r,\Omega)$. This set consists of the eigenfunctions of the angular operator

$$\left\{-\frac{\hbar^2}{m}[L^2(\Omega)-(D-1)(D-3)/4]r^{-2}+V(r,\Omega)\right\}B_\lambda(r,\Omega)=U_\lambda(r)B_\lambda(r,\Omega)\ , \tag{2.2}$$

where $r$ is a parameter.

The eigenvector $B_\lambda(r,\Omega)$ is for the eigenpotential $U_\lambda(r)$. In using the complete $B_\lambda$ basis for the expansion of the wave function $U(r,\Omega)$, the equations obtained in projecting (2.1) on this basis become coupled by the first and second derivatives of $B_\lambda(r,\Omega)$ with respect to $r$. When the variation of $B_\lambda$ with $r$ is small we expect to generate nearly decoupled equations which can be easily solved using the eigenpotential $U_\lambda(r)$.

For improving the solution in which only one equation is taken into account one uses the concept of optimal subset; i.e., one assumes that only the equations directly coupled to the main equation contribute significantly to the solution.

Let us write Eq. (2.1) using

$$u(r,\Omega)=B_\lambda(r,\Omega)u_\lambda(r)\ ,$$
$$\left[-\frac{\hbar^2}{m}\frac{d^2}{dr^2}+U_\lambda(r)-E\right]B_\lambda(r,\Omega)u_\lambda(r)=0\ , \tag{2.3}$$

where

$$\int |B_\lambda(r,\Omega)|^2 d\Omega=1\ . \tag{2.4}$$

Taking the derivatives of $B_\lambda(r,\Omega)$ one generates two new orthogonal elements $B_\lambda^{(1)}$ and $B_\lambda^{(2)}$:

$$\frac{dB_\lambda}{dr}=C_1B_\lambda^{(1)}\ ,$$
$$\frac{d^2B_\lambda}{dr^2}=C_1C_2B_\lambda^{(2)}+\frac{dC_1}{dr}B_\lambda^{(1)}-C_1^2B_\lambda\ , \tag{2.5}$$

where the normalization constants are chosen in such a way that

$$\langle B_\lambda^{(i)}(r,\Omega)\,|\,B_\lambda^{(j)}(r,\Omega)\rangle=\delta_{ij}\ ,$$
$$B_\lambda^{(0)}\equiv B_\lambda\ . \tag{2.6}$$

The bracket means an integration over the surface of the unit sphere in the $D$ dimensional space as in Eq. (2.4).

One expands the wave function according to

$$\Psi(r,\Omega)=r^{-(D-1)/2}\sum_{i=0}^{2}B_\lambda^{(i)}(r,\Omega)u_\lambda^{(i)}(r)\ . \tag{2.7}$$

The Schrödinger equation (2.1) is transformed into a set of three coupled equations

$$-\left[\frac{\hbar^2}{m}\frac{d^2}{dr^2}+E\right]u_\lambda^{(i)}(r)+\sum_{j=0}^{2}U_\lambda^{(i,j)}(r)u_\lambda^{(j)}(r)=0\ , \tag{2.8}$$

where

$$U_\lambda^{(i,j)}(r)=\left\langle B_\lambda^{(i)}\left|-\frac{\hbar^2}{mr^2}\left[L^2(\Omega)-\frac{(D-1)(D-3)}{4}\right]+V(r,\Omega)\right|B_\lambda^{(j)}\right\rangle+W_\lambda^{(i)}\delta ij+\frac{\hbar^2}{m}P_j^{(i)}\ , \tag{2.9}$$

