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Contribution of three-body force to the trinucleon problem by an essentially exact calculation

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The hyperspherical harmonic method has been used to calculate the effect of two-pion exchange three-body force (Fujita-Miyazawa type) on trinucleon systems, with the N-N Afnan-Tang S3 potential. The Coulomb force and three-body force have been taken into account nonperturbatively. Simplification in the numerical calculation has been achieved through the use of an adiabatic approximation in order to decouple the system of differential equations. Binding energy, charge form factor, and pointlike proton density of both ^{3}H and ^{3}He have been calculated for various values of the cutoff parameter in the three-body force. Results indicate that the inclusion of the three-body force improves agreement with experiment. An interesting feature is the possible appearance of nodes near origin in the hyper-radial wave function.

[NUCLEAR STRUCTURE Trinucleon systems; three-body force; bound states of ^{3}H and ^{3}He using hyperspherical harmonic method; adiabatic approximation; charge form factors.]

I. INTRODUCTION

The progress in the numerical approach enabling one to solve the few-body problem in physics has induced physicists to attempt to understand the data obtained from ^{3}He and ^{3}H. The technical part of the problem has now been well established with accurate methods such as the hyperspherical harmonic (HH) method[1,2] (applied mostly for bound systems) and Faddeev equations[3–5] (a correlation between the two exists[6]) at least, to treat the dominant nonrelativistic part of the two-body forces (2BF). However, even with the advance in these studies, standard information, such as binding energies, charge form factors, etc., cannot be satisfactorily obtained in terms of realistic 2BF. Hence, the problem still resides in the understanding of the mechanism responsible for the binding of such physical systems.

Calculations for trinucleon systems using standard 2BF [for example, Reid soft core[7] (RSC) potential] have been performed by many authors using Faddeev equations[3–5,8–16] or the HH approach.[1,2,17–19] The results quoted by various authors agree within reasonable limits. The triton is underbound by about 1.3 ± 0.3 MeV according to the investigated two-body potentials while the charge form factor $F_{ch}(q)$ reported in these calculations shows a striking disagreement with the experiment: The first diffraction minimum is around $q_{min}^2\approx16$ fm^{-2} compared to an experimental value of $q_{min}^2\approx11.8$ fm^{-2}. Also the first maximum of $|F_{ch}(q)|$ (henceforth called F_{max}) is $\lesssim 1.0\times10^{-3}$, whereas the experimental value for ^{3}He is about 6.0×10^{-3}. Sick *et al.*[20] have shown that the calculated pointlike proton density $\rho(r)$ in ^{3}He should present a central depression in order to reproduce the experimental values of both the position of the first diffraction minimum and the value of F_{max}. In fact, no central depression is obtained from most of the calculations using realistic 2BF.[4,5,8–19] But Sick and collaborators claim that various effects such as relativistic corrections, exchange currents, and finite sizes of the nucleons were taken into account in their calculations of the proton density.

The discrepancy between the experimental binding energy (BE) and theoretical calculations with realistic 2BF (Ref. 21) seems to indicate that at least something is missing in the interaction. Ballot and Fabre de la Ripelle pointed out[22] that inclusion of two-pion exchange (TPE) three-body force (3BF) might cut out the Gordian knot by introducing three-body correlation, such that the central depression in the pointlike proton density can, in principle, be explained. However, that calculation was incomplete,[22] since the full effect of the 3BF was not

taken correctly into account. Workers in this field have recently focused their attention toward the investigation of the importance of 3BF and other effects in the elucidation of this thus far endless problem. Sato *et al.*[23] performed variational calculations including TPE-3BF and obtained an enhancement of BE by 1.16—1.50 MeV (depending on the choice of 2BF) for ^{3}H and $R_{\rm rms}=1.323$ fm (the experimental value is 1.68 fm). Yang[24] also performed a variational calculation including TPE-3BF and reported an increase of 2.32 MeV in BE of ^{3}H and $R_{\rm rms}=1.6$ fm. None of these calculations attempted to calculate either $F_{\rm ch}(q)$ or the pointlike proton density. Hadjimichael,[25] on the other hand, used impulse approximation to investigate various effects, including Δ resonance on $F_{\rm ch}(q)$ (no calculation of BE). He reported that the contribution coming only from Δ resonance is too small to reproduce the experimental data and that the contribution of other effects [meson exchange currents (MEC), recoil] is necessary in order that $F_{\rm ch}(q)$ become comparable with experimental data. Recently Nogami *et al.*[26] performed a model and variational calculation to estimate the effect of TPE-3BF on ^{3}He and ^{4}He. They calculated the pointlike proton density directly from a simple trial function and showed that a central depression indeed yielded results, but only about one third of that reported by McCarthy *et al.*[20] for the theoretically estimated strength of TPE-3BF. Two more works using Faddeev equations by Hajduk *et al.*[27,28] and Torre *et al.*,[29] which include contributions from the Δ-nucleon resonance and 3BF, have been reported. The magnitudes of $F_{\rm max}$ and of the central depression in $\rho(r)$ obtained by Hadjimichael[25] with the inclusion of MEC and recoil effects are larger than those reported by Hajduk[27,28] and Haftel and Kloet.[30] (This last reference includes 3BF generated by an *ad hoc* unitary transformation.)

Our goal in this work is to include the TPE-3BF using a basically complete and accurate method of calculation for trinucleon systems, viz., the HH method, in which inclusion of 3BF and Coulomb force do not alter the structure of the equations. The HH method transforms the Schrödinger equation in a system of coupled differential equations, which can be solved to a desired degree of precision numerically. However, due to practical limits of available computer time and memory, we have adopted the adiabatic approximation (AA) to decouple the system. Earlier investigations[31] have shown that indeed AA is applicable to soft core potentials with an accuracy of about 1—2%. Computer limitations have also restricted us to the totally symmetric S state of the trinucleon system, which is responsible for about 90% of the trinucleon ground state.

Since there are many versions of 3BF, as well as of 2BF, some ambiguity in the results is expected. In this work we use a 3BF which results from the contribution of the lowest order Feynman diagram of the TPE.[32] The effect of higher order diagrams,[32] corresponding to still shorter range forces, should be, to some extent, shadowed by the two-body repulsive core in nuclei. Among the various above mentioned versions[33,34] of 3BF, here the classic Fujita and Miyazawa[35] interaction is used.

In Sec. II the theoretical calculations are explained in five subsections. Some formulas are derived in the Appendix. In Sec. III the results are discussed and finally, in Sec. IV, we summarize and conclusions are drawn.

II. THE THEORY

This section is divided into four subsections devoted to: (A) the three-body force; (B) the theoretical method to handle the Schrödinger equation; (C) calculation of needed matrix elements; (D) the calculation of the charge form factor.

