Relativistic few-body problem. I. Two-body equations

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This paper begins with an explanation of the implications of the requirement that a two-body relativistic equation should approach a one-body equation when one of the masses becomes very large. It is found that the Bethe-Salpeter equation does not satisfy this requirement. An infinite family of three-dimensional equations depending on a parameter $-1 \leq \nu \leq 1$ is constructed, all of which do satisfy this limit. When $|\nu| = 1$ one of the particles is on its mass shell; when $\nu = 0$ both particles are equally off mass shell. The fourth order irreducible kernel for this family is studied in the expanded static limit for all ν. It is found, *both* for scalar theories *and* for a realistic chiral theory of spin $\frac{1}{2}$ nucleons interacting with isovector pions, that the leading order terms in the static limit cancel for any ν, and that the nonleading terms are independent of energy only for the $|\nu| = 1$ equation. Other criteria for the selection of a relativistic two-body equation and implications for the form of the two-pion exchange potential are briefly discussed.

[NUCLEAR STRUCTURE Family of two body equations. One body limit. Energy independence of fourth order kernel. Chiral symmetry and nuclear forces.]

I. SUMMARY AND CONCLUSIONS

There exists a variety of two-body relativistic wave equations. Perhaps the best known, and the first to be introduced, is the Bethe-Salpeter (BS) equation[1] in which all four components of the relative four-momentum of the two off-mass-shell particles are variables in the Green's function. More recently a number of simpler, so called "three dimensional" equations have been introduced.[2–5] In these, the time component of the relative momentum is fixed in some covariant way, so that it no longer appears as a separate variable in the Green's function. There have been attempts to compare some of these equations with each other in order to determine which is most suitable for nuclear physics.[6,7] It has also been realized that an infinite number of such equations exist.[8]

The principal goals of this paper are (1) to develop criteria by which relativistic two body equations can be judged and (2) to apply these to a family of such equations in a realistic case with spin $\frac{1}{2}$ nucleons interacting in a chirally invariant way with isovector pions.

One criterion which is often used (sometimes implicitly) is that any equation should reproduce the results obtained from the BS equation (in ladder approximation). This approach *assumes* that the BS equation is the "best" equation for nuclear physics. A different viewpoint will be taken in this paper. The criteria by which equations are to be compared will be developed from *physical* considerations. Any relativistic two body equation (including the BS equation) can then be evaluated. Proceeding in this way, there are compelling reasons for concluding that the BS equation is *not* the best equation for nuclear physics. The situation is paradoxial; the extra work involved in keeping the relative energy as a variable seems to reward one with a poorer answer, rather than the improved result expected.

All of the results in this paper are summarized in this section. The criteria by which it is proposed to evaluate equations is discussed in Sec. I A. The most important one, that the equation should have a one body limit, is developed in Sec. II.

The equations included in this study are the BS equation and a family of three-dimensional equations characterized by a continuous parameter $-1 \leq \nu \leq 1$. When $|\nu| = 1$, the relative energy is fixed by restricting one of the two particles to its mass shell ($\nu = +1$ corresponds to particle 1 on shell, while $\nu = -1$ puts particle 2 on shell). When $|\nu| \neq 1$, neither particle is on shell, but for $\nu > 0$ particle 1 is "closer" to being physical than is particle 2, and conversely for $\nu < 0$. The case $\nu = 0$ is particularly interesting; it corresponds to the case when the particles are equally off shell and the relative energy is zero in the center of mass (c.m.) system (for equal mass particles). The choices $|\nu|=1$ correspond to the equation proposed in Ref. 3, while $\nu=0$ is identical to that proposed by Todorov.[4] The Green's function for this continuous family is developed in detail in Sec. II A.

These relativistic equations are evaluated using two illustrative theories and the results of this evaluation are summarized in Sec. I B. The first is a completely spinless theory, in which two heavy particles of mass M exchange a lighter particle of mass μ. The results for this are worked out in detail in Sec. III, and are a generalization of results previously presented in Ref. 3. The second illustrative theory is a realistic one including spin and isospin consistent with chiral symmetry; the interactions include a purely γ^5 πNN coupling and the required σ-like $NN2\pi$ contact term. The results for this case are presented in detail in Sec. IV. Finally, the results of the entire paper are discussed in Sec. I C.

A. Criteria for evaluating relativistic equations

Probably everyone would agree that the most important requirement for a two body equation is that its solutions give the correct answer. While the correct answer is not known in the general case, there is one limiting case where it is known. Suppose the two body system includes two particles, one of mass m and spin s, the other of mass $M \gg m$ with no internal degrees of freedom (spin zero). The particles interact by exchanging a meson of mass μ. In the limiting case when the mass of the heavy particle becomes very large, so that it neither gives up nor absorbs energy, it should be possible to describe the behavior of the remaining particle by a one-body relativistic equation (appropriate to the spin s of the particle) with an instantaneous potential. The fact that the potential is instantaneous follows from the fact that it cannot transfer energy to or from the particle.

The requirement that a two body equation reduce to a one body equation when one of the particles becomes very massive will be referred to as the "one body limit," and makes good physical sense. If it were not so, then it would not be possible to isolate a physical system from the rest of the world.

The implications of this requirement are studied in Sec. II. There it is shown that the sum of relativistic ladder diagrams does *not* generate the desired one-body equation in the one body limit. It is shown that the sum of all crossed ladder diagrams must be added to the ladders in order to give a minimal set of Feynman diagrams which will generate the correct limit. This result has been known for a number of years,[9] and has its analog in the derivation of the eikonal limit in high energy scattering.[10] It appears that the sum of all ladder and crossed ladder diagrams is the smallest set of Feynman diagrams which gives reasonable results in both the low and high energy limits.