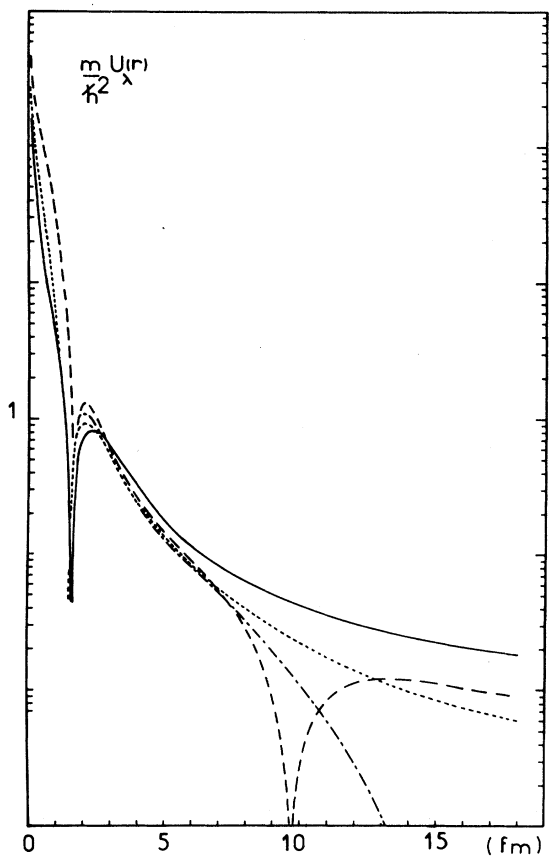


FIG. 1. Comparison of the eigenpotentials $|U_\lambda(r)|$ determined from the two-body interactions: Volkov (continuous line), $G2$ (dashed line), $S4$ (dotted line), and $S3$ (dotted-dashed line).

and $W_\lambda^{(i)}$ is given in terms of the normalization functions $C_i(r)$

$$W_\lambda^{(0)}=\frac{\hbar^2}{m}C_1^2,\quad W_\lambda^{(2)}=\frac{\hbar^2}{m}C_2^2,$$
$$W_\lambda^{(1)}=W_\lambda^{(0)}+W_\lambda^{(2)}, \tag{2.10}$$

while $P_j^{(i)}$ are the following differential operators:

$$P_1^{(0)}=-P_0^{(1)}=2\sqrt{C_1}\frac{d}{dr}\sqrt{C_1}$$
$$=\frac{dC_1}{dr}+2C_1\frac{d}{dr},$$
$$P_2^{(1)}=-P_1^{(2)}=2\sqrt{C_2}\frac{d}{dr}\sqrt{C_2}, \tag{2.11}$$
$$P_2^{(0)}=P_0^{(2)}=-C_1C_2.$$

The elements $B(r,\Omega)$ orthogonal to $B_\lambda^{(i)}$ $(i=0,1,2)$ do not give an equation coupled to the main $(i=0)$ equation and are neglected in the optimal subset approximation.

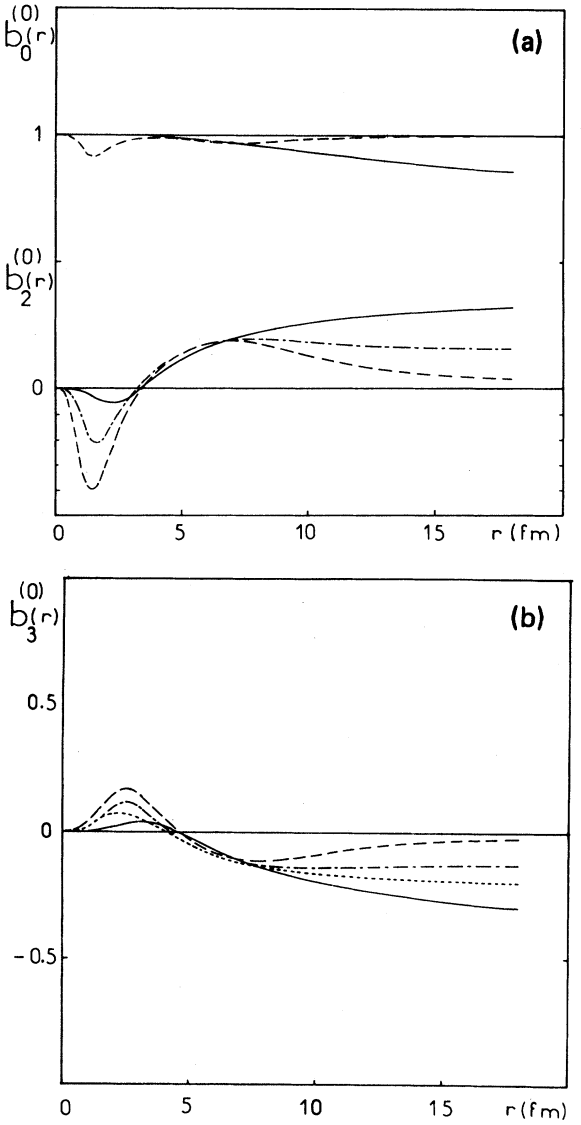


FIG. 2. Comparison of (a) the eigenfunctions $b_0^{(0)}(r)$ and $b_2^{(0)}(r)$ determined from two-body interactions: Volkov (continuous line), $G2$ (dashed line), and $S3$ (dotted-dashed line), and of (b) the eigenfunction $b_3^{(0)}(r)$ determined from two-body potentials: Volkov (continuous line), $G2$ (dashed line), $S4$ (dotted line), and $S3$ (dotted-dashed line).

