

Uncoupled adiabatic approximation for the hyperspherical harmonic approach

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The accuracy of the uncoupled adiabatic approximation is investigated for bound trinucleon systems. The quality of this approximation is related to the strength of the repulsive core of the nuclear potential.

[NUCLEAR STRUCTURE Quantum few-body problem.]

I. INTRODUCTION

The few-body problem in physics is still an area of intensive investigation. Powerful and exact approaches, such as the hyperspherical harmonics formalism,^{1,2} Faddeev³ equations, and Yakubowski equations⁴ were developed in order to set up the problem. It seems that in most cases, for bound state calculations, the hyperspherical harmonic approach is becoming adequate and quite popular. For the three-body problem, for certain kinds of physical interactions, one cannot use the Faddeev equations³ in the t -matrix form because the Schmidt norm of the kernel of these equations diverges. But they could be used as well and the results are comparable with the hyperspherical harmonic (HH) approach, if they are written in the configuration space involving differential equations.⁵ All approaches will eventually result in difficult numerical calculations.¹⁻¹² In configuration space the HH expansion method leads to the solution of an infinite system of coupled differential equations.¹⁻⁶ Many papers^{1,2,6,9} have been published in order to handle this difficulty. For practical purposes the system is truncated according to certain mathematical procedures leading to the solution of a desired accuracy.^{1,2} However, the resulting numerical problem is still not so simple. Our goal in this paper is to test an approximate method (which we call the uncoupled adiabatic approximation) which decouples the truncated system to a single differential equation. A technique for the reduction of the system of coupled differential equations to three coupled equations for the hyperspherical harmonics formalism was proposed by Fabre¹³ in 1972. But the technique was not followed with a numerical application. In this work we calculate the first of the three coupled equations.

The same approximation has already been used

for atoms.^{14,15} But in that case the problem is simpler because in hyperspherical coordinates the Coulomb potential appears as a product of an angular function times a function of the hyper-radius. The smooth variation of the potential does not then generate any significant coupling between the three coupled equations of Ref. 13, which leads to a good accuracy when only the first equation is used.

Our purpose in this work is to investigate the quality of this approximation when we have to deal with nuclear potentials which exhibit rapid variations in short range, inducing coupling between the three equations of Ref. 13 which, this time, could not be negligible. The results are extremely encouraging. The binding energies obtained using current physical potentials found in the literature are quite close to the exact values.^{1,2,16} An achievement of less mathematical and numerical complexity is by itself rewarding. The reason for choosing the name adiabatic approximation has its roots in a basically similar approximation used in molecular physics.

In Sec. II the method is recalled and applied to the three-body case. In Sec. III, numerical results are given, leading to the conclusions.

II. THE METHOD: ($N=3$)-BODY CASE

The nonrelativistic Schrödinger equation for three particles with masses m_1 , m_2 , and m_3 , respectively, may be written as

$$\left[-\frac{\hbar^2}{2m} (\nabla_{\vec{x}_i}^2 + \nabla_{\vec{y}_i}^2) + V_{123}(\vec{x}_i, \vec{y}_i) \right] \Psi(\vec{x}_i, \vec{y}_i) = E\Psi(\vec{x}_i, \vec{y}_i), \quad (1)$$

where the c.m. motion is removed using the Jacobi coordinates (not unique)

$$\begin{aligned}\bar{x}_i &= \left[\frac{m_j m_k}{(m_j + m_k)m} \right]^{1/2} (\bar{r}_j - \bar{r}_k), \\ \bar{y}_i &= \left[\frac{m_i(m_j + m_k)}{mM} \right]^{1/2} \\ &\times \left[\bar{r}_i - \frac{m_j \bar{r}_j + m_k \bar{r}_k}{m_j + m_k} \right],\end{aligned}\quad (2)$$

where \bar{r}_i are the particle coordinates while V_{123} is the interaction potential among the three particles, $M = m_1 + m_2 + m_3$, and

$$m = \frac{1}{M} \sum_{i < j = 2}^3 m_i m_j.$$

For pairwise interactions

$$V_{123} = \sum_{i < j = 2}^3 V_{ij}(\bar{r}_i - \bar{r}_j) \equiv \sum_{k=1}^3 V_k. \quad (3)$$