The principal role of any relativistic two body equation is to evaluate this sum for finite masses where the one body equation is not sufficient. To accomplish this the equation defines an infinite series of irreducible kernels, of increasingly higher order n. The definition of irreducibility appropriate for an equation depends specifically on the structure of the Green's function which defines that equation. The sum of these kernels up to order N can be denoted by $\mathscr{K}(N)$

$$\mathscr{K}(N)=\sum_{n=2}^{N} K^{(n)} \tag{1.1}$$

and the solution of the equation with kernel $\mathscr{K}(N)$ can be denoted $\mathscr{M}(N)$. We assume that all equations will give the same answer for $\mathscr{M}(\infty)$, and that this answer, if expanded, would be equivalent to the sum of all ladder and crossed ladder diagrams. However, since the evaluation of the exact kernel $\mathscr{K}(\infty)$ involves computing an infinite sum of kernels $K^{(n)}$ which increase enormously in complexity with order n, the exact kernel is *never* used and the issues of practical importance concern how well the series $\mathscr{M}(N)$ for some small finite N approximates the full result $\mathscr{M}(\infty)$. Once again, since $\mathscr{M}(\infty)$ is not known, it is only possible to make an educated guess about how rapidly $\mathscr{M}(N)\rightarrow\mathscr{M}(\infty)$.

The criteria by which it is proposed to evaluate two body equations will now be stated.

(1) For each order N, the equation must reduce to a one body equation in the one body limit.

(2) The series for $\mathscr{K}(N)$ should converge rapidly as $N\rightarrow\infty$.

(3) $\mathscr{K}(N)$ should be independent of energy.

(4) $\mathscr{K}(N)$ should be well defined and contain no spurious singularities.

No attempt will be made to satisfy these criteria in their most general form. Rather, it must be recognized that practical calculations in nuclear physics will rarely go beyond $N=4$. Taking this into account, the second criterion is restated in a more modest form:

(2′) The fourth order kernel, $K^{(4)}$, should be as small as possible, and in particular, should vanish in the one body limit.

B. Summary of results

The principal result of Sec. II is that the smallest set of Feynman diagrams which will give a one body limit is the sum of all ladder and crossed ladder diagrams. A corollary to this result is that the Bethe-Salpeter equation in ladder approximation ($N=2$) does *not have a one body limit* and hence does not satisfy requirement (1) above. The family of three dimensional equations discussed above and in Sec. III A all have a one body limit.

In Secs. III and IV the fourth order kernel is evaluated for the BS equation, and the family of relativistic three dimensional equations discussed above. Section III examines the fourth order kernel for a spinless theory, in which case the kernel is limited to the first three diagrams shown in Fig. 1. In this figure, the line with a cross means that the particle is on shell, so that Fig. 1 shows the kernel specifically for the $\nu=1$ equation, although the kernel for equations with $\nu\neq 1$ will still have terms similar to those shown. Figure 1(a) is the box, Fig. 1(b) the iteration of the one pion exchange (OPE) to fourth order, so that the *difference* between 1(a) and 1(b), called the *subtracted* box, is that part of the full box not included by the iteration of the second order kernel. Addition of the crossed box, Fig. 1(c), ensures that all contributions from ladders and crossed ladders have been included to fourth order.

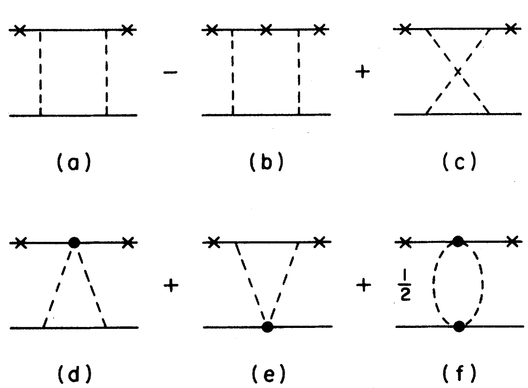

FIG. 1. The six diagrams which contribute to the fourth order kernels discussed in this paper. (a) and (b) subtracted box, (c) crossed box, (d) and (e) triangles, and (f) bubble. The heavy dot in the last three is the σ-like $NN2\pi$ contact term and these are present only for the realistic theory with spin. The cross denotes a particle on mass shell.

For the BS equation, (a) and (b) cancel exactly, leaving only (c).

Section IV examines the fourth order kernel in a realistic theory with spin, isospin, and approximate chiral symmetry. The fourth order kernel now includes all six diagrams shown in Fig. 1. The πNN vertex is assumed to have a pure $g\gamma^5\tau^i$ structure, and the σ-like $NN2\pi$ contact term (represented by a heavy dot in the figure) has the structure

$$\frac{g^2}{M}\delta_{ij}$$

as required by chiral symmetry. The bubble diagram, (f), must be multiplied by $\frac{1}{2}$ as required by the Feynman rules.

For both the spinless and realistic theory the kernel is evaluated in the expanded static limit, where it is assumed that $\mu/M \ll 1$ and terms of order $(\mu/M)^2$ are neglected, but those of order unity and μ/M are retained. For the spinless theory this limit exists without regularization at the vertices, while for the realistic theory with spin the vertices must be regularized, and it is assumed that it is also true that the regularization mass $\Lambda \ll M$. Again terms up to order Λ/M are retained. The following almost identical results are obtained for both theories:

(a) For the entire family of three dimensional equations, the leading order terms in the fourth order kernel cancel, leaving the kernel of order μ/M in the static limit. Specific expressions for this kernel are given in Eqs. (3.34) and (3.37) for the spinless case and Eqs. (4.37) and (4.39) for the case with spin. In particular, this ensures that condition (2′) is satisfied. For the BS equation, the leading order term does *not* cancel, and condition (2′) is not satisfied.

(b) Only the equations with $|\nu|=1$, in which one particle is restricted to its mass shell, have a kernel with no energy dependence in the static limit. (The static BS kernel is also energy independent for the case with spin, but not for the spinless case.)

(c) The equations with $\nu\neq 0$ all have spurious singularities arising from the crossed box diagram. These are cancelled in higher order, occurring at very high internal momenta, $p > MW/\nu + W^2/4$, where W is the total energy, and are not present in the static limit. The location, origin, and cancellation of these singularities are discussed in Appendix B.

The following conclusions can be drawn from these results.

(a) The BS equation is by no means the optimal equation for nuclear physics. It does not have a one body limit and its fourth order kernel has a nonzero term of leading order, suggesting that it does not give a rapidly converging series of irreducible kernels. The extra work involved in retaining the relative energy as a fourth variable is not justified by the quality of the results.