The $B_\lambda^{(i)}$ constitute the optimal adiabatic subset. The main equation without coupling is called the uncoupled adiabatic approximation (UAA)

$$\left\{-\frac{\hbar^2}{m}\frac{d^2}{dr^2}+U_\lambda^{(0,0)}(r)-E\right\}u_\lambda^{(0)}(r)=0. \tag{2.12}$$

TABLE I. Convergence of the triton binding energy in terms of $K$ for the Volkov potential for each adiabatic approximation EAA, UAA, CAA, and the corresponding coupled equation (CE) results.

| | Volkov potential (Ref. 31) | | | |
|---|---|---|---|---|
| $K$ | EAA | UAA | CAA | CE |
| 2 | 8.190 | 8.036 | 8.074 | 8.079 |
| 3 | 8.475 | 8.278 | 8.330 | 8.330 |
| 4 | 8.524 | 8.324 | 8.375 | 8.376 |
| 5 | 8.568 | 8.363 | 8.414 | 8.416 |
| 6 | 8.598 | 8.389 | 8.440 | 8.443 |
| 7 | 8.607 | 8.397 | 8.448 | 8.451 |
| 8 | 8.615 | 8.402 | 8.453 | 8.457 |
| 9 | 8.619 | 8.406 | 8.457 | 8.461 |
| 10 | 8.621 | 8.408 | 8.458 | 8.463 |
| 11 | 8.623 | 8.4086 | 8.4591 | 8.464 |
| 12 | 8.624 | 8.4092 | 8.4597 | 8.465 |

Here

$$U_\lambda^{(0,0)}(r)=U_\lambda(r)+\frac{\hbar^2}{m}C_1{}^2(r)\ . \tag{2.13}$$

The neglect of $C_1{}^2$ in (2.13) leads to the extreme adiabatic approximation (EAA). The extreme adiabatic approximation provides a lower limit to the ground state binding energy, as shown below.

Let us assume that the exact solution of (2.1) is given by

$$u_X(r,\Omega)=B_X(r,\Omega)u_X(r)\ .$$

The projection of the Schrödinger equation on any $B(r,\Omega)$ orthogonal to $B_X(r,\Omega)$ disappears and the Schrödinger equation becomes

TABLE II. Convergence of the triton binding energy in terms of $K$ for the Afnan-Tang ($S3$) potential for each adiabatic approximation EAA, UAA, CAA, and the corresponding coupled equation (CE) results.

| | $S3$ potential (Ref. 33) | | | |
|---|---|---|---|---|
| $K$ | EAA | UAA | CAA | CE |
| 2 | 2.396 | 2.050 | 2.112 | 2.12 |
| 3 | 4.976 | 4.480 | 4.593 | 4.594 |
| 4 | 5.602 | 5.072 | 5.191 | 5.196 |
| 5 | 6.252 | 5.655 | 5.785 | 5.775 |
| 6 | 6.733 | 6.076 | 6.211 | 6.208 |
| 7 | 6.899 | 6.219 | 6.355 | 6.355 |
| 8 | 7.033 | 6.334 | 6.471 | 6.470 |
| 9 | 7.130 | 6.417 | 6.554 | 6.557 |
| 10 | 7.172 | 6.453 | 6.589 | 6.592 |
| 11 | 7.203 | 6.480 | 6.615 | 6.620 |
| 12 | 7.226 | 6.501 | 6.634 | 6.640 |

TABLE III. Convergence of the triton binding energy in terms of $K$ for the Eikemeier-Hakenbroich ($S4$) potential for each adiabatic approximation EAA, UAA, CAA, and the corresponding coupled equation (CE) results.