In Eq. (2), i, j, k are (1,2,3). The sign of \bar{x}_i is fixed by the condition that (i, j, k) should form a cyclic permutation of (1,2,3). Equation (2) defines three equivalent sets of coordinates ($i=1,2,3$) for the description of the three-body problem.⁸ Equation (1) can be solved using, for example, the hyperspherical harmonic approach^{1,2} in which the wave function Ψ is expanded in a complete orthonormal set of hypergeometric polynomials in the following way:

$$\Psi(\bar{x}_i, \bar{y}_i) = \rho^{-5/2} \sum_{K\alpha_i} \Phi_{K\alpha_i}(\rho) P_{K\alpha_i}(\hat{x}_i, \hat{y}_i, \phi_i), \quad (4)$$

where $\rho^2 = x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2$, $x_i = \rho \cos \phi_i$, and $y_i = \rho \sin \phi_i$ ($0 \leq \phi_i \leq \pi/2$). The notation \hat{x}_i and \hat{y}_i means $\hat{x}_i \equiv (\theta_{x_i}, \varphi_{x_i})$, and $\hat{y}_i \equiv (\theta_{y_i}, \varphi_{y_i})$, respectively. The complete orthonormal sets of angular functions $\{P_{K\alpha_i}(\Omega_i)\}$ are the angular part of the homogeneous harmonic polynomials of degree K in the six-dimensional space. The label $K\alpha_i$ stands for the five quantum numbers related to the five degrees of freedom Ω_i . Substitution of Eq. (4) into Eq. (1) leads to the system of coupled differential equations¹

$$\left[-\frac{d^2}{d\rho^2} + \frac{\mathcal{L}_K(\mathcal{L}_K + 1)}{\rho^2} + k^2 \right] \Phi_{K\alpha_i}(\rho) + \sum_{K'\alpha'_i} \langle K\alpha_i | v | K'\alpha'_i \rangle \Phi_{K'\alpha'_i}(\rho) = 0, \quad (5)$$

where $\mathcal{L}_K = K + \frac{3}{2}$, $k^2 = (-2m/\hbar^2)E$ ($E < 0$ for bound states), $v = (2m/\hbar^2)V_{123}$, and $\langle v \rangle$ is integrated over the five angles, resulting in a function of ρ . The elements of the angular basis with total angular momentum (L, M) related to the angular coordinates $(\bar{x}_i, \bar{y}_i, \phi_i)$ are

$$\begin{aligned}P_{K\alpha_i}(\Omega_i) &= \left[\frac{2n!(K+2)(n+l_{x_i}+l_{y_i}+1)!}{\Gamma(n+l_{x_i}+\frac{3}{2})\Gamma(n+l_{y_i}+\frac{3}{2})} \right]^{1/2} \left[(\cos \phi_i)^{l_{y_i}} (\sin \phi_i)^{l_{x_i}} P_n^{(l_{x_i}+(1/2), l_{y_i}+(1/2))}(\cos 2\phi_i) \right] \\ &\times \sum_{m_{x_i} m_{y_i}} \langle l_{x_i} l_{y_i} m_{x_i} m_{y_i} | LM \rangle Y_{l_{x_i}, m_{x_i}}(\hat{x}_i) Y_{l_{y_i}, m_{y_i}}(\hat{y}_i),\end{aligned}\quad (6)$$