(b) The $|\nu|=1$ equation has attractive advantages. It enjoys the features of all the three dimensional equations in that it is simple, with a one body limit, and with a fourth order kernel which is local, energy independent, and free of spurious singularities *in the static limit.* The presence of spurious singularities in the exact crossed box is a serious disadvantage, however, in that it makes it practically difficult to treat the fourth order kernel exactly.[11]

(c) The $\nu=0$ equation is very attractive also. It has no singularities, and enjoys the advantages common to all the three-dimensional equations. Unfortunately, its fourth order kernel has a significant energy dependence in leading, nonvanishing order. This is an indication of the fact that the equation has not succeeded in confining the energy dependence to the iteration of the second order kernel, where it most naturally belongs, and where it can be taken into account without the need to worry about nonorthogonal wave functions and the other complexities which accompany energy dependent potentials.

C. Discussion

The most extensive comparison of three-dimensional relativistic wave equations was carried out by Woloshyn and Jackson,[6] who studied six different equations for spinless particles. Their case A is identical to our $|\nu|=1$, but our $\nu=0$ case is not included in their discussion. Nevertheless, some comparisons are possible. They calculated the second Born approximation at *threshold*, and observed that other equations gave a better approximation to the fourth order ladder sum (box plus crossed box) than case A ($|\nu|=1$). This conclusion can also be drawn from our results, Eqs. (3.37) and (4.37). If these general expressions are evaluated at threshold ($t=\vec{p}^{\,2}=\vec{p}^{\,\prime 2}=\epsilon=0$) it is readily seen that the $\nu=0$ case gives the smallest fourth order irreducible kernel, and hence the second Born approximation for this equation must be closest to the exact fourth order diagrams. Our results also show, however, that as the size of the fourth order kernel at threshold decreases as ν approaches 0, the accompanying energy dependence (and nonlocality) grows by a compensating amount, so that the $\nu=0$ equation also has the fourth order kernel with the largest energy dependence. Since this energy dependence introduces many technical complications, it is not clear that the threshold value of the fourth order kernel is the best indicator of the efficiency of the equation.

This paper focuses primarily on the effect of treating two equal mass particles in an unsymmetrical way. The amount of asymmetry introduced is characterized by the parameter ν, and could be varied from $\nu=0$ where the particles were equally off mass shell to $|\nu|=1$, where one particle was on shell and the other off shell, giving maximum asymmetry. While both symmetry and asymmetry offer advantages and disadvantages as discussed in Sec. I B above, it would be difficult to argue, *from the viewpoint of the two-body problem in isolation,* that either offers decisive advantages over the other. This conclusion is also supported by Woloshyn and Jackson, whose six equations included three which were symmetric and three which were unsymmetric (one particle on shell). They calculated numerical S wave phase shifts obtained using second and fourth order kernels with each equation. They found that all the equations gave similar results when calculated with fourth order kernels, and that while a symmetric equation gave the smallest difference between second and fourth order results, some of the other asymmetric equations (but not the $|\nu|=1$ case) also gave small results. In recent work with separable potentials,[7] a similar conclusion was found. It may be that additional convergence factors of energy in the propagator are more important than the asymmetry,[7] and these may be less important, in turn, when regularized propagators are used.

The advantages of the unsymmetric approach become more apparent when consideration is given to the ease with which two body wave functions and amplitudes with one particle on shell can be used in interactions with external probes[12] and in the three body system.[13]

The treatment in Sec. IV of the realistic case of two spin $\frac{1}{2}$ nucleons interacting with an isovector pion is sufficiently detailed to shed some light on cancellations in the nuclear force. When γ^5 coupling is used, a realistic theory requires a σ interaction of some kind, and we used a $NN2\pi$ contact term, such as one obtains naturally from the lowest order expansion of a nonlinear chirally invariant Lagrangian, or from a theory with a massive σ meson which satisfies the relation

$$\frac{g_\sigma f_\sigma}{m_\sigma{}^2}=\frac{g^2}{M}\,, \tag{1.2}$$

where g_σ and f_σ are the σNN and $\sigma\pi\pi$ coupling constants and m_σ is the σ mass. We found that, *in the static limit,* the leading contribution of the box and crossed box combine to give a large scalar contribution, which is in turn canceled by the leading contributions from the triangle diagrams, Figs. 1(d) and (e), and the σ bubble, Fig. 1(f). The cancellation of the box and crossed box has been known for many years,[14] and the cancellation of the "two pair" terms by the σ was emphasized by Lomon.[15] As long as a three-dimensional equation is used, this cancellation is largely independent of the kind of equation (i.e., the parameter ν), except for the terms of order (Δ/M) which are energy independent if $|\nu|=1$. Furthermore, for $|\nu|=1$ the pair terms included in the iteration of the OPE are just the right size to cancel the final remaining term left from the fourth order kernel (Sec. IV E). If the other important contributions to the two pion exchange (TPE) force (such as that obtained from Δ diagrams not included here) also tend to cancel that part of the fourth order kernel coming from Fig. 1, then it may be useful to retain the pair terms coming from the iteration of OPE, and the use of γ^5 with the $|\nu|=1$ equation would give a good description of the NN interaction in the OBE approximation. If these other Δ contributions do not cancel the diagrams in Fig. 1, then the γ^5 coupling is inefficient, and it may be better to use $\gamma^5\gamma^\mu$ coupling where these terms are absent from the start.

In a recent series of impressive papers, Zuilhof and Tjon[11,16] have studied the NN interaction using the BS, the Blankenbecler and Sugar, and the $|\nu|=1$ equation. All three are studied in the one boson exchange (OBE) approximation, and TBE contributions to the two three dimensional equations are also studied. They find large differences between the different equations, and point out that the three dimensional equations do not do a good job approximating the BS equation (in ladder approximation). We also would expect this, but believe that this difference should be seen more as a difficulty with the BS equation. It is the sum of all ladders *and crossed ladders* which should be calculated, and while the BS equation can calculate the ladders exactly, it is expected to do a *poorer* job with the ladders and crossed ladders, order by order, than any of the three dimensional equations.

Zuilhof and Tjon also find that adding the fourth order subtracted box [Figs. 1(a) and (b) *only*] to the OBE kernel will not always improve the convergence of the three dimensional equations to the BS equation in ladder approximation. From the viewpoint of this paper, this seems to be a test of doubtful value since the leading term of the subtracted box is in no sense "small," reflecting the fact that the three dimensional equations are not designed to converge to the ladder sum.