A. The three-body force

The Fujita-Miyazawa[35] force consists of two terms W_s and W_p, which are generated by the s and p waves of virtual pions, respectively. The effect of the W_s component, which is drastically reduced by the σ-meson exchange,[36] will not be considered here. The W_p component refers to the Feynman diagram in which the intermediate state of the k nucleon (the others are i and j nucleons) is a Δ resonance,[35] and is given by:

$$W_p(k)=-\left[\frac{C_p}{8\mu^4}\right]\frac{1}{9}[5(\vec{\tau}_i\cdot\vec{\tau}_k)(\vec{\tau}_k\cdot\vec{\tau}_j)+3(\vec{\tau}_j\cdot\vec{\tau}_k)(\vec{\tau}_k\cdot\vec{\tau}_i)]\times[S_{ik}U_{(2)}(x_j)+(\vec{\sigma}_i\cdot\vec{\sigma}_k)U_{(0)}(x_j)]$$

$$\times[S_{kj}U_{(2)}(x_i)+(\vec{\sigma}_k\cdot\vec{\sigma}_j)U_{(0)}(x_i)]+(i\leftrightarrow j)\ , \tag{1}$$

where

$$S_{ij}=3(\vec{\sigma}_i\cdot\hat{x}_k)(\vec{\sigma}_j\cdot\hat{x}_k)-\vec{\sigma}_i\cdot\vec{\sigma}_j \ . \tag{2}$$

The $\vec{\sigma}$'s and $\vec{\tau}$'s are the usual Pauli operators for spin and isospin, respectively, and

$$U_{(0)}(x)=\frac{e^{-\mu x}}{\mu x},$$
$$U_{(2)}(x)=\left[1+\frac{3}{\mu x}+\frac{3}{(\mu x)^2}\right]\cdot\frac{e^{-\mu x}}{\mu x} \ , \tag{3}$$

where $\vec{x}_k\equiv\vec{r}_{ij}=\vec{r}_i-\vec{r}_j$ (cyclic permutations). C_p is a coupling coefficient ranging in the interval 0.46–1.3 MeV, according to the $\pi N\Delta$ coupling constants used in the calculations.[35,37]

Equation (1) contains three different kinds of terms: (i) the product of two central potentials; (ii) the product of one central and one tensor potential; (iii) the product of two tensor potentials. The first two cases do not contribute to the space completely symmetrical S state of the trinucleon bound state used in our calculation. The effective 3BF acting on the S state is given by[36]

$$V^{(3)}\equiv\sum_{k=1,2,3}W_p(k)=\sum_{\text{cyclic}}C_p(3\cos^2\theta_k-1)\times U_{(2)}(x_i)U_{(2)}(x_j) \ , \tag{4}$$

where θ_k is the angle between the directions $\vec{x}_j$ and $\vec{x}_i$. Since $U_{(2)}(x)$ is a positive, monotonically decreasing function, $V^{(3)}$ is negative for the equilateral triangle configuration and positive for the aligned configuration.[22] Thus the triangle configuration is favored compared to the aligned configuration. This might generate a central depression in the proton density if the 3BF is strong enough.

B. The hyperspherical harmonic approach

In this work we use a HH expansion of the wave function for solving the nonrelativistic Schrödinger equation for three nucleons of mass m

$$\left[-\frac{\hbar^2}{m}(\nabla_{\vec{x}_i}^2+\nabla_{\vec{y}_i}^2)+V_{123}(\vec{x}_i,\vec{y}_i)\right]\Psi(\vec{x}_i,\vec{y}_i)=E\Psi(\vec{x}_i,\vec{y}_i) \ , \tag{5}$$

written in terms of the Jacobi coordinates (not unique)

$$\vec{x}_i=\vec{r}_j-\vec{r}_k \ ,$$
$$\vec{y}_i=\frac{2}{\sqrt{3}}\left[\vec{r}_i-\frac{\vec{r}_j+\vec{r}_k}{2}\right] \ , \tag{6}$$

where $\vec{r}_i$ are the particle coordinates while V_{123} is the interaction between the three nucleons. This interaction consists of a sum of a two-body force (2BF), $V^{(2)}$ (which is the sum of three pairwise interactions), and a three-body force (3BF), $V^{(3)}$ (constituted by three terms, each representing a Δ resonance on the kth nucleon line, for $k=1, 2, 3$). Equation (6) defines three equivalent sets of coordinates ($i=1, 2, 3$) for the description of the three-body problem. Equation (5) can be solved using, for example, the hyperspherical harmonic approach[1,2] in which the wave function Ψ is expanded in a complete orthonormal set of hyperspherical functions in the following way:

$$\Psi(\vec{x}_i,\vec{y}_i)=r^{-5/2}\sum_{K\alpha_i}\Phi_{K,\alpha_i}(r)\mathscr{P}_{2K,\alpha_i}(\hat{x}_i,\hat{y}_i,\phi_i) \ , \tag{7}$$

where

$$r^2=x_1^2+y_1^2=x_2^2+y_2^2=x_3^2+y_3^2, \quad x_i=r\cos\phi_i, \quad y_i=r\sin\phi_i, \quad (0\le\phi_i\le\pi/2) \ .$$

The notations $\hat{x}_i$ and $\hat{y}_i$ mean $\hat{x}_i\equiv(\theta_{x_i},\phi_{x_i})$, $\hat{y}_i\equiv(\theta_{y_i},\phi_{y_i})$, respectively. The complete orthonormal sets $[\mathscr{P}_{2K,\alpha_i}(\Omega_i)]$ are the angular part of homogeneous harmonic polynomials of degree $2K$ ($K=0, 1, 2,\ldots,\infty$) in the six-dimensional space. The label $K\alpha_i$ stands for the five quantum numbers related to the five degrees of freedom contained in Ω_i. Substitution of Eq. (7) into Eq. (5) leads to a system of coupled differential equations[1]

$$\left[-\frac{d^2}{dr^2}+\frac{\mathscr{L}_K(\mathscr{L}_K+1)}{r^2}+k^2\right]\Phi_{K\alpha_i}(r)+\sum_{K'\alpha_i'}\langle K\alpha_i\,|\,v\,|\,K'\alpha_i'\rangle\Phi_{K'\alpha_i'}(r)=0 \ , \tag{8}$$

where $\mathscr{L}_K=K+\frac{3}{2}$, $k^2=-(m/\hbar^2)E$ ($E<0$ for bound states), $v=(m/\hbar^2)V_{123}$, and $\langle v\rangle$ is integrated over the five angles, resulting in a function of r. The elements of the angular basis with total angular momentum