where $n = \frac{1}{2}(K - l_{x_i} - l_{y_i})$, and $P_n^{(a,b)}(x)$ is a Jacobi polynomial. One should notice that the $P_{K\alpha_i}(\Omega_i)$ constitute for any i an infinite and complete denumerable set of orthogonal functions. Since the hyperspherical basis $P_{K\alpha_i}$ is complete for any i , one can choose this index arbitrarily, and for notational convenience we have dropped the indices i in the quantum numbers and variables. For practical purposes, the solution of Eq. (5) is obtained numerically by truncating the sum over the K' and α' quantum numbers. This procedure is justified by the

fact that the convergence of the solution is reached for the first few K', α' values.¹ The rate of convergence is determined by the shape of the interactions itself.^{1,12} In many realistic calculations the number of coupled equations needed for good accuracy becomes large enough to make the solution hard to obtain.^{1,2,9} In the literature one finds approaches, such as an optimal subset^{1,2} generated from potential harmonics, which reduce the number of coupled equations, but still the number of coupled equations needed for a good accuracy may be large.

For reducing again the number of significant

coupled equations we used (in Ref. 13) a diagonalization procedure in the angular space and generated a set of three coupled equations. The first of our equations is precisely the one proposed by Macek¹⁴

for atomic problems. This equation is transformed in this paper in order to be used in connection with nuclear interactions. Let us write Eq. (1) in hyperspherical coordinates:

$$\left\{ - \left[\frac{d^2}{d\rho^2} + \frac{L^2(\Omega) - (D-1)(D-3)/4}{\rho^2} \right] + v(\rho, \Omega) + k^2 \right\} \Phi(\rho, \Omega) = 0 \quad (7)$$

where

$$\Psi = \rho^{-(D-1)/2} \Phi(\rho, \Omega).$$

The procedure is the following: Instead of Eq. (4), one can expand $\Psi(\vec{x}, \vec{y})$ in the complete orthonormal basis functions $B_\lambda(\rho, \Omega)$, which are the eigenfunctions of the equation¹³

$$\{ - [L^2(\Omega) - (D-1)(D-3)/4] \rho^{-2} + v(\rho, \Omega) \} B_\lambda(\rho, \Omega) = \omega_\lambda(\rho) B_\lambda(\rho, \Omega), \quad (8)$$

where $L^2(\Omega)$ is the grand orbital operator in the D dimensional space (six for the three-body case). Namely,

$$\Psi(\vec{x}, \vec{y}) = \rho^{-(D-1)/2} \sum_\lambda \zeta_\lambda(\rho) B_\lambda(\rho, \Omega). \quad (9)$$

Next, each of the eigenfunctions $B_\lambda(\rho, \Omega)$ can be expanded for any given value of ρ in the complete set of hyperspherical harmonics $P_{K\alpha}(\Omega)$:

$$B_\lambda(\rho, \Omega) = \sum_{K\alpha} \chi_{K\alpha, \lambda}(\rho) P_{K\alpha}(\Omega), \quad (10)$$

so that Eq. (8) holds. The $\chi_{K\alpha, \lambda}(\rho)$ are real numbers which for a given λ constitute the coordinates of a vector B_λ in the Hilbert space and fulfill the orthonormalization conditions

$$\sum_{K\alpha} \chi_{K\alpha, \lambda} \chi_{K\alpha, \lambda'} = \delta_{\lambda, \lambda'}. \quad (11)$$

We can multiply Eq. (8) by $P_{K\alpha}^*(\Omega)$ and integrate over the unit hypersphere leading to the matrix equation:

$$\sum_{K'\alpha'} \left[\frac{\mathcal{L}_K(\mathcal{L}_K + 1)}{\rho^2} \delta_{K\alpha, K'\alpha'} + \langle K\alpha | v | K'\alpha' \rangle \right] \chi_{K'\alpha', \lambda}(\rho) = \omega_\lambda(\rho) \chi_{K\alpha, \lambda}(\rho), \quad (12)$$

where the eigenvalues $\omega_\lambda(\rho)$ and the eigenvector $\chi_{K\alpha, \lambda}(\rho)$ are parametric functions of ρ . Using Eqs. (10) and (9) the partial waves in Eq. (4) are given by

$$\Phi_{K\alpha}(\rho) = \sum_\lambda \zeta_\lambda(\rho) \chi_{K\alpha, \lambda}(\rho). \quad (13)$$