| | $S4$ potential (Ref. 32) | | | |
|---|---|---|---|---|
| $K$ | EAA | UAA | CAA | CE |
| 0 | 4.596 | 4.596 | 4.596 | 3.667 |
| 2 | 4.919 | 4.697 | 4.738 | 4.741 |
| 3 | 6.106 | 5.842 | 5.891 | 5.891 |
| 4 | 6.400 | 6.133 | 6.181 | 6.182 |
| 5 | 6.722 | 6.443 | 6.493 | 6.490 |
| 6 | 6.980 | 6.686 | 6.736 | 6.735 |
| 7 | 7.075 | 6.773 | 6.824 | 6.824 |
| 8 | 7.158 | 6.846 | 6.899 | 6.898 |
| 9 | 7.222 | 6.902 | 6.955 | 6.957 |
| 10 | 7.251 | 6.927 | 6.980 | 6.982 |
| 11 | 7.275 | 6.946 | 7.000 | 7.002 |
| 12 | 7.293 | 6.961 | 7.015 | 7.018 |

$$\left\{-\frac{\hbar^2}{m}\frac{d^2}{dr^2}+U_X(r)+\frac{\hbar^2}{m}C_X{}^2(r)-E_X\right\}u_X(r)=0\ ,$$

where $U_X(r)$ is the matrix element (2.9) in which $\lambda=X$ for $i=j=0$, while $C_X$ is taken from (2.5). Let us label $\lambda=0$ the ground state.

The inequality $U_0(r)<U_X(r)$ results from the property that $B_0(r,\Omega)$ is associated with the lowest eigenpotential $U_0(r)$.

Therefore, the extreme adiabatic equation

$$\left\{-\frac{\hbar^2}{m}\frac{d^2}{dr^2}+U_0(r)-E_E\right\}u_E(r)=0$$

TABLE IV. Convergence of the triton binding energy in terms of $K$ for the Bell-Delves (BD) potential for each adiabatic approximation EAA, UAA, and the corresponding coupled equation (CE) results.

| | Bell-Delves Potential (Ref. 34) | | |
|---|---|---|---|
| $K$ | EAA | UAA | CE |
| 0 | | | 2.541 |
| 2 | 3.148 | 3.137 | 3.140 |
| 3 | 3.559 | 3.539 | 3.542 |
| 4 | 3.652 | 3.629 | 3.633 |
| 5 | 3.739 | 3.713 | 3.717 |
| 6 | 3.808 | 3.780 | 3.785 |
| 7 | 3.835 | 3.806 | 3.810 |
| 8 | 3.858 | 3.829 | 3.833 |
| 9 | 3.878 | 3.848 | 3.852 |
| 10 | 3.888 | 3.857 | 3.862 |
| 11 | 3.897 | 3.865 | 3.871 |
| 12 | 3.905 | 3.873 | 3.878 |

TABLE V. Convergence of the triton binding energy in terms of $K$ for the $G2$ potential for each adiabatic approximation EAA, UAA, CAA, and the corresponding coupled equation (CE) results compared to the interpolated formula values.

| | $G2$ potential (Ref. 12) | | | | |
|---|---|---|---|---|---|
| $K$ | EAA | UAA | CAA | CE | Interpolated (formula $X$) |
| 0 | | | | | |
| 2 | | | | | |
| 3 | 3.098 | 2.303 | 2.502 | 2.504 | 2.46 |
| 4 | 4.864 | 3.870 | 4.116 | 4.125 | 4.07 |
| 5 | 6.517 | 5.293 | 5.572 | 5.589 | 5.54 |
| 6 | 7.618 | 6.254 | 6.534 | 6.560 | 6.52 |
| 7 | 7.976 | 6.564 | 6.844 | 6.873 | 6.846 |
| 8 | 8.253 | 6.808 | 7.084 | 7.117 | 7.097 |
| 9 | 8.447 | 6.982 | 7.253 | 7.290 | 7.275 |
| 10 | 8.527 | 7.056 | 7.323 | 7.362 | 7.35 |
| 11 | 8.588 | 7.113 | 7.377 | 7.417 | 7.408 |
| 13 | 8.633 | 7.156 | 7.416 | 7.458 | 7.451 |

provides a lower limit to the binding energy. We have the following sequence:

$$E_E < E_X < E_C < E_U ,$$

where the ground state binding energies $E$ refer, respectively, to the extreme adiabatic ($E$), exact ($X$), coupled adiabatic ($C$), and uncoupled adiabatic ($U$) equations. Of course the inequalities for $E_C$ and $E_U$ follow immediately from the variational principle.

## III. DETERMINATION OF THE EIGENBASIS $B_\lambda(r,\Omega)$

Let us assume that we have to solve a problem in which the wave function is expanded on the h.h. basis $P_{2K}(\Omega)$:

$$u(r,\Omega)=\sum_{K=0}^{K_{\max}} P_{2K}(\Omega)u_{2K}(r) , \tag{3.1}$$

where $K$ varies from zero to $K_{\max}$.