(L,M) related to the angular coordinates $(\hat{x}_i, \hat{y}_i, \phi_i)$ are

$$\mathscr{P}_{2K,\alpha_i}(\Omega_i)=\sum_{m_{x_i}m_{y_i}}\langle l_{x_i}l_{y_i}m_{x_i}m_{y_i}\,|\,LM\rangle Y_{l_{x_i}m_{x_i}}(\hat{x}_i)Y_{l_{y_i}m_{y_i}}(\hat{y}_i)\,{}^{(2)}P_{2K}^{l_{y_i}l_{x_i}}(\phi_i)\ , \tag{9}$$

where

$${}^{(2)}P_{2K}^{l_{y_i}l_{x_i}}(\phi_i)=\left[\frac{4(K+1)n!(n+l_{x_i}+l_{y_i}+1)!}{\Gamma(n+l_{x_i}+\frac{3}{2})\Gamma(n+l_{y_i}+\frac{3}{2})}\right]^{1/2}(\cos\phi_i)^{l_{y_i}}(\sin\phi_i)^{l_{x_i}}\times P_n^{(l_{x_i}+(1/2),\,l_{y_i}+(1/2))}(\cos 2\phi_i)\ , \tag{10}$$

where n is the integer $\frac{1}{2}(2K-l_{x_i}-l_{y_i})$ and $P_n^{(a,b)}(x)$ is the usual Jacobi polynomial. One should notice that $\mathscr{P}_{2K,\alpha_i}(\Omega_i)$ constitute, for any i, an infinite and complete set of orthonormal functions. Since the hyperspherical basis $\mathscr{P}_{2K,\alpha_i}$ is complete for any i, one can choose this index arbitrarily. For the sake of simplicity, we will drop from now on the subscripts in the quantum numbers and variables. When v contains spin, isospin, etc., and operators, the associated quantum numbers are included into the α label and in the sum in the matrix elements of Eq. (8).

C. Matrix elements of the interactions

In this paper we restrict ourselves to the space symmetrical S state (for which $L=M=0 \Longrightarrow l_x=l_y=l$; $\{\alpha\}\rightarrow\{0,\,0,\,l,\,l\}$), and we select the optimal subset for further reducing the number of significant coupled equations. Let $A(s,t)$ be the fully antisymmetric spin-isospin state for total spin (isospin) $S=\frac{1}{2}$ $(T=\frac{1}{2})$. The wave function of the space completely symmetric S state is given by

$$\Psi(\vec{x},\vec{y})=A(s,t)r^{-5/2}\sum_{K=0}^{\infty}\mathscr{P}_{2K}(\Omega)\Phi_K(r)\ , \tag{11a}$$

where the normalized symmetric HH are

$$\mathscr{P}_{2K}(\Omega)=C_{2K}\sum_{l=0,\text{even}}\{\hat{S}_0^{(2)}P_{2K}^{ll}(\phi)\}\sqrt{2l+1}\times\mathscr{P}_{2K,0,0,l,l}(\Omega)\ . \tag{11b}$$

The operator $\hat{S}_0$ is defined by

$$\hat{S}_0 f(\phi)\equiv\frac{1}{3}\left[f(0)+f\left[\frac{2\pi}{3}\right]+f\left[-\frac{2\pi}{3}\right]\right],$$

and

$$(C_{2K})^{-2}=\sum_{\substack{l'=0\\ \text{even}}}^{K}(2l'+1)[\hat{S}_0({}^{(2)}P_{2k}^{l'l'}(\phi))]^2\ . \tag{11c}$$

In practice we are obliged to consider two separate expansions, one for the 2BF and another for the 3BF. The matrix element of 2BF between two HH is given by

$$\begin{aligned}\langle K\alpha\,|\,v^{(2)}\,|\,K'\alpha'\rangle&=\langle K\alpha\,|\,\sum_{i<j}v^{(2)}(r_{ij})\,|\,K'\alpha'\rangle\\&=3\langle K\alpha\,|\,v^{(2)}(x_1)\,|\,K'\alpha'\rangle\delta_{\alpha'\alpha}\\&=3\sum_{K''=0}^{\infty}(-)^{K''}\langle K\,|\,K''\,|\,K'\rangle v_{2K''}^{(2)}(r)\ ,\end{aligned} \tag{12}$$

where

$$v_{2K''}^{(2)}(r)=\frac{8}{\sqrt{\pi}}\frac{K''!}{\Gamma(K''+\frac{3}{2})}\int_0^1 v^{(2)}(ur)P_{K''}^{1/2,\,1/2}(1-2u^2)\sqrt{1-u^2}u^2\,du\ , \tag{13}$$

are the so-called potential multipoles,[1] and

$$\begin{aligned}\langle K\,|\,K''\,|\,K'\rangle=&\frac{\pi}{16}C_{2K}C_{2K'}\,{}^{(2)}P_{2K''}^{00}(0)\\&\times\sum_{l'=0}^{\min(K,K')}(2l'+1)\hat{S}_0({}^{(2)}P_{2K}^{l'l'}(\phi))\hat{S}_0({}^{(2)}P_{2K'}^{l'l'}(\phi))\langle\,{}^{(2)}P_{2K}^{l'l'}\,|\,{}^{(2)}P_{2K''}^{00}\,|\,{}^{(2)}P_{2K'}^{l'l'}\rangle\ .\end{aligned} \tag{14}$$

The last $3P$ matrix element in Eq. (14) is defined as[1]

$$\langle {}^{(2)}P^{l_2'l_1'}_{2K''+2} \,|\, {}^{(2)}P^{ll}_{2K} \,|\, {}^{(2)}P^{l_2'l_1'}_{2K'+2} \rangle = \int_0^{\pi/2} d\phi \sin^2\phi \cos^2\phi \, {}^{(2)}P^{l_2'l_1'}_{2K''+2}(\phi) \, {}^{(2)}P^{ll}_{2K}(\phi) \, {}^{(2)}P^{l_2'l_1'}_{2K'+2}(\phi) \; . \tag{15}$$

The effective potential $v^{(2)}(x_1)$ is taken to be half the sum of the singlet and triplet even central potentials for the space symmetric S state. The difference between the singlet and triplet even central potential generates the mixed symmetry S state which contributes around one percent to the wave function[1] and that we neglected.

The factors $\langle K \,|\, K'' \,|\, K' \rangle$ are geometrical coefficients independent of the shape of the interaction. A similar procedure is used to obtain the matrix elements of 3BF. It is given by

$$\langle K\alpha | \, v^{(3)} \, | K'\alpha' \rangle = 3 \sum_{K''=0}^{\infty} (-)^{K''} \langle K | \, K'' \, | K' \rangle v^{(3)}_{2K''}(r) \; , \tag{16}$$

where the potential multipoles are

$$\begin{aligned} v^{(3)}_{2K''}(r) = \frac{\pi^{3/2}}{128} c_p \frac{(-1)^{K''}}{\langle 0 \,|\, K'' \,|\, K'' \rangle} C_{2K''} \sum_{l=0,\text{even}}^{K''} \sqrt{2l+1} \{ \hat{S}_0 \, {}^{(2)}P^{ll}_{2K''}(\phi) \} \\ \times \sum_{K_1,K_2} (-1)^{K_1+K_2} U_{2K_1}(r) U_{2K_2}(r) \sum_{\lambda} (2\lambda+1) \begin{bmatrix} 2 & l & \lambda \\ 0 & 0 & 0 \end{bmatrix}^2 \\ \times \langle {}^{(2)}P^{2,0}_{2K_1+2} \,|\, {}^{(2)}P^{ll}_{2K''} \,|\, {}^{(2)}P^{\lambda,l}_{2K_2+2} \rangle \\ \times {}^{(2)}P^{2,0}_{2K_1+2}(0) \, {}^{(2)}P^{\lambda,l}_{2K_2+2}(-2\pi/3) \; , \end{aligned} \tag{17}$$

where $c_p = (m/\hbar^2) C_p$ and

$$U_{2K}(r) = \frac{2^7}{15\pi} (K+\tfrac{5}{2})(K+\tfrac{3}{2}) \int_0^1 U_{(2)}(r'r) \, {}_2F_1(-K, K+4; \tfrac{7}{2}; r'^2) \sqrt{1-r'^2} \, r'^4 \, dr' \; . \tag{18}$$

In the Appendix we explain how to obtain the 3BF multipoles.