Substituting Eq. (13) into Eq. (5) using Eq. (12), and taking the inner product with $\chi_{K\alpha, \lambda}$, one obtains:

$$\left\{ - \frac{d^2}{d\rho^2} + [\omega_\lambda(\rho) + k^2] \right\} \zeta_\lambda(\rho) - 2 \sum_{\lambda' K\alpha} \frac{d\zeta_{\lambda'}(\rho)}{d\rho} \left[\chi_{K\alpha, \lambda}(\rho) \frac{d\chi_{K\alpha, \lambda'}(\rho)}{d\rho} \right] - \sum_{\lambda' K\alpha} \zeta_{\lambda'}(\rho) \left[\chi_{K\alpha, \lambda}(\rho) \frac{d^2 \chi_{K\alpha, \lambda'}(\rho)}{d\rho^2} \right] = 0.$$

Differentiating Eq. (11) with respect to ρ once one obtains

$$\sum_{K\alpha} \left[\chi_{K\alpha, \lambda}(\rho) \frac{d\chi_{K\alpha, \lambda'}}{d\rho} + \chi_{K\alpha, \lambda'} \frac{d\chi_{K\alpha, \lambda}}{d\rho} \right] = 0. \quad (15)$$

Differentiating once again for $\lambda = \lambda'$ leads to

$$\sum_{K\alpha} \chi_{K\alpha, \lambda}(\rho) \frac{d^2 \chi_{K\alpha, \lambda}(\rho)}{d\rho^2} = - \sum_{K\alpha} \left| \frac{d\chi_{K\alpha, \lambda}(\rho)}{d\rho} \right|^2. \quad (16)$$

Substituting Eqs. (15) and (16) into Eq. (14), one obtains

$$\left[-\frac{d^2}{d\rho^2} + \omega_\lambda(\rho) + k^2 + \sum_{K\alpha} \left| \frac{d\chi_{K\alpha,\lambda}(\rho)}{d\rho} \right|^2 \right] \xi_\lambda(\rho) + \sum_{\lambda' \neq \lambda} \left[2 \frac{d\xi_{\lambda'}(\rho)}{d\rho} \sum_{K\alpha} \chi_{K\alpha,\lambda'}^*(\rho) \frac{d\chi_{K\alpha,\lambda}(\rho)}{d\rho} + \xi_{\lambda'}(\rho) \sum_{K\alpha} \chi_{K\alpha,\lambda'}^*(\rho) \frac{d^2\chi_{K\alpha,\lambda}(\rho)}{d\rho^2} \right] = 0. \quad (17)$$

Up to now Eq. (17) is still exact and equivalent to Eq. (5). When one neglects the sum over λ' in Eq. (17) we have to deal with the so called uncoupled adiabatic approximation. Hence for the lowest eigenpotential ω_0 one obtains the approximate uncoupled equation

$$\left[-\frac{d^2}{d\rho^2} + \omega_0(\rho) + k^2 + \sum_{K\alpha} \left| \frac{d\chi_{K\alpha,0}}{d\rho} \right|^2 \right] \xi_0(\rho) = 0. \quad (18)$$

This equation is similar to the one of Macek when we have to deal with a Coulomb interaction, but in using relation (16) it immediately appears that the variation with ρ of the basis B_λ [i.e., the last term in the brackets of Eq. (18)] weakens the binding. On the other hand, by truncating expansion (9) to one term only, one reduces the Hilbert space. Consequently, according to the Rayleigh-Ritz principle, Eq. (18) provides an upper limit of the ground state binding energy.