Equation (2.8) is solved in using the same truncated h.h. basis for the expansion of $B_\lambda^{(i)}$:

$$B_\lambda^{(i)}(r,\Omega)=\sum_{K=0}^{K_{\max}} b_{K,\lambda}^{(i)}(r)P_{2K}(\Omega) . \tag{3.2}$$

The set $b_{K,\lambda}^{(0)}(r)$ is a solution of the system of linear equations

$$\left[\frac{\hbar^2}{m}L_K(L_K+1)r^{-2}-U_\lambda(r)\right]b_{K,\lambda}^{(0)}(r)+\sum_{K'=0}^{K_{\max}} U_K^{K'}(r)b_{K',\lambda}^{(0)}(r)=0 . \tag{3.3}$$

$L_K=2K+(D-3)/2$ for bosons ground state and $D=6$ for the triton. $U_K^{K'}(r)$ is the potential matrix

$$U_K^{K'}(r)=\langle P_{2K}(\Omega)\,|\,V(r,\Omega)\,|\,P_{2K'}(\Omega)\rangle$$
$$=3\sum_{K''}(-1)^{K''}\langle K\,|\,K''\,|\,K'\rangle V_{2K''}(r) ,$$

where $V_{2K''}(r)$ are the multipole of the potential (see Ref. 17).

According to Eqs. (2.4) and (2.5) the $b_{k,\lambda}^{(i)}(r)$ and the normalization constants are given by

$$b_{K,\lambda}^{(1)}(r)=C_1{}^{-1}\frac{db_{K,\lambda}^{(0)}(r)}{dr} ,$$
$$b_{K,\lambda}^{(2)}(r)=(C_1C_2)^{-1}\left[\frac{d^2}{dr^2}-\frac{1}{C_1}\frac{dC_1}{dr}\frac{d}{dr}+C_1{}^2\right]b_{K,\lambda}^{(0)}(r) , \tag{3.4}$$

with

$$\sum_K b_{K,\lambda}^{(i)}b_{K,\lambda}^{(j)}=\delta_{ij} ,$$

$$C_1{}^2=\sum_K\left[\frac{d}{dr}b_{K,\lambda}^{(0)}(r)\right]^2 ,$$

$$[C_1C_2]^2=\sum_K\left[\left[\frac{d^2}{dr^2}-\frac{1}{C_1}\frac{dC_1}{dr}\frac{d}{dr}+C_1{}^2\right]\times b_{K,\lambda}^{(0)}(r)\right]^2 .$$

Using the expansion (3.2) the bracket in Eq. (2.9) becomes

$$\left\langle B_\lambda^{(i)} \left| -\frac{\hbar^2}{m}\left[L^2(\Omega)-\frac{(D-1)(D-3)}{4}\right]r^{-2}+V(r,\Omega)\right| B_\lambda^{(j)}\right\rangle$$

$$=\sum_{KK'} b_{K,\lambda}^{(i)}(r)b_{K',\lambda}^{(j)}(r)\left[\frac{\hbar^2}{mr^2}L_K(L_K+1)\delta_{KK'}+U_K^{K'}(r)\right] . \quad (3.5)$$

## IV. NUMERICAL RESULTS

In order to get some idea of how the results of the different adiabatic approximations should look with different kinds of interactions, we perform calculations on the triton binding energy with five central two-body potentials having typical behavior: Volkov ($V$) (Ref. 31), Afnan-Tang ($S3$) (Ref. 33), Eikemeier-Hachenbroich ($S4$) (Ref. 32), Bell-Delves (BD) (Ref. 34), and G2 (Ref. 12).

For a given potential we calculate the matrix elements of the bracket of formula (3.3) which reads:

$$M_K^{K'}(r)=(2K+\tfrac{3}{2})(2K+\tfrac{5}{2})r^{-2}+3\frac{m}{\hbar^2}$$
$$\times\sum_{K''}(-1)^{K''}\langle K\,|\,K''\,|\,K'\rangle V_{2K''}(r) ,$$

where $V_{2K''}(r)$ and $\langle K\,|\,K''\,|\,K'\rangle$ stand for the multipoles of the potential and the three-body hyperspherical coefficients,[17] respectively. We diagonalize the matrix $M_K^{K'}(r)$ for every value of $r$. To construct the eigenpotential, we select after the diagonalization at the lowest $r$ value $r_0$, and among the eigenvectors $b_{0\lambda}^{(0)}(r_0)$, that one which is completely decoupled and characterized by the unity for $b_{0\lambda}^{(0)}(r_0)$ and zero for all the $b_{K\lambda}^{(0)}(r)$ with $K\neq 0$. We select the corresponding eigenvalue which is the value of our eigenpotential at $r_0$. Then for all the next steps in $r$ we have to follow this eigenvalue by continuity and select its corresponding eigenvector. Indeed we have to be careful with the crossing of the eigenvalues in the successive diagonalizations, also the choice at each $r$ step of the lowest eigenvalue may be dangerous for instance if we introduce the mixed symmetry state with the $G2$ potential.