From Eq. (8) we can write the system of coupled equations where 2BF and 3BF are taken into consideration:

$$\left[-\frac{d^2}{dr^2} + \frac{\mathscr{L}_K(\mathscr{L}_K+1)}{r^2} + k^2 \right] \Phi_K(r) + 3 \sum_{K',K''=0}^{\infty} (-)^{K''} \langle K \,|\, K'' \,|\, K' \rangle (v^{(2)}_{2K''}(r) + v^{(3)}_{2K''}(r)) \Phi_{K'}(r) = 0 \; . \tag{19}$$

D. Charge form factor

For the S state, the charge form factor is given by[1]

$$F^{T_3}_{\text{ch}}(q) = \frac{3G_{\text{ES}}(q) + 2T_3 G_{\text{EV}}(q)}{T_3 + \frac{3}{2}} F(q/\sqrt{3}) \; , \tag{20}$$

where $T_3 = \frac{1}{2}$ for ^{3}He and $T_3 = -\frac{1}{2}$ for ^{3}H, and

$$\begin{aligned} F(q) = 8 \sum_{KK'K''} \langle K \,|\, K'' \,|\, K' \rangle \\ \times \int_0^{\infty} \Phi_K(r) \Phi_{K'}(r) \\ \times \frac{J_{2K''+2}(qr)}{(qr)^2} dr \; . \end{aligned} \tag{21}$$

$G_{\text{ES}}(q)$ and $G_{\text{EV}}(q)$ are the scalar and vector electric form factors of the nucleons and are given in Refs. 38 and 39. The pointlike proton density for the fully symmetric S state is given by[20,40]

$$\rho(r) = \frac{1}{2\pi^2 r} \int_0^{\infty} F(q/\sqrt{3}) \sin(qr) q \, dq \; . \tag{22}$$

It is obviously the inverse Fourier transform of Eq. (20) when $G_{\text{ES}} = G_{\text{EV}} = \frac{1}{2}$.

III. RESULTS AND DISCUSSION

To reduce the complexity of the numerical calculations we use the adiabatic approximation to approximately decouple the system of equations. Justification of this approximation has already been advanced.[31] Comparison between exact and approxi-

mate calculations showed that the approximate binding energy is reliable to within 1–2% even for a relatively strong soft core.[31] Table I illustrates a comparison of adiabatic approximation with exact calculation using the *NN* Afnan-Tang S3 potential.[41] Since our primary aim in this work is to investigate the enhancements in BE and charge form factor (CFF) due to the inclusion of 3BF, rather than to try to reproduce experimental results with realistic forces, we have chosen for two-body interaction the Afnan-Tang S3 central potential,[41] which is reasonably realistic, although quite simple in structure.

Calculations are performed with twelve partial waves ($K'=0, 2, 3, 4, \ldots, 12$) ensuring the convergence for the S3 potential.[43] Only five multipoles of 3BF [$K''=0$, 1, 2, 3, 4, and 5 for $v^{(3)}_{2K''}(r)$ in Eq. (19)] have been used since the increment of BE from four multipoles to five multipoles for a typical calculation is about 0.01 MeV, which is about the limit of accuracy of the adiabatic approximation.[31] This also sets the accuracy of the present calculations to be about 0.03 MeV in BE. Table II shows that CFF attains a convergence with about five 3BF multipoles, even though the BE is still increasing slowly. However, at very short distances, the effect of πN vertex factor, ρ exchange, etc., becomes important and makes the form of W_P and hence $V^{(3)}$ uncertain. Furthermore, the 2BF repulsive core prevents particles from reaching short distances. Since we use a soft-core 2BF, we introduce a purely phenomenological cutoff parameter x_0 to restrict the 3BF at extremely short separations:

$$U_{(2)}(x)=\begin{cases} U_{(2)}(x_0), & x<x_0\ , \\ U_{(2)}(x), & x\geq x_0\ . \end{cases} \tag{23}$$

Treating x_0 as a parameter , we investigate the effect of the 3BF on the trinucleon systems. However, according to the previous discussion, very small values of x_0 must be forbidden. A physically reasonable value of x_0 should be around the 2BF hard core radius.

TABLE I. Comparison of adiabatic approximation with exact calculation (Ref. 2) using S3 potential for ^{3}H. N represents the various numbers of coupled differential equations.

N	Binding energy (MeV) Adiabatic approximation	Exact
1	0.3647	0.346
2	2.0495	3.120
4	5.0712	5.196
6	6.0582	6.208
8	6.3153	6.470
10	6.4381	6.5923
12	6.4889	6.6403

Our calculations show that both BE and $F_{\rm ch}(q)$ depend strongly on x_0. In Fig. 1, we show a preliminary plot of BE and $F_{\rm max}$ [value of the first maximum of $|F_{\rm ch}(q)|$] as functions of x_0 for ^{3}H, calculated with twelve 2BF multipoles and only two 3BF multipoles and $C_p=0.90$ MeV. Figure 2 shows similar results for ^{3}He (calculated with $C_p=0.46$ MeV). Both curves show the same qualitative behavior: BE increases gradually with decreasing x_0, until a critical value, then decreases discontinuously and starts increasing again, simultaneously, giving rise to a node near origin (NNO) in the hyper-radial wave function $\zeta_0(r)$. Decreasing x_0 further, the BE again suffers a discontinuity at a

TABLE II. Convergence behavior for various 3BF multipoles. ($C_p=0.9$ MeV, $\mu=0.7$ fm^{-1}, $x_0=0.340$ fm, no node in hyper-radial wave function.)