For bound states the hyper-radial wave function $\xi_\lambda(\rho)$ is normalized according to

$$\int_0^\infty \sum_{K\alpha} |\chi_{K\alpha,\lambda}(\rho) \xi_\lambda(\rho)|^2 d\rho = \int_0^\infty |\xi_\lambda(\rho)|^2 d\rho = 1, \quad (19)$$

leading to a normalized wave function $\Psi(\vec{x}, \vec{y})$ [Eq. (4)].

The partial waves $\Phi_{K\alpha}(\rho)$ in Eq. (4) are given in the uncoupled adiabatic approximation by

$$\Phi_{K\alpha}(\rho) = \chi_{K\alpha,\lambda}(\rho) \xi_\lambda(\rho). \quad (20)$$

The procedure for solving Eq. (7) is reduced in two numerical steps when using the uncoupled adiabatic approximation:

- (i) perform the diagonalization of the matrix equation (12);
- (ii) integrate the single second order differential equation (18) for the suitable potential $\omega_\lambda(r)$.

For both calculations a standard accurate numerical algorithm exists.

The name "adiabatic approximation" used for this method originates from the decoupling operated between the hyperorbital and hyper-radial motions. This procedure is similar to the one used in solving the molecular problems in which the electrons and nuclei motions are nearly decoupled: First the average field in which the nuclei move is deduced from the eigenenergy of the electrons for fixed nuclei and then the problem is solved introducing the differential equation for the nuclei only.

The method proposed in Ref. 13 contains three coupled equations. The most important is Eq. (18). The other equations are generated by taking the derivative of the basis $B_\lambda(\rho, \Omega)$ with respect to ρ , which is a parameter in the diagonalization of the matrix appearing in Eq. (12), which in turn is equivalent to Eq. (8). The approximation of using the single Eq. (18) is justified if the variation with ρ of the eigenvectors $B_\lambda(\rho, \Omega)$ is small enough that the coupling introduced by the first and second derivations of B_λ with respect to ρ (Ref. 13) can be neglected.

The term $\sum_{\lambda' \neq \lambda}$ in Eq. (17) contains precisely the projection of the derivatives of $\{\chi_\lambda\}$ on the vectors $\{\chi_{\lambda'}\}$ orthogonal to $\{\chi_\lambda\}$. They disappear, as they should, when $B_\lambda(\rho, \Omega)$ becomes independent of ρ [e.g., for a potential of the form $U(\Omega)\rho^{-2} + V(\rho)$]. The magnitude of the coupling can be estimated, for instance, for the ground state in calculating the first derivative

$$\left\langle \frac{dB_0}{d\rho} \left| \frac{dB_0}{d\rho} \right\rangle = \sum_{K\alpha} \left[\frac{d\chi_{K\alpha,0}}{d\rho} \right]^2.$$

The integral

$$(\bar{\chi}'_0)^2 = \int_0^\infty \left\{ \sum_{K\alpha} \left[\frac{d\chi_{K\alpha,0}}{d\rho} \xi_0(\rho) \right]^2 \right\} d\rho \quad (21)$$

is a measure of the inaccuracy of the decoupling in the equation describing the ground state $\lambda=0$. The larger $(\bar{\chi}'_0)^2$, the larger the difference between the exact binding energy and the one given by the adiabatic approximation.

Equations (8)–(13) are quite general and can be applied to any system of coupled differential equations (5). For many-body problems ($N > 3$) in which B_λ is expanded in the hyperspherical harmonic basis $\{P_{K\alpha}\}$ one now has to deal with a set $K\alpha$ of $3N-4$ quantum numbers where K is the grand orbital quantum number, while

$$\mathcal{L}_K = K + \frac{D-3}{2}.$$

We did not discuss up to now the symmetries required for describing the physical states in expansion (4). As a matter of fact the basis $P_{K\alpha}$ does not have any definite symmetry in the exchange of particles, and for either boson or fermion systems the wave function Ψ must exhibit the required symmetry. This task is accomplished in combining appropriate spin and isospin states with the $P_{K\alpha}$ functions for fixed K , in order to obtain a linear combination of spin, isospin, and harmonics exhibiting a definite symmetry under any exchange of two identical particles.