The significance of the cancellations described in this paper can be easily questioned, when it is recalled that they hold only in the *expanded static limit,* and for a theory with spin this requires form factor masses much less than the nucleon mass, M, of about 1 GeV. Form factor masses typically used in fits to NN phase shifts are larger than M, so the static limit may be irrelevant. To this there are two responses. First, the existence of the one body limit may be a guide to the choice of wave equation, so that even if the cancellations are only imperfect in real cases, the very existence of the correct limit may be telling us that the three dimensional equations are a more correct starting point for a realistic calculation than is the BS equation. Second, the recent realization that the high momentum structure of the nuclear force is probably determined by new mechanisms depending on quarks, gluons, and quantum chromodynamics (QCD), suggests that perhaps only the low momentum (large distance) behavior of πN boxes is of physical significance. If the correct quark confinement radius is about 0.8 fm, then we can expect QCD effects for momenta larger than 250 MeV, suggesting a cutoff at about this momentum, which is considerably smaller than a nucleon mass. The accuracy of the static limit is examined in Sec. III E.

The final test of any theory is its ability to calculate results, of course, and until we have numerical results these speculations are of limited value. Work in progress on a OBE model of NN scattering using the $\nu=1$ equation suggests that a good fit to the NN phase shifts is possible with only the four mesons essential to any theory; π, ρ, σ, and ω.[17] This will be published elsewhere. The detailed calculations which serve as a foundation for the results and conclusions discussed above are presented in the following sections.

II. THE ONE BODY LIMIT

As discussed in Sec. I above, any reasonable approach to the relativistic two body problem should have a satisfactory one body limit. The one body limit occurs whenever the second particle becomes so massive that its kinetic energy becomes a negligible part of the total energy and its internal degrees of freedom become unimportant. In this case the problem reduces to the motion of the light particle in a static potential field created by the massive particle, and we should require that in this limit the two body equation reduce to the correct relativistic one particle equation for the light particle. In this section, the implications of this requirement will be examined.

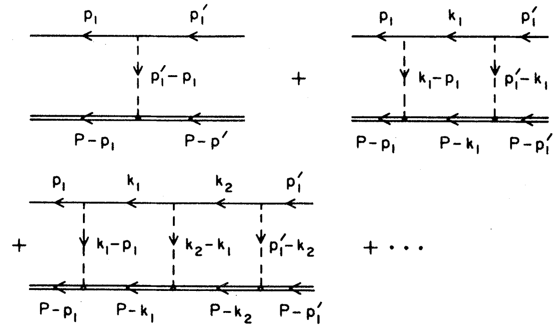

FIG. 2. All terms up to sixth order in the ladder sum. The double line is the heavy particle of mass M.

A. Ladder sums

Consider the ladder sum of one particle exchanges, as shown in Fig. 2. In this figure, the upper line represents a light particle of mass m and spin s. The lower particle is a heavy particle of mass M and in order that its internal degrees of freedom be unimportant, spin zero. Then, the first three terms in this ladder sum can be written

$$\mathscr{M}_L(p_1p_1',P)=V(p_1p_1')+i\int\frac{d^4k_1}{(2\pi)^4}\frac{V(p_1k_1)\Delta(k_1)V(k_1p_1')}{[M^2-(P-k_1)^2-i\epsilon]}$$
$$-\int\frac{d^4k_1d^4k_2}{(2\pi)^8}\frac{V(p_1k_1)\Delta(k_1)V(k_1k_2)\Delta(k_2)V(k_2p_1')}{[M^2-(p-k_1)^2-i\epsilon][M^2-(P-k_2)^2-i\epsilon]}\,, \tag{2.1}$$

where $V(p_1p_1')$ is the relativistic kernel describing the exchange of the particle of mass μ which generates the interaction and $\Delta(k_1)$ is the propagator of the particle of mass m and spin s. In the case where all particles have spin zero, then

$$V(p_1p_1')=\frac{-g^2}{\mu^2-(p_1-p_1')^2-i\epsilon}\,,$$
$$\Delta(k_1)=[m^2-k_1{}^2-i\epsilon]^{-1}\,. \tag{2.2}$$

It is clear that the ladder sum is just the iteration of the following equation

$$\mathscr{M}_L(p_1p_1',P)=V(p_1p_1')+i\int\frac{d^4k_1}{(2\pi)^4}\frac{V(p_1k_1)\Delta(k_1)\mathscr{M}_L(k_1p_1',P)}{[M^2-(P-k_1)^2-i\epsilon]}\,. \tag{2.3}$$

This equation is the Bethe-Salpeter equation[1] in the ladder approximation and is often regarded as a good starting point for a relativistic two body theory.

We now wish to test this popular ladder sum to see whether the equation it generates, Eq. (2.3), satisfies the requirement that the one body limit exist. To do this, we first let $M\to\infty$, and write, in the center of mass of the two particles,

$$[M^2-(P-k_1)^2-i\epsilon]^{-1}=(2E_{k_1})^{-1}\{[E_{k_1}-P_0+k_{10}-i\epsilon]^{-1}+[E_{k_1}+P_0-k_{10}-i\epsilon]^{-1}\}$$
$$\to(2M)^{-1}\{[M-P_0+k_{10}-i\epsilon]^{-1}+[M+P_0-k_{10}-i\epsilon]^{-1}\}\,, \tag{2.4}$$

where we rely on the fact that V and Δ provide sufficient convergence in $\vec{k}_1$ so that $\langle\vec{k}_1{}^2\rangle\ll M^2$ and

$$E_{k_1}=(M^2+\vec{k}_1{}^2)^{1/2}\to M\,.$$

In the same approximation, $P_0=M+e$, where $e=p_{10}'=p_{10}$ is the energy of the light particle of mass m and is also much less than M. Then, it is clear that the second term in Eq. (2.4) will give contributions which are much smaller than the first (since it is large only when $k_{10}\simeq 2M$, in which case V and Δ are small), and we obtain

$$[M^2-(P-k_1)^2-i\epsilon]^{-1}\to(2M)^{-1}[k_{10}-e-i\epsilon]^{-1}\,. \tag{2.5}$$

Note that this denominator depends only on the energy $k_{10}-e$ of the incoming meson. This is as far as we can carry the $M\to\infty$ limit.