Calculation	Number of 3BF multipoles	BE (MeV)	$\lvert F_{\rm ch}(q^2)\rvert$ Value at $q=1$ fm^{-1}	Position of first zero (fm^{-2})	$F_{\rm max}\times 10^{+3}$
^{3}H	1	6.86	0.613	16.10	1.52
	2	7.63	0.632	16.53	1.86
	3	7.63	0.631	16.41	1.96
	4	7.65	0.631	16.42	1.96
	5	7.66	0.632	16.47	1.94
^{3}He	1	6.15	0.590	16.68	1.08
	2	6.89	0.609	16.46	1.34
	3	6.90	0.609	16.33	1.41
	4	6.91	0.609	16.34	1.41
	5	6.92	0.609	16.38	1.39

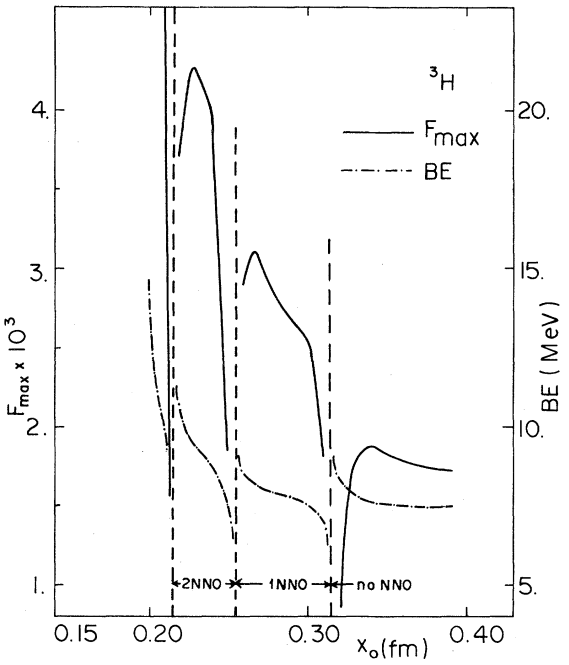

FIG. 1. Calculated binding energies and F_{max} as a function of x_0 for triton.

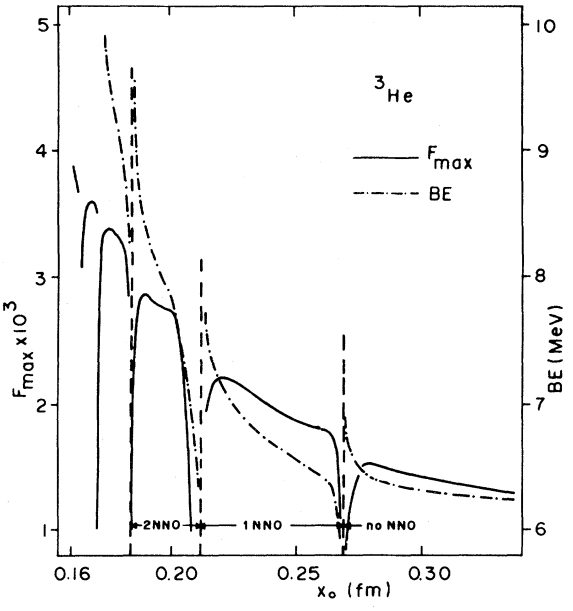

FIG. 2. Same as Fig. 1 for ^{3}He.

still smaller value of x_0, simultaneously, giving rise to an extra NNO. Discontinuity in BE proceeds from an extra node in the wave function which increases the kinetic energy. F_{max} also exhibits discontinuity at the same value of x_0, however, it reaches a maximum between two consecutive discontinuities.

As x_0 decreases, the contribution of 3BF increases very rapidly, generating a sharp fluctuation in the effective "lowest eigenpotential" $\omega_0(r)$. Figures 3 and 4 show typical plots of $\omega_0(r)$. A very narrow and deep attractive part in $\omega_0(r)$ causes nodes in that region. Typical plots of $\zeta_0(r)$ with 0, 1, or 2 NNO have been shown in Figs. 5 and 6.

TABLE III. Results of calculation for the bound states of trinucleon system and comparison with experimental data.

Description	C_p	x_0 (fm)	Number of nodes	BE (MeV)	$\lvert F_{ch}(q^2)\rvert$ Value at $q=1$ fm^{-1}	$\lvert F_{ch}(q^2)\rvert$ Position of first zero (fm^{-2})	$F_{max}\times 10^{+3}$	rms charge radius (fm)
^{3}H (2BF) (calc.)			0	6.489	0.590	15.98	1.50	1.82
^{3}H (2BF + 3BF) (calc.)	0.9	0.340	0	7.658	0.617	16.47	1.94	1.74
		0.270	1	8.274	0.626	15.43	3.01	1.71
		0.240	2	8.992	0.635	14.81	3.89	1.68
^{3}He (2BF) (calc.)			0	5.789	0.565	15.91	1.06	1.89
^{3}He (2BF + 3BF) (calc.)	0.9	0.340	0	6.922	0.592	16.39	1.39	1.81
	0.46	0.277	0	6.485	0.581	15.54	1.58	1.84
		0.220	1	7.356	0.599	15.20	2.23	1.79
		0.188	2	9.031	0.628	15.57	2.86	1.70
^{3}H (expt.)				8.482	0.622			1.70±0.05
^{3}He (expt.)				7.718	0.576	11.8	~6	1.84±0.03

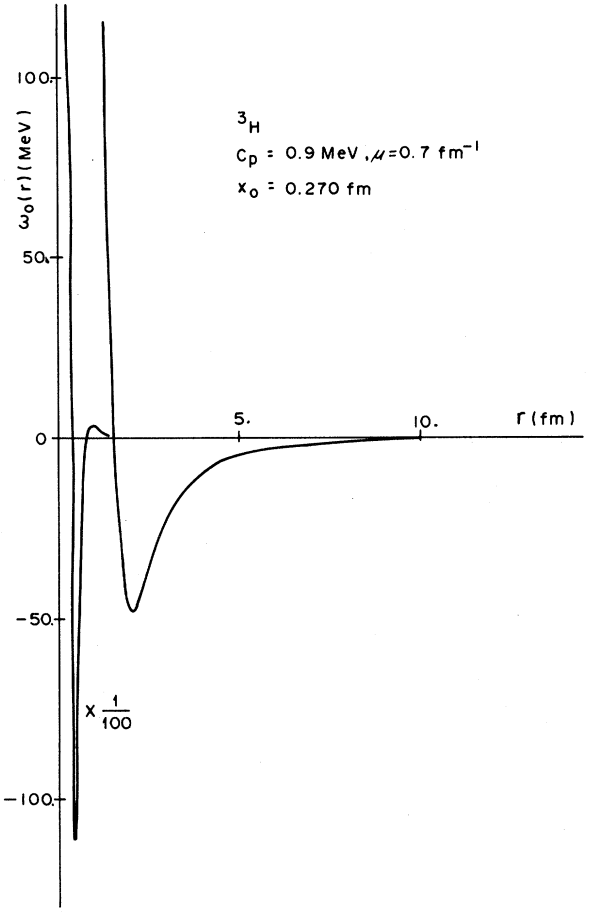

FIG. 3. Calculated $\omega_0(r)$ for ^{3}H.