For instance, when we have to deal with the trinucleon system one generates the so called S state ($L=M=0$) occurring in the description of the ground state by combining the completely antisymmetric spin-isospin state $A(s,t)$ with the fully space symmetric S state.^{1,2}

The ground state wave function of the trinucleon system becomes

$$\Psi(\rho, \Omega) = A(s, t) \rho^{-5/2} \sum_{K=0}^{\infty} \mathcal{P}_K(\Omega) \Phi_K(\rho), \quad (22)$$

where $\mathcal{P}_K(\Omega)$ is a linear combination of the $P_{K\alpha}(\Omega)$ symmetric in any exchange of two particles. The grand orbital K is even, in agreement with the parity of the ground state.

When the so called optimal subset is chosen for the expansion as in Refs. 1 and 2, only one $\mathcal{P}_K(\Omega)$ occurs for each K (even) in the expansion of the wave function and the potential $v(\rho, \Omega)$.

III. NUMERICAL RESULTS

As an application we have chosen to solve the triton (${}^3\text{H}$) with four different simple potentials for which exact calculations by the hyperspherical harmonic method exist.¹ As in Ref. 1 we used the “optimal subset” corresponding to the hyperspherical harmonics which are directly connected by the potential to the first term $K=0$ in the expansion. The wave function is given by Eq. (22) while the potential v is expanded in the same hyperspherical har-

monics basis:

$$v(\rho, \Omega) = \sum_K \hat{v}_K(\rho) P_K(\Omega), \quad (23)$$

where \hat{v}_K are the “potential multipoles” and the matrix element $\langle K | v | K' \rangle$ can be expanded in the form

$$\langle K | v | K' \rangle = \sum_K (-)^{K''} \langle K | K'' | K' \rangle \hat{v}_{K''}(\rho), \quad (24)$$

where $\langle K | K'' | K' \rangle$ are geometrical coefficients, independent of the shape of the interaction.

With this expansion in potential multipoles the matrix equation (12) is diagonalized, with ρ as a parameter. Then Eq. (18) is solved numerically, subject for the bound states to the appropriated

TABLE I. Comparison of the adiabatic approximation with exact calculations for chosen potentials and various numbers (N) of coupled differential equations for ${}^3\text{H}$.

Potential	N	Binding energy (MeV)	
		Adiabatic approximation	Exact (1)
Baker	1	9.2048 ^a	9.2077
	2	9.5982	9.6150
	4	9.7409	9.7661
	6	9.7532	9.7795
	8	9.7544	9.7808
	10	9.7546	9.7811
Volkov	12	9.7547	9.7812
	1	7.7076 ^a	7.708
	2	8.0348	8.079
	4	8.3227	8.376
	6	8.3876	8.443
	8	8.4013	8.4575
S3	10	8.4065	8.4630
	12	8.4082	8.4648
	1	0.3647 ^a	0.346
	2	2.0495	2.120
	4	5.0712	5.196
	6	6.0582	6.208
S4	8	6.3153	6.470
	10	6.4381	6.5923
	12	6.4889	6.6403
	1	3.6670 ^a	3.667
	2	4.6958	4.741
	4	6.1319	6.182
	6	6.6793	6.735
	8	6.8396	6.898
	10	6.9217	6.9820
	12	6.9583	7.0182

^aThe small discrepancy is due to the choice of step size in the ρ mesh.