If we now substitute (2.5) into Eq. (2.1), we see that the required one body limit *does not exist.* What we expect is an equation of the form

$$\widetilde{\mathscr{M}}_L(p_1p_1',P)=\widetilde{V}(p_1p_1') -\int \frac{d^3k_1}{(2\pi)^3}\widetilde{V}(p_1k_1) \times\widetilde{\Delta}(k_1)\widetilde{M}_L(k_1p_1',P)\ , \tag{2.6}$$

where in every amplitude the energy of the light particle is e, reflecting the fact that the heavy particle is fixed and cannot absorb or give up energy. The potential V should be instantaneous, that is, it can no longer depend on the energy difference $k_{10}-p_{10}$, which is zero. However, Eq. (2.6) cannot be obtained from Eq. (2.1) with (2.5). The k_{10} integration in the second Born term, for example, includes not only the desired pole at $k_{10}=e+i\epsilon$ which comes from (2.5), but also other poles in the upper half plane which come from the terms in V and from the negative energy poles in Δ (which are not negligible if m is small).

We conclude that the Bethe-Salpeter equation in the ladder approximation does not reduce to the correct one body limit, and that the ladder sum is too small a class of Feynman diagrams for this purpose.

B. Ladders and crossed ladders

If the ladder sum is enlarged to include all crossed ladders (with all possible crossings), the correct one body limit is obtained. In effect, the crossed ladders cancel all of the singularities except the pole at $k_{10}=e$, so that the resulting sum would generate Eq. (2.6) when iterated to all orders.

The additional crossed ladder diagrams up to sixth order are shown in Fig. 3. Note that it is possible to label the momenta in each diagram so that all propagators have the same labeling as in the corresponding ladder diagrams except for the propagators of the heavy particle. Then, it is easy to combine ladders and crossed ladders. In fourth order, for example, we have

$$\begin{aligned}\mathscr{M}^{(4)}&=i\int\frac{d^4k}{(2\pi)^4}I(k_1)\left[\frac{1}{M^2-(P-k_1)^2-i\epsilon}+\frac{1}{M^2-(P+k_1-p_1-p_1')^2-i\epsilon}\right]\\ &\underset{M\to\infty}{\longrightarrow} i\int\frac{d^4k}{(2\pi)^4}\frac{I(k_1)}{2M}\left[\frac{1}{k_{10}-e-i\epsilon}+\frac{1}{e-k_{10}-i\epsilon}\right]\\ &=\int\frac{d^4k}{(2\pi)}\frac{I(k_1)}{2M}\delta(k_{10}-e)\ .\end{aligned} \tag{2.7}$$

We see immediately that only the term with $k_{10}=e$ survives; all other singularities are canceled by the crossed box. If a factor of $(2M)$ is absorbed into both $\mathscr{M}$ and V, precisely the correct fourth order term is obtained to be consistent with (2.6).

The same cancellation can be shown to work to *all* orders. To carry out the argument efficiently, write a typical $(2n)$th order diagram as

$$\mathscr{M}^{(2n)}=\sum{}^{(n)}[(\omega_1+i\epsilon)(\omega_1+\omega_2+i\epsilon)\cdots(\omega_1+\omega_2+\cdots+\omega_{n-1}+i\epsilon)]^{-1}\ , \tag{2.8}$$

where $\sum^{(n)}$ includes the integral over all internal four momenta and all factors of V and Δ. Only the positive energy denominators for the massive particle are written explicitly in the brackets, and we have used the fact that in the $M\to\infty$ limit the positive energy denominators of the M particle depend only on the energies of the exchanged meson which are now represented by the new notation ω_i, as shown in Fig. 4(a).

There are $n+1$ different $(2n+2)$th order diagrams generated from each typical $(2n)$th order diagram. These correspond to the extra insertion of an additional meson energy ω_n in all possible places inside the diagram, as shown schematically in Figs. 4(b)—(d). Hence we can write

$$\begin{aligned}\mathscr{M}^{(2n+2)}=\sum{}^{(n+1)}\{&[(\omega_n+i\epsilon)(\omega_1+\omega_n+i\epsilon)\cdots(\omega_1+\omega_2+\cdots+\omega_n+i\epsilon)]^{-1}\\ &+[(\omega_1+i\epsilon)(\omega_1+\omega_n+i\epsilon)\cdots(\omega_1+\omega_2+\cdots+\omega_n+i\epsilon)]^{-1}\\ &+\cdots+[(\omega_1+i\epsilon)(\omega_1+\omega_2+i\epsilon)\cdots(-\omega_n+i\epsilon)]^{-1}\}\ ,\end{aligned} \tag{2.9}$$

where the three terms shown correspond to (b)—(d) in Fig. 4. Adding the first two terms together gives

$$\mathscr{M}^{(2n+2)}=\sum{}^{(n+1)}\{[(\omega_1+i\epsilon)(\omega_n+i\epsilon)\cdots(\omega_1+\omega_2+\cdots+\omega_n+i\epsilon)]^{-1}+\cdots+[(\omega_1+i\epsilon)(\omega_1+\omega_2+i\epsilon)\cdots(-\omega_n+i\epsilon)]^{-1}\}\ . \tag{2.10}$$

In a similar way, the sum of the first two terms may be added to the third term, giving

$$\{[(\omega_1+i\epsilon)(\omega_n+i\epsilon)(\omega_1+\omega_2+\omega_n+i\epsilon)\cdots]^{-1}+[(\omega_1+i\epsilon)(\omega_1+\omega_2+i\epsilon)(\omega_1+\omega_2+\omega_n+i\epsilon)\cdots]^{-1}\}=[(\omega_1+i\epsilon)(\omega_1+\omega_2+i\epsilon)(\omega_n+i\epsilon)]^{-1}\ . \tag{2.11}$$