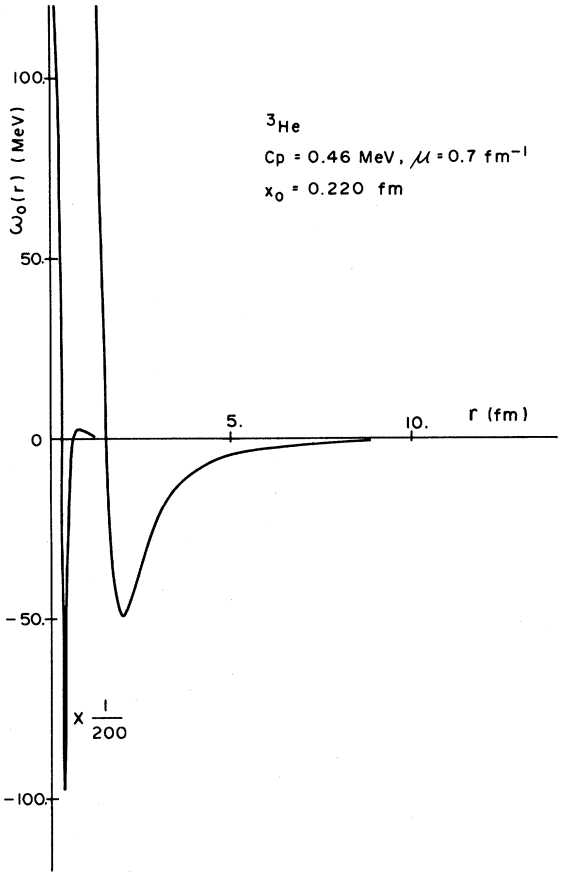

FIG. 4. Calculated $\omega_0(r)$ for ^{3}He.

Nodes appear in $\zeta_0(r)$ for $r \lesssim 0.5$ fm.

In Table III we present the results of a more complete calculation with five multipoles of 3BF, for several values of C_p and x_0. The largest values of $F_{\max}$ are given for a definite number of NNO. Table III, together with Figs. 1 and 2, gives an idea about the enhancement in BE and $F_{\max}$ due to the inclusion of 3BF. The obvious question is: What should be the value of x_0? From the nature of Figs. 1 and 2, it is clear that x_0 cannot be chosen as a *free phenomenological parameter* to fit experimental data. The NNO are generated by the very narrow and extremely deep attractive part in $\omega_0(r)$ (Figs. 3 and 4) introduced by the 3BF, which is very singular at the origin and *attractive* for the equilateral triangle configuration. It is conceivable that the extremely narrow and attractive well may bind one or more states with lower energy and the lowest one without nodes. But such a state will be extremely peaked at that point (resembling a δ function) and consequently not "physical." Hence, we tentatively designate the states exhibiting one or two nodes (indicated in Table III and Figs. 1, 2, 5, and 6) as "physical states." It is important to notice from Figs. 5 and 6 that these physical states, exhibiting one or two NNO, differ very little from the zero

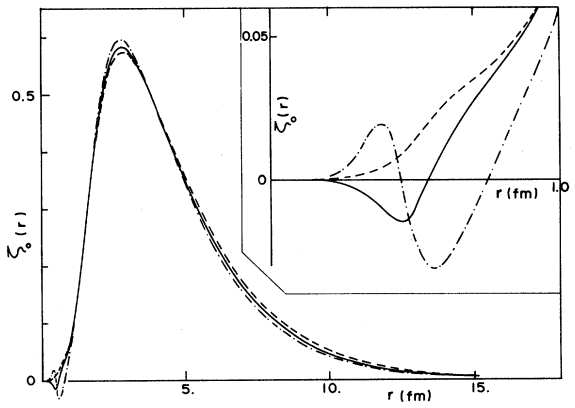

FIG. 5. Calculated wave function $\zeta_0(r)$ for ^{3}H. Parameter values are x_0=0.34 fm (dotted curve), x_0=0.27 fm (continuous curve), x_0=0.24 fm (dashed-dotted curve); C_p=0.9 MeV and μ=0.7 fm^{-1} for all curves.

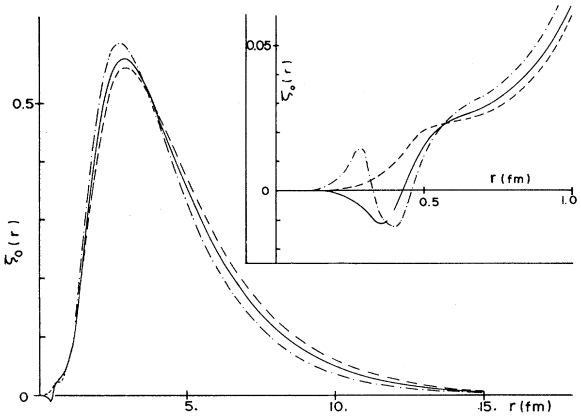

FIG. 6. Calculated wave function $\zeta_0(r)$ for ^{3}He. Parameter values are $x_0=0.270$ fm (dotted curve), $x_0=0.220$ fm (continuous curve), $x_0=0.188$ fm (dashed-dotted curve); $C_p=0.46$ MeV, and $\mu=0.7$ fm^{-1} for all curves.

node solution (for larger x_0) except for $r \lesssim 0.6$ fm where nodes appear.

Even then, one cannot admit an arbitrary number of NNO, which will increase rapidly as x_0 decreases. The problem is that 3BF is attractive and extremely singular in the equilateral triangle configuration and hence, with any soft core 2BF, will cause the same behavior. We can bypass this problem using a hard core 2BF. For example, with the hard core radius of 0.42 fm of the Reid hard core (RHC) potential no NNO will appear in $\zeta_0(r)$. For a soft core 2BF one can choose x_0 to be either the hard core radius or the value corresponding to the maximum of $F_{\max}$ for zero NNO. However, neither is completely satisfactory, since the former is artificial, while the latter is an (arbitrary) attempt to fit experimental data. This leads to a more fundamental and profound problem: The value of x_0 (or more precisely, the form of 3BF for extremely short separations) must come from more complete consideration of the origin of 3BF.

In the absence of an accurate theory of 3BF for extremely short separations, we choose the value of x_0 corresponding to the maximum of $F_{\max}$ for zero NNO. The enhancement of both BE and $F_{\max}$, seen in Table III, is insufficient to explain experimental data.

In Figs. 7 and 8 $|F_{\rm ch}(q)|$ are plotted as functions of q^2 for several values of x_0. The improvement of $F_{\max}$ and of the position of the first zero of the $F_{\rm ch}$ toward the experimental data is much too small, although a slight tendency to move towards experimental data can be seen. The comparison (Fig. 9) between the calculated pointlike proton density of ^{3}He for three values of x_0, and the "experimental" curves of McCarthy *et al.*,[20] shows that there is a small "central hole" in $\rho(r)$ only for two NNO solution.