If we compare the selected eigenpotentials $U_\lambda(r)$ in accordance with the analyzed potential, one can see in Fig. 1, that eigenpotentials $U_\lambda(r)$ have very different asymptotic behavior according to the hardness of the interaction, but all of them go through zero near $r=1.5$ fm.

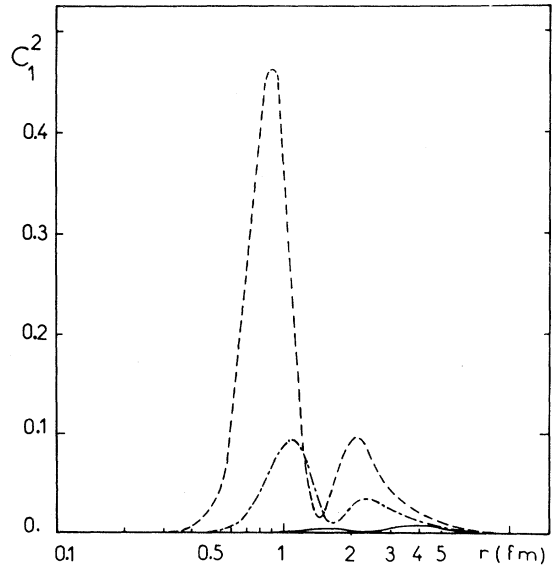


FIG. 3. Shape of the normalization function $C_1^2(r)$ for three potentials: Volkov (continuous line), $G2$ (dashed line), and $S3$ (dotted-dashed line).

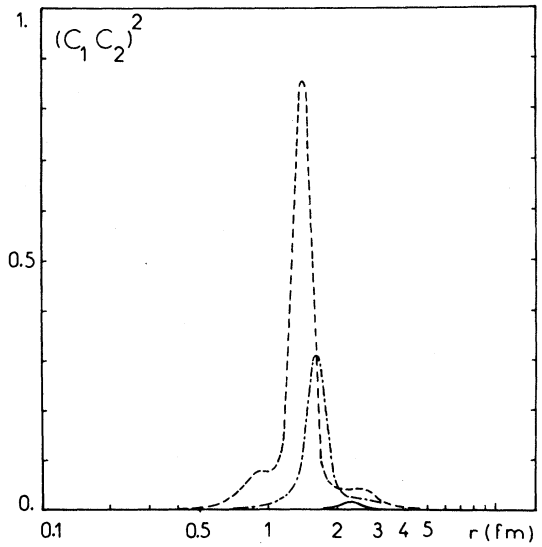


FIG. 4. Shape of the normalization function $[C_1C_2]^2$ for three potentials: Volkov (continuous line), $G2$ (dashed line), and $S3$ (dotted-dashed line).

TABLE VI. Comparison of the percentages of the partial waves obtained for the five potentials [$V$, $S3$, $S4$, BD, $G2$] in the uncoupled and coupled adiabatic approximation with the coupled equations (CE) ones.