In this way, the terms involving ω_n in the sum are accumulated and canceled leaving finally only two terms:

$$\begin{aligned}\mathscr{M}^{(2n+2)}&=\sum{}^{(n+1)}[(\omega_1+i\epsilon)(\omega_1+\omega_2+i\epsilon)\cdots(\omega_1+\omega_2+\cdots+\omega_{n-1}+i\epsilon)]^{-1}\{(\omega_n+i\epsilon)^{-1}+(-\omega_n+i\epsilon)^{-1}\}\\&=-2\pi i\sum{}^{(n+1)}\delta(\omega_n)[(\omega_1+i\epsilon)(\omega_1+\omega_2+i\epsilon)\cdots(\omega_1+\omega_2\cdots+\omega_{n-1}+i\epsilon)]^{-1}\ .\end{aligned} \tag{2.12}$$

The energy of the new meson must be zero. Since the remaining $(2n)$th order diagram is only one typical $(2n)$th order diagram, we can now consider the other $(2n)$th order diagrams, and repeat the argument, showing eventually by the same combinations that the energies of each meson must be zero. This gives a Born series of instantaneous potentials, the iteration of Eq. (2.6). We conclude that both ladders and crossed ladders are needed to give the correct one body limit.

III. SCALAR THEORY

A family of relativistic two body equations, all of which give the correct one body limit, will be defined and discussed in this section. The fourth order irreducible kernel for all members of this family will be calculated. The singularities of this kernel are studied, and the accuracy of the static limit is discussed.

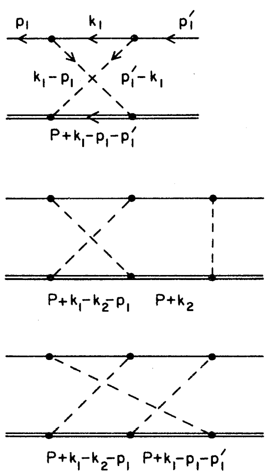

FIG. 3. Additional terms which must be added to the terms shown in Fig. 2 to obtain all ladder and crossed ladder diagrams up to sixth order. All lines not labeled have the same momenta as the corresponding diagram in Fig. 2.

A. The two body propagator

The key to the construction of any two body wave equation is the two body propagator. The propagator to be discussed here is written (for spin-

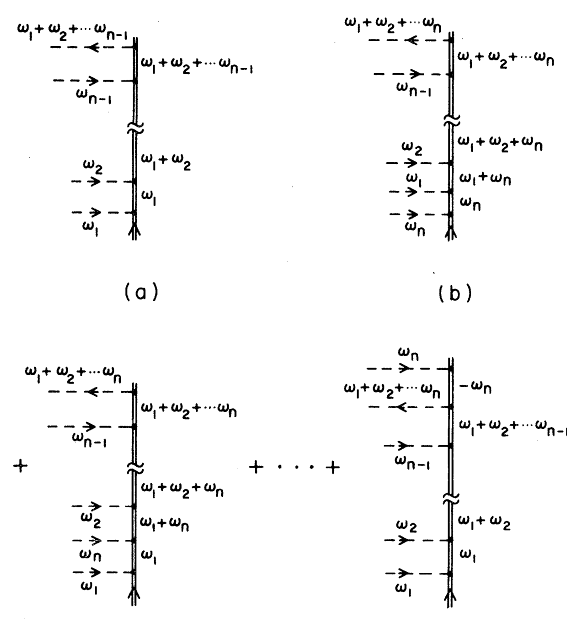

FIG. 4. (a) The interactions with the heavy particle in a typical $(2n)$th order diagram. (b)—(d) Representation of the series of $n+1$ terms arising from adding an extra meson to (a) in all possible ways.

less particles)

$$G(k,P;\nu)=\frac{(2\pi)\delta_+[\Delta(\nu)]}{(A_+ + A_-)}\ , \tag{3.1}$$

where $A_\pm{}^{-1}$ are the relativistic propagators of the two particles,

$$\Delta(\nu)=A_+\tfrac{1}{2}(1+\nu)-A_-\tfrac{1}{2}(1-\nu) \tag{3.2}$$

and ν is a continuous parameter which will vary from

$$-1\le\nu\le 1\ . \tag{3.3}$$

If the two particles have equal mass M, then in the center of mass

$$\begin{aligned} A_\pm &= M^2+\vec{k}^2-(W/2\pm k_0)^2-i\epsilon \\ &= E^2-W^2/4-k_0{}^2\mp Wk_0-i\epsilon\ , \end{aligned} \tag{3.4}$$

where k_1 and k_2 are the four momenta of the two particles, and

$$\begin{aligned} &P=k_1+k_2=(W,\vec{0})\ , \\ &k=\tfrac{1}{2}(k_1-k_2)\ , \\ &E=(M^2+\vec{k}^2)^{1/2}\ . \end{aligned} \tag{3.5}$$

If $\nu=1$, particle 1 is on the positive energy mass shell; if $\nu=-1$, particle 2 is on the positive energy mass shell; and if $\nu=0$, the particles are equally off shell and $k_0=0$ in the c.m. More generally, the δ_+ function fixes k_0 in the c.m. system at

$$k_0(\nu)=\frac{1}{2\nu}[-W+(W^2+\nu^2(4E^2-W^2))^{1/2}]\ , \tag{3.6}$$

where (3.6) is to be expanded near $\nu=0$ to give $k_0=0$ as $\nu=0$. Both of the choices $|\nu|=1$ and $\nu=0$ have been previously investigated.[3,4] The propagator (3.1) can also be written explicitly

$$G(\nu)=\frac{2\pi\nu\delta(k_0-k_0(\nu))}{2k_0(\nu)W(W^2(1-\nu^2)+4\nu^2E^2)^{1/2}}\ , \tag{3.7}$$

which gives for the special cases of particular interest

$$\begin{aligned} G(\pm 1)&=\frac{2\pi\delta(k_0\mp(E-W/2))}{2EW[2E-W]}\ , \\ G(0)&=\frac{2\pi\delta(k_0)}{2W[E^2-W^2/4]}\ . \end{aligned} \tag{3.8}$$

The two-body equation which follows from any of these propagators is (for spinless particles)

$$\begin{aligned} \mathcal{M}(p_1p_1',P)=&V(p_1p_1') \\ &-\int\frac{d^4k_1}{(2\pi)^4}V(p_1k_1)G(k,P;\nu) \\ &\times\mathcal{M}(k_1p_1',P)\ , \end{aligned} \tag{3.9}$$

where the notation is the same as in Sec. II. Note that for any ν, the propagator fixes k_0 at $k_0(\nu)$ [Eq. (3.6)], which approaches

$$\begin{aligned} k_0(\nu)\underset{M\to\infty}{\longrightarrow}&\frac{\nu}{W}(E^2-W^2/4) \\ =&\frac{\nu}{W}(\vec{k}^2-\alpha^2)\to 0\ . \end{aligned} \tag{3.10}$$

Thus all of the equations give the correct one body limit.