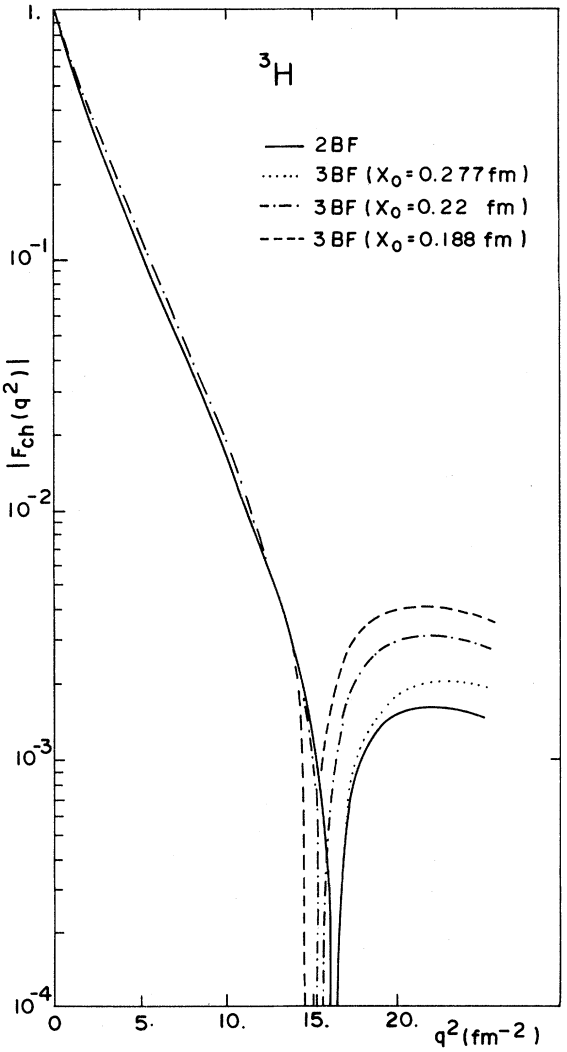

FIG. 7. Calculated charge form factor for ^{3}H ($C_p=0.9$ MeV and $\mu=0.7$ fm^{-1}).

IV. SUMMARY AND CONCLUSION

The understanding of trinucleon properties (especially binding energy and charge form factor) using realistic 2BF has been a stumbling block for a long time. The 2BF which well reproduces the properties to two-nucleon systems, obstinately fails to reproduce the BE (by about 1.5 MeV) and the first maximum of $|F_{\rm ch}(q)|$ by about a factor of 6, among other things, for three-nucleon systems. An obvious suggestion made by various authors[21,22,26]

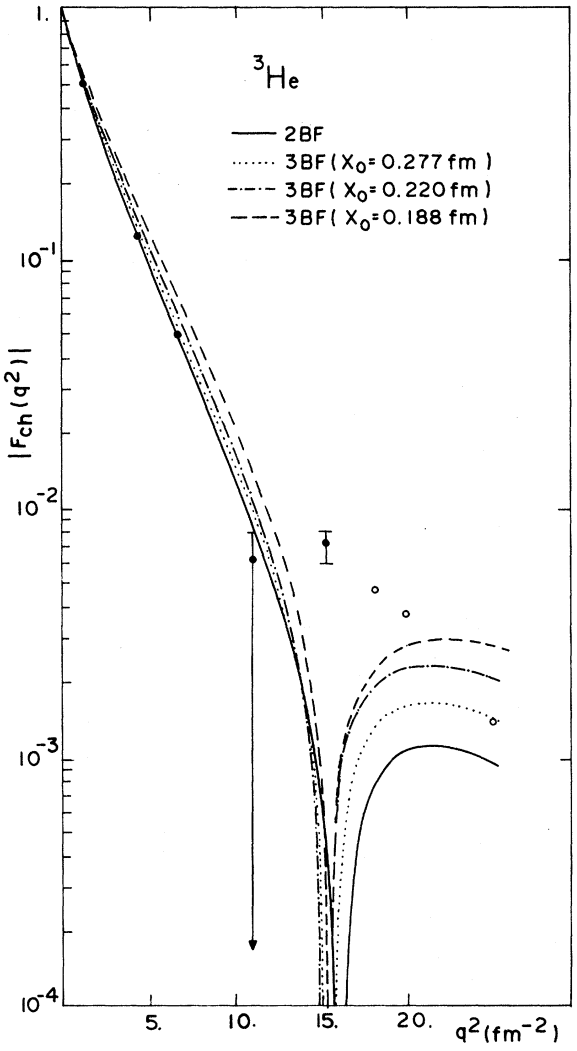

FIG. 8. Calculated charge form factor for ^{3}He (C_p=0.46 MeV and μ=0.7 fm^{-1}). The experimental points are taken from Refs. 20 and 39 (closed circles) and from Ref. 42 (open circles).

was that the 3BF is partially responsible for such discrepancies. In this work, we set out to examine the validity of this thesis.

We have solved the Schrödinger equation with a semirealistic 2BF (Afnan Tang S3 potential[41]) and the dominant 3BF (given by Fujita and Miyazawa[35]) using the HH method.

For practical limitations of computer time and memory we have restricted ourselves to use: (i) the space fully symmetric S state of the trinucleon system; (ii) a semirealistic 2BF (S3 of Afnan and Tang[41]); and (iii) an adiabatic approximation.

From the results quoted in Sec. III it is evident that the inclusion of 3BF, as expected from general

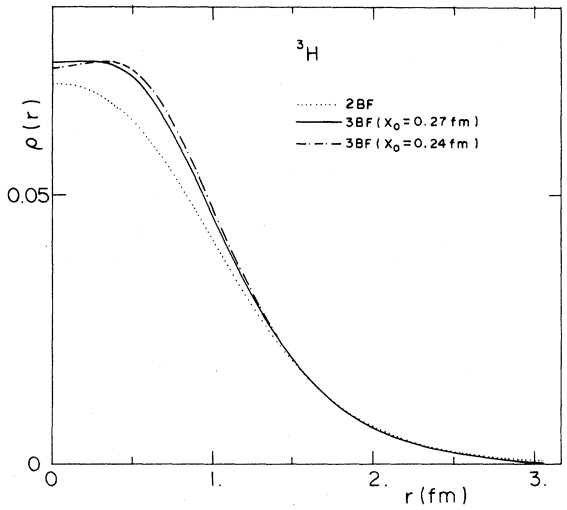

FIG. 9. Calculated pointlike charge density for ^{3}H (for 3BF, C_p=0.9 MeV and μ=0.7 fm^{-1}).

considerations, enhances both BE and F_{max} from the values calculated with 2BF only towards experimental values, and that both quantities depend strongly on the cutoff parameter x_0 of the 3BF soft core [Eq. (23)]. Phenomenologically one would expect that x_0 should be around the hard core radius of the 2BF. For the x_0=0.42 fm (hard core radius of RHC potential), the enhancements due to the inclusion of 3BF are small ($\sim$10% and 50% of the discrepancies in F_{max} and BE, respectively). It cannot explain the differences between 2BF calculation and experimental data. No central hole in $\rho(r)$ occurs for this value of x_0.

On the other hand, for smaller values of x_0, both

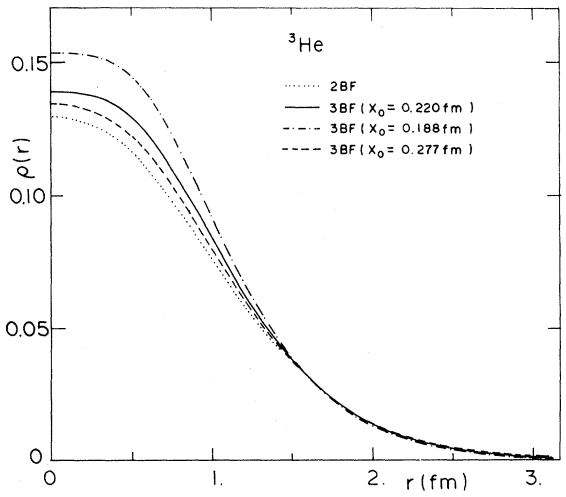

FIG. 10. Calculated pointlike charge density for ^{3}He (for 3BF, C_p=0.46 MeV and μ=0.7 fm^{-1}).