| | $K$ | 0 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | UAA | 98.317 | 1.0680 | 0.4143 | 0.0623 | 0.0533 | 0.0396 | 0.0142 | 0.0119 | 0.0092 | 0.0040 | 0.0032 | 0.0024 |
| $0S4$ | CAA | 98.70 | 0.8075 | 0.3248 | 0.0535 | 0.0468 | 0.0336 | 0.0114 | 0.0091 | 0.0067 | 0.0028 | 0.0022 | 0.0016 |
| | CE | 98.82 | 0.6899 | 0.3162 | 0.0532 | 0.0478 | 0.0359 | 0.0126 | 0.0103 | 0.0079 | 0.0034 | 0.0027 | 0.0020 |
| | UAA | 97.418 | 1.4533 | 0.7116 | 0.1220 | 0.1149 | 0.872 | 0.0307 | 0.0251 | 0.0186 | 0.0078 | 0.0059 | 0.0042 |
| $S3$ | CAA | 97.868 | 1.1441 | 0.6346 | 0.1134 | 0.1022 | 0.0718 | 0.0235 | 0.0179 | 0.0126 | 0.0051 | 0.0038 | 0.0027 |
| | CE | 98.07 | 0.9753 | 0.5947 | 0.1066 | 0.0990 | 0.0748 | 0.0255 | 0.0201 | 0.0152 | 0.0061 | 0.0046 | 0.0034 |
| | UAA | 98.802 | 0.874 | 0.261 | 0.0285 | 0.0174 | 0.0097 | 0.0028 | 0.0019 | 0.0013 | 0.0005 | 0.0003 | 0.0002 |
| $V$ | CAA | 99.186 | 0.61 | 0.1649 | 0.0183 | 0.0112 | 0.0059 | 0.0015 | 0.0099 | 0.0006 | 0.0002 | 0.0001 | 0.0001 |
| | CE | 99.29 | 0.5036 | 0.1618 | 0.0198 | 0.0132 | 0.0077 | 0.0022 | 0.0015 | 0.0009 | 0.0004 | 0.0002 | 0.0002 |
| | EAA | 98.134 | 1.1817 | 0.5025 | 0.0754 | 0.0518 | 0.0307 | 0.0090 | 0.0064 | 0.0043 | 0.0017 | 0.0013 | 0.0009 |
| BD | UAA | 98.129 | 1.1839 | 0.5038 | 0.0756 | 0.0519 | 0.0308 | 0.0090 | 0.0065 | 0.0044 | 0.0017 | 0.0013 | 0.0009 |
| | CE | 98.34 | 1.047 | 0.4484 | 0.0677 | 0.0468 | 0.0267 | 0.0082 | 0.0059 | 0.0040 | 0.0016 | 0.0012 | 0.0009 |
| | UAA | 95.554 | 2.1716 | 1.343 | 0.2655 | 0.2652 | 0.2013 | 0.0694 | 0.0549 | 0.0393 | 0.0159 | 0.0116 | 0.0081 |
| $G2$ | CAA | 96.079 | 1.795 | 1.333 | 0.2569 | 0.2337 | 0.1614 | 0.0516 | 0.0382 | 0.0261 | 0.0105 | 0.0077 | 0.0055 |
| | CE | 96.30 | 1.659 | 1.252 | 0.2472 | 0.2286 | 0.1641 | 0.0542 | 0.0416 | 0.0291 | 0.0117 | 0.0085 | 0.0059 |

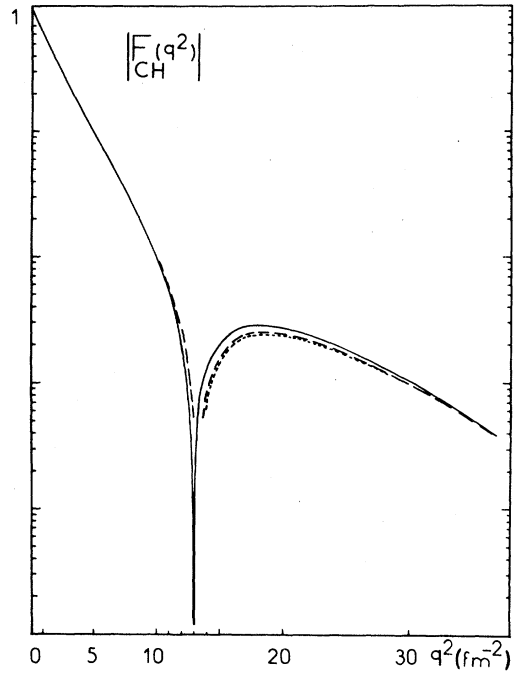


FIG. 5. The $^3$H charge form factor calculated with three wave functions CE (continuous line), CAA (dashed line), and UAA (dotted line) determined from the $G2$ potential.

The set of eigenfunctions $b^{(0)}_{K,\lambda}(r)$ shows structure at short distance and has different straight asymptotes for large $r$ values as shown in Figs. 2(a) and 2(b). The determination of the basis $b^{(1)}_{K\lambda}(r)$ and $b^{(2)}_{K\lambda}(r)$ is done numerically with a five points differentiation formula from the basis $b^{(0)}_{K\lambda}(r)$ according to the Eqs. (3.4). At short distance up to $r=0.8$ fm we use a Stoer-Padé extrapolation and interpolation of the basis $b^{(1)}_{K\lambda}(r)$ and $b^{(2)}_{K\lambda}(r)$ to remove spurious numerical oscillations. Then each basis, $b^{(1)}_{K\lambda}(r)$ and $b^{(2)}_{K\lambda}(r)$, is reorthogonalized to obtain accurate orthogonality relations [Eq. (2.6)]. These two last procedures are necessary to minimize the errors in the determination of the coupling amplitude $U^{(1,2)}_{\lambda}(r)$ [Eq. (2.9)].