B. The fourth order kernel

As discussed in Sec. I, the approach to the one body limit will now be studied. The method will be to look at the fourth order kernel derivable from the subtracted box and crossed box graphs shown in Figs. 1(a)–(c). This potential will be a function of ν, and we expect that it will be smaller for certain values of ν than for others.

The full box ($+$) and crossed box ($-$) diagrams are

$$M_{4\pm}=\frac{ig^4}{(2\pi)^4}\int\frac{d^4k'}{D_1D_2A_+A_-(\pm)}\ , \tag{3.11}$$

where, allowing the meson masses to be unequal (useful for the eventual introduction of form factors later)

$$\begin{aligned} &D_1=\mu_1{}^2-(k'-\Delta)^2-i\epsilon\ , \\ &D_2=\mu_2{}^2-(k'+\Delta)^2-i\epsilon\ , \\ &A_+=M^2-(\tfrac{1}{2}W+Q+k')^2-i\epsilon\ , \\ &A_-(\pm)=M^2-(\tfrac{1}{2}W-Q\mp k')^2-i\epsilon\ , \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &p=\tfrac{1}{2}(p_1-p_2)\ ,\quad \Delta=\tfrac{1}{2}(p-p')\ , \\ &p'=\tfrac{1}{2}(p_1'-p_2')\ ,\quad Q=\tfrac{1}{2}(p+p')\ , \\ &k'=k-Q\ , \end{aligned} \tag{3.13}$$

where k, p_1, and p_2 were defined in Eq. (3.5) and Figs. 2 and 3.

Examine the box first. Note that the fully off shell two nucleon propagator can be written

$$G_0 \equiv \frac{1}{A_+A_-}$$

$$= \int_{-(1+\nu)^{-1}}^{(1-\nu)^{-1}} \frac{d\alpha}{2}[\Delta(\nu)\alpha + \tfrac{1}{2}(A_+ + A_-)]^{-2} . \tag{3.14}$$

While this form makes its appear that G_0 depends on ν, the transformation

$$\alpha = \frac{\beta}{1-\nu\beta}; \quad \beta = \frac{\alpha}{1+\nu\alpha} \tag{3.15}$$

shows that the apparent ν dependence is indeed absent. Nevertheless, the form (3.14) is convenient, as will be seen shortly. Next, using the identity

$$(A^2BC)^{-1} = \int_{-1}^{1} d\gamma \int_0^\infty 3x\, dx[Ax + \tfrac{1}{2}B(1+\gamma) + \tfrac{1}{2}C(1-\gamma)]^{-4} \tag{3.16}$$

and performing the d^4k' integration gives

$$M_{4+} = \frac{-g^4}{64\pi^2} \int_{-1}^{1} d\gamma \int_0^\infty x\, dx \int_{-(1+\nu)^{-1}}^{(1-\nu)^{-1}} d\alpha\, \eta_+^{-4} , \tag{3.17}$$

where

$$\eta_+^2 = M^2x^2\alpha^2 + \tfrac{1}{2}[1+x(1+\nu\alpha)][\mu_1^2(1+\gamma)+\mu_2^2(1-\gamma)] - \tfrac{1}{4}t(1-\gamma^2)$$
$$+ \left[M^2 - \frac{W^2}{4}\right][x^2(1+\nu\alpha)^2 - x^2\alpha^2] + \tfrac{1}{2}xW\left[\frac{p_0(\nu)}{\nu}(1+\gamma) + \frac{p_0'(\nu)}{\nu}(1-\gamma)\right] , \tag{3.18}$$

where $t=(p-p')^2=4\Delta^2$ and $p_0(\nu)$ and $p_0'(\nu)$ are given by Eq. (3.6) with $\vec{k}$ replaced by $\vec{p}$ and $\vec{p}\,'$. Note that much of the ν dependence of (3.17) can be removed by the transformation

$$\alpha = \alpha'[1-\nu\alpha']^{-1}; \quad x = x'(1-\nu\alpha') . \tag{3.19}$$

The amplitude M_{4+} only depends on ν through the relation (3.6).

As expected, M_{4+} has a singularity whenever

$$W^2 > 4M^2 ,$$

which comes from the elastic cut. This can be readily seen from (3.18) by noting that $p_0(\nu)/\nu$ and $p_0'(\nu)/\nu \geq 0$, and hence every term in (3.18) is positive definite if $W^2 < 4M^2$ and $t<0$. If $W^2 > 4M^2$, the fourth term is negative, and for small α and large x this term will produce a zero in η^2, giving rise to the singularity.

Now M_{4+} is not the correct contribution to the fourth order kernel, since the iteration of the second order kernel must first be subtracted to avoid double counting. In Appendix A it is shown that the iteration of the second order kernel can be obtained from (3.17) merely by extending the range of the α integration in the correct way. Specifically,

$$M_{4\nu} = \frac{-g^4}{64\pi^2} \int_{-1}^{1} d\gamma \int_0^\infty x\, dx \int_{-\left|\frac{1+x}{\nu x}\right|}^{\infty} d\alpha\, \eta_+^{-4} . \tag{3.20}$$

If $\nu \leq 0$, the integral over α runs from $-\infty$ to $-(1+x)/\nu x$. The subtracted box is then simply ($\nu \geq 0$)

$$V_{4s} \equiv M_{4+} - M_{4\nu} = \frac{g^4}{64\pi^2} \int_{-1}^{1} d\gamma \int_0^\infty x\, dx \left\{ \int_{-\frac{1+x}{\nu x}}^{-(1+\nu)^{-1}} + \int_{(1-\nu)^{-1}}^{\infty} \right\} d\alpha\, \eta_+^{-4} . \tag{3.21}$$

As Charap and Fubini[14] first pointed out a long time ago, this subtraction has eliminated the elastic cut and rendered the kernel (or potential) far less energy dependent than would otherwise be the case. In Eq. (3.21) the ν dependence can now no longer be transformed away.