BE and F_{max} depend strongly on x_0, exhibiting discontinuities, and simultaneous appearance of one extra NNO in $\zeta_0(r)$ as x_0 decreases (Figs. 1 and 2). It is produced by the highly singular 3BF, which is attractive in the equilateral triangle configuration. The solutions containing one or two NNO correspond to a substantial increase of BE and F_{max} (Table III). The solution with two NNO produces a small central hole in $\rho(r)$ (Fig. 9, but almost none in Fig. 10).

In conclusion, it is evident from this work where the Coulomb effect as well as 3BF has been taken into account nonperturbatively, such that: (a) enhancements of BE and $F_{ch}(q)$ are in the right direction compared with experimental data, although they are too small. This conclusion confirms previous approximate or model estimations[26–30]; (b) the hyper-radial wave function $\zeta_0(r)$ has NNO's for small values of the cutoff parameter x_0 of 3BF; and (c) x_0 cannot be arbitrarily small, for a given soft core 2BF. This requires a further study of 3BF for very small particle separations.

ACKNOWLEDGMENTS

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APPENDIX: MATRIX ELEMENTS FOR THE 3BF

To calculate the matrix element

$$\langle \mathscr{P}_{2K}(\Omega) \mid v^{(3)} \mid \mathscr{P}_{2K'}(\Omega) \rangle$$

of the three-body force between two space fully symmetric HH one must expand the 3BF in terms of the same basis.

One notices that there is a single HH in this symmetry for each of the lowest values $2K$ of the grand orbital for $K=0$, 2, 3, 4, and 5 (for $K=1$ there are only mixed symmetry states). Therefore, the HH decomposition is unique for $K<6$. In agreement with Eq. (11) the HH expansion of 2BF is given by:

$$\sum_{i,j>i} v(r_{ij}) = 3\pi^{3/2} \sum_K (-1)^K \langle 0 \mid K \mid K \rangle \mathscr{P}_{2K}(\Omega) v_{2K}(r) \ . \tag{A1}$$

The coefficients $\langle 0 \mid K \mid K \rangle$ can be found in Ref. 1:

$$\langle 0 \mid K \mid K \rangle = (K+1/3)^{1/2} \times \begin{cases} \sqrt{K+3}, & K=3n, \\ \sqrt{K-1}, & K=3n+1, \\ \sqrt{K+1}, & K=3n+2, \end{cases} \quad (n \text{ integer}) \ . \tag{A2}$$

In order to obtain an expansion of the 3BF similar to (A1), one defines the 3BF multipoles:

$$v_{2K}^{(3)}(r) = \frac{(-1)^K}{\langle 0 \mid K \mid K \rangle} \pi^{-3/2} \langle \mathscr{P}_{2K}(\Omega) \mid \tilde{w}_p(k) \rangle \ , \tag{A3}$$

where $\tilde{w}_p(k)$ may be any one of the terms in Eq. (4) because $\mathscr{P}_{2K}(\Omega)$ is space fully symmetric. To calculate (A3) let us start from Eq. (4), where we write

$$3\cos^2\theta_k - 1 = \frac{8\pi}{5} \sum_{m=-2}^{2} Y_{2m}^*(\hat{r}_{ik}) Y_{2m}(\hat{r}_{jk}) \ .$$

Hence,

$$\tilde{w}_p(k) = \frac{8\pi}{5} c_p \sum_{m=-2}^{2} (Y_{2m}^*(\hat{r}_{ik}) U_{(2)}(r_{ik}))(Y_{2m}(\hat{r}_{jk}) U_{(2)}(r_{jk})) \ . \tag{A4}$$

Each term in parentheses can be further expanded in terms of hyperspherical harmonics[1]

$$(Y^*_{2m}(\hat{r}_{ik})U_{(2)}(r_{ik}))=\frac{\pi^2}{8}\sum_{K=0}^{\infty}(-)^K U_{2K}(r)$$
$$\times\sum_{\substack{l_1 m_1\\ l_2 m_2}}(-)^m\left[\frac{5(2l_1+1)(2l_2+1)}{4\pi}\right]^{1/2}\begin{bmatrix}2 & l_1 & l_2\\ 2 & 0 & 0\end{bmatrix}\begin{bmatrix}2 & l_1 & l_2\\ m & -m_1 & -m_2\end{bmatrix}$$
$$\times{}^{(2)}P^{l_2 l_1}_{2K+2}(\phi_{ik})Y_{l_1 m_1}(\omega_1)Y_{l_2 m_2}(\omega_2){}^{(2)}P^{l_2 l_1}_{2K+2}(\phi)\,, \tag{A5}$$

where $\phi_{ik}=0$, $-2\pi/3$ and $2\pi/3$ correspond to the interparticle distances r_{12}, r_{23}, and r_{31}, respectively, and $U_{2K}(r)$ are the tensor multipole in the six-dimensional space given by Eq. (18). By using the expansion (A5) for the terms between parentheses in Eq. (A4), the result projected on the basis $\mathscr{P}_{2K,0,0,l,l}(\Omega)=\mathscr{P}_{2K,l}(\Omega)$ is given by

$$\langle\mathscr{P}_{2K,00,ll}(\Omega)\,|\,\tilde{w}_p(k)\rangle=\frac{\pi^3}{128}c_p\sum_{K_1,K_2}(-1)^{K_1+K_2}U_{2K_1}(r)U_{2K_2}(r)$$
$$\times\sum_{\substack{l_1,l_2\\ l'_1,l'_2}}(2l_1+1)(2l_2+1)(2l'_1+1)(2l'_2+1)\begin{bmatrix}l_1 & l_2 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}l'_1 & l'_2 & 2\\ 0 & 0 & 0\end{bmatrix}$$
$$\times\begin{bmatrix}l_1 & l'_1 & l\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}l_2 & l'_2 & l\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}l_2 & l'_2 & l\\ l'_1 & l_1 & 2\end{Bmatrix}$$
$$\times\langle{}^{(2)}P^{l_2 l_1}_{2K_1+2}\,|\,{}^{(2)}P^{ll}_{2K}\,|\,{}^{(2)}P^{l'_2 l'_1}_{2K_2+2}\rangle{}^{(2)}P^{l_2,l_1}_{2K_1+2}(\phi_{ik})(2)P^{l'_2,l'_1}_{2K_2+2}(\phi_{kj})\,. \tag{A6}$$

In choosing $i=1$, $k=2$, and $j=3$ which corresponds to $\phi_{ik}=0$ and $\phi_{kj}=-2\pi/3$, the above equation is simplified because $l_1=0$; thereby one obtains the 3BF multipole given by Eq. (17).

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