In Tables I—V we give the results obtained for the binding energy of the triton for each adiabatic approximation EAA, UAA, and CAA, the accuracy of which is found by comparison with the coupled equation (CE) results for the same two-body potential. The UAA results agree with independent calculations by Das *et al.*[27] The extreme adiabatic approximation gives an energy below that of the coupled equations because in the EAA one defines a lower limit of the eigenpotential. The energy difference between the EAA and the UAA depends on the hardness of the interaction and it increases with the strength of the core. The UAA effective potential differs from the EAA one by the normalization function $C_1{}^2(r)$. This function contains all the information about the variation of the eigenfunction $b^{(0)}_{K\lambda}(r)$ and its amplitude increases with the hardness of the interaction. In Fig. 3 we show the shape of $C_1{}^2(r)$ for three different potentials. The great difference between the amplitude of $C_1{}^2(r)$ for Volkov and $G2$ potentials accounts for the energy difference between the EAA and UAA of these two potentials. This energy gap is characteristic of the hardness of the interaction. Using the value of this gap it is possible to estimate the CE energy with the very simple rule

$$E_{(\mathrm{CE})}=E_{(\mathrm{UAA})}+0.20\,[E_{(\mathrm{EAA})}-E_{(\mathrm{UAA})}](X)\ .$$

The normalization function $[C_1C_2]^2$ has a shape similar to that of $C_1{}^2(r)$ (Fig. 4) and a more important amplitude; however, its role in the coupled equations (1.8) is decreased by the presence of the coupling interactions $P^i_j(r)$. The uncoupled and the coupled adiabatic approximation both give energies above the coupled equation energy because we are using trial functions with the complete Hamiltonian. The CAA energy is lower than UAA since CAA trial function has greater flexibility. Lower limits that are reasonably close to the exact energy are hard to find in quantum mechanics. For the triton problem we do have the use of the Hall-Post lower bound. But even for a simple two body potential, Humberston[29] finds that the Hall-Post lower bound falls 1 MeV below the triton energy. However, Brady[30] finds that the Hall-Post lower bound misses by tens or even hundreds of MeV for more realistic potentials. The EAA should prove useful both for its simplicity and for providing a lower bound relatively close to the true energy.

TABLE VII. Percentage of the three partial waves [Eq. (2.7)] in the CAA.

| | $P(u^0)$ | $P(u^1)$ | $P(u^2)$ |
|---|---|---|---|
| $V$ | 99.954 | 0.039 | 0.007 |
| $S3$ | 99.951 | 0.037 | 0.011 |
| $S4$ | 99.962 | 0.032 | 0.006 |
| $G2$ | 99.919 | 0.062 | 0.018 |

In Table VI we compare the percentages of the partial waves obtained for the five potentials in the uncoupled and coupled adiabatic approximation compared to the coupled equation ones. We see significant differences among the percentages for the first partial waves despite the fact that all the energies are very close. This difference is due to a slight change of the partial wave amplitude but not a

change of the shape, then it does not modify the shape of the charge form factor $F_{ch}(q^2)$ (Fig. 5).

In Table VII we give, for comparison, the percentage of the three partial waves occurring in the expansion (2.7). As expected, the most important is the first wave $B_\lambda^{(0)}(r)$, which accounts for more than 99.9% in the wave function, the amplitude of the two others $B_\lambda^{(1)}(r)$ and $B_\lambda^{(2)}(r)$ increasing with the hardness of the interaction $S3$ and $G2$.

## V. SUMMARY AND OUTLOOK

The ground state of three particles interacting with central forces is susceptible to be found with a discrepancy about 0.5% by means of the coupled adiabatic approximation. The amount of computer time used is rather modest and it is reasonable to view realistic calculations with this method, or at least to use the coupled adiabatic wave function as a trial wave function for the coupled differential equations. At present an extension of the CAA method using a higher number of coupled equations is not interesting, because of the complicated nature of coupling amplitudes. Of course the CAA is quite general and can be applied to a large variety of problems using coupled differential equations in single variable.


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