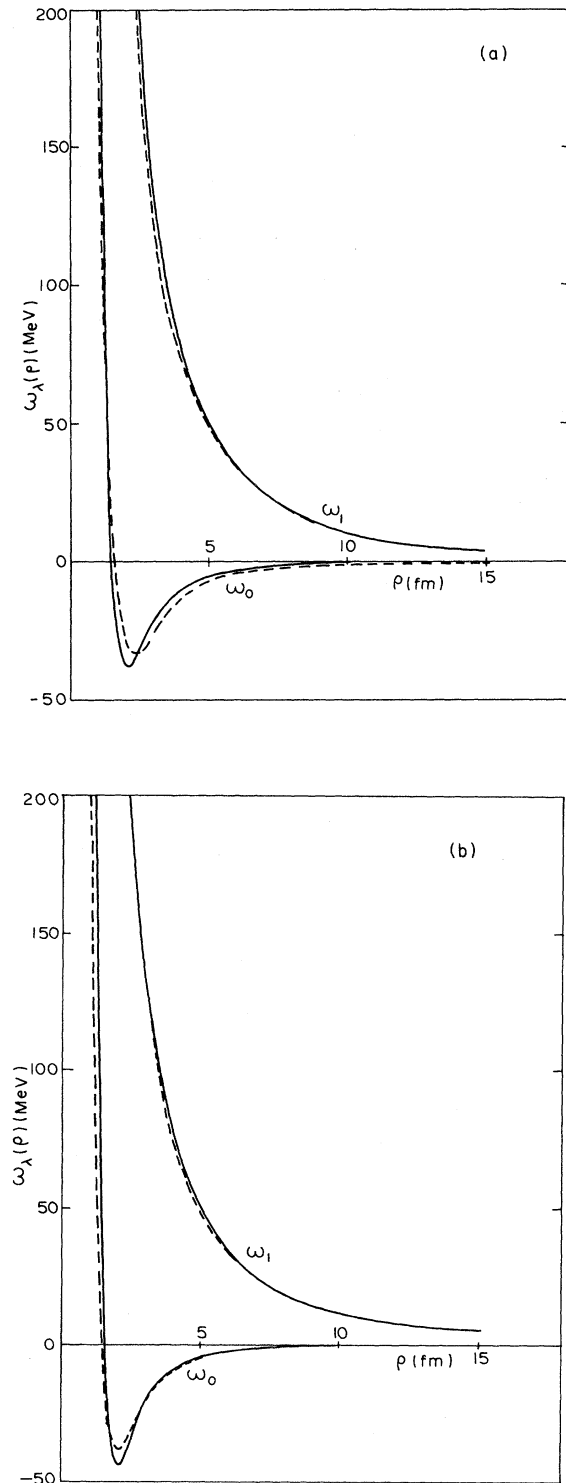


FIG. 1. (a) Plot of the lowest (ω_0) and first excited (ω_1) "eigenpotentials" against ρ for a Baker potential (continuous curve) and a Volkov potential (dashed curve). (b) Same for the S3 potential (continuous curve) and the S4 potential (dashed curve).

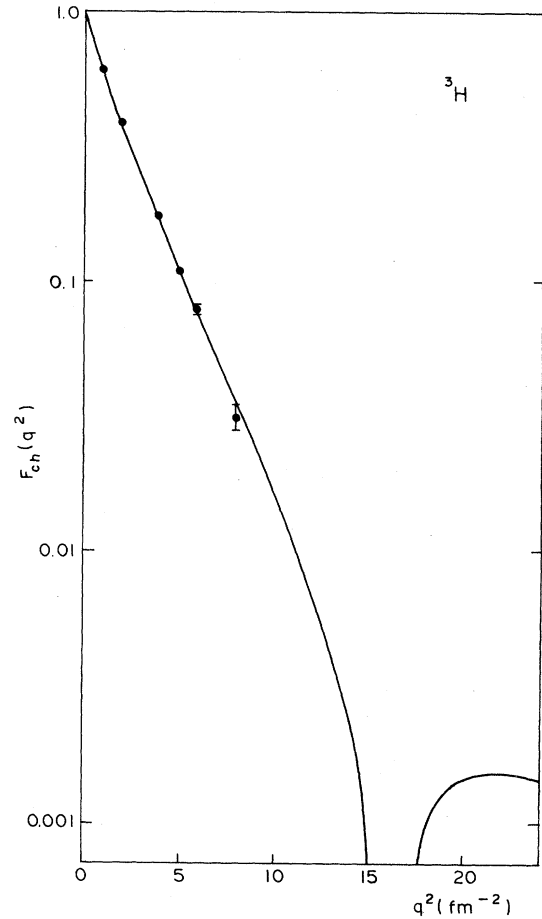


FIG. 2. Plot of $|F_{ch}(q^2)|$ for ³H against q^2 calculated for the S3 potential with an adiabatic approximation. Exact calculations by Afnan and Tange (Ref. 16) agree closely with ours. Some of the available experimental points, with their error bars, have been indicated.

boundary conditions,

$$\xi_0(\rho) \rightarrow \begin{cases} e^{-k\rho}, & \text{for } \rho \rightarrow \infty, \\ 0, & \text{for } \rho \rightarrow 0. \end{cases} \quad (25)$$

The lowest "eigenpotentials" $\omega_0(\rho)$ and $d\chi_{\kappa\alpha,0}/d\rho$ are calculated numerically. In Table I the comparison between the results given by the adiabatic approximation and the exact one is shown. The adiabatic approximation seems very good for very smooth Baker or Volkov potentials^{1,2} leading to a binding energy differing from the exact calculation by about 0.3% and 0.7%, respectively. But even for potentials which have a rather strong soft core (i.e., a rather sharp change in a short range) like^{1,2} S4 or S3, we find that the binding energy differs from the "exact" result^{1,9} by 0.8% and 2.2%, respectively. Furthermore, Table I shows

that the convergence trend for the adiabatic approximation is the same as the one of the exact calculation.¹²

Besides $\omega_0(\rho)$, we have also calculated $\omega_1(\rho)$, the first excited eigenpotential. They do not differ much from one potential to another, as can be seen from Fig. 1. The accuracy of the adiabatic approximation is therefore certainly not connected to the difference $[\omega_1(\rho) - \omega_0(\rho)]$. Instead it clearly appears that the inaccuracy is largest for the potential exhibiting the strongest variation at short range (i.e., S3). In Fig. 2, we plot the charge form factor $|F_{\text{ch}}(q)|$ calculated for the S3 potential; we also indicate the experimental points. The approximate result compares remarkably well with good variational calculations.¹⁶

IV. CONCLUSION

From the foregoing it appears that the adiabatic approximation is generally good whenever the hyper-radial dependence of $\langle K | v | K' \rangle$ is smooth and does not have sharp singularities. The calculation of the eigensolution of a large number of coupled differential equations is extremely difficult. This task is remarkably simplified (with much less

computer time and memory requirements) by using the adiabatic approximation. The price that we pay for this is not so critical for the binding energy, and is certainly acceptable for the charge form factor when $q^2 < 20 \text{ fm}^{-2}$, for most soft core potentials.

Finally the calculation of the ground state is obtained by the following procedure:

- (1) calculate the potential multipoles $[\hat{v}_{K\alpha}(\rho)]$;
- (2) construct the auxiliary matrix $\langle K\alpha | v | K'\alpha' \rangle$;
- (3) solve the matrix Eq. (12); select the lowest eigenvalue $\omega_0(\rho)$ and the lowest eigenfunction $\chi_{K\alpha,0}(\rho)$;
- (4) substitute $\omega_0(\rho)$ and $[d\chi_{K\alpha,0}(\rho)]/d\rho$, calculated in steps (1)–(3) for various values of ρ , in Eq. (18) and find the eigensolution.

Although we have taken the hyperspherical harmonic approach for the triton problem as an example, this method can be used in other areas of physics where coupled differential equations are generated by an expansion of the wave function in a suitable basis. Macek has shown¹⁴ that it should work well in atomic physics.

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