A similar parametrization of the crossed box is possible. We make no subtraction of the crossed box, and the result is

$$V_{4-} \equiv M_{4-} = \frac{-g^4}{64\pi^2} \int_{-1}^{1} d\gamma \int_0^\infty x\, dx \int_{-1}^{1} d\alpha\, \eta_-^{-4} , \tag{3.22}$$

where

$$\eta_-{}^2=M^2x^2+\tfrac{1}{2}(1+x)[\mu_1{}^2(1+\gamma)+\mu_2{}^2(1-\gamma)]-\tfrac{1}{4}t[1-\gamma^2-x^2(1-\alpha^2)]$$
$$-(M^2-\tfrac{1}{4}W^2)x^2(1-\alpha^2)+\tfrac{1}{2}xW(p_0(\nu)+p_0'(\nu))[\nu^{-1}-\alpha+x\nu^{-1}(1-\alpha^2)]$$
$$-\tfrac{1}{2}xW(p_0(\nu)-p_0'(\nu))\gamma[1-\alpha\nu^{-1}]\ . \qquad (3.23)$$

C. Singularities of the fourth order kernel

Before we discuss the static limit of the fourth order kernel, we discuss the singularity structure of the subtracted box and crossed box diagrams.

It is shown in Appendix B that V_{4s} has a singularity whenever

$$W^2>\frac{2}{1+\nu}\left\{(M+\mu)^2+\nu M^2+\left[((M+\mu)^2+\nu M^2)^2-\frac{\nu^2}{4}(2\mu M+\mu^2)^2\right]^{1/2}\right\}. \qquad (3.24)$$

For the two special cases of interest,

$$\begin{aligned}\nu=0\quad & W^2>(2M+2\mu)^2\ ,\\ \nu=1\quad & W^2>(2M+\mu)^2\ .\end{aligned} \qquad (3.25)$$

This singularity is due to the presence of inelastic cuts in the subtracted box.

The crossed box has singularities[11] whenever

$$u=(p+p')^2>4M^2\ . \qquad (3.26)$$

If both particles in *either* the initial *or* final state are restricted to their mass shell, this condition cannot be realized. However, if *both* initial and final states are unphysical, there is a singularity in $\vec{p}^{\,2}$ (or $\vec{p}^{\,\prime 2}$) for *all* W. The precise location of the singularity depends on $\vec{p}$ and $\vec{p}^{\,\prime}$. To obtain the minimum value of $\vec{p}^{\,2}$, or $\vec{p}^{\,\prime 2}$, at which the V_{4-} kernel is singular, choose $\vec{p}^{\,\prime}+\vec{p}=0$ (to minimize u) and obtain

$$\vec{p}^{\,2}=\vec{p}^{\,\prime 2}\geq\frac{MW}{\nu}+\tfrac{1}{4}W^2\ . \qquad (3.27)$$

Note that only the case $\nu=0$ is free of this singularity; for other values of ν the singularity occurs at very large (but finite) three momenta. For $\nu=1$, with $W=2M$, the singular region starts at

$$|\vec{p}\,|\cong\sqrt{3}M\simeq 1700\ \text{MeV}/c\ . \qquad (3.28)$$

It is shown in Appendix B that this singularity is canceled in higher order, and is an artifact of the prescription (3.6). It arises from the overlapping of positive and negative energy nucleon poles which seems to be characteristic of crossed ladders, and would be absent if all negative energy nucleon poles were ignored. Since the singularity is eventually canceled in higher order, no errors would result from taking only the principal value contribution. Nevertheless, the presence of this explicit singularity is a nuisance; it would considerably complicate any attempt to calculate V_{4-} exactly by numerical methods.[11] This difficulty presents us with a strong reason to favor the $\nu=0$ equation.

On the other hand, this singularity is very distant, and in a region of very high momentum where many physical phenomena other than two meson exchange are important. It seems unwise to let the choice of equation be dictated exclusively by such behavior.

In the next section we will consider the fourth order kernels in the static limit. In this case, the singularities of V_{4s} and V_{4-} can be ignored.

D. The fourth order kernel in the static limit

To obtain the static limit, we scale (3.21) by introducing the variable

$$y=M\alpha x,\quad z=Mx.$$

This gives

$$V_{4s}=\frac{g^4}{64\pi^2M^2}\int_{-1}^{1}d\gamma\int_0^\infty dz\left\{\int_{-\left|\frac{M+z}{\nu}\right|}^{-\frac{z}{1+\nu}}+\int_{\frac{z}{1-\nu}}^{\infty}\right\}dy\,\xi_+{}^{-4}\ , \qquad (3.29)$$

where

$$\xi_+{}^2=y^2+\tfrac{1}{2}\left[1+\frac{z+\nu y}{M}\right][\mu_1{}^2(1+\gamma)+\mu_2{}^2(1-\gamma)]-\tfrac{1}{4}t(1-\gamma^2)$$
$$+\left[1-\frac{W^2}{4M^2}\right][(z+\nu y)^2-y^2]+\frac{1}{2}\frac{zW}{M}\left[\frac{p_0(\nu)}{\nu}(1+\gamma)+\frac{p_0'(\nu)}{\nu}(1-\gamma)\right]. \tag{3.30}$$

For V_{4-} we introduce

$$y=Mz$$

into (3.22), giving

$$V_{4-}=\frac{-g^4}{64\pi^2M^2}\int_{-1}^{1}d\gamma\int_0^{\infty}y\,dy\int_{-1}^{1}d\alpha\,\xi_-{}^{-4}, \tag{3.31}$$

where

$$\xi_-{}^2=y^2+\tfrac{1}{2}\left[1+\frac{y}{M}\right][\mu_1{}^2(1+\gamma)+\mu_2{}^2(1-\gamma)]-\tfrac{1}{4}t(1-\gamma^2)$$
$$-\left[1-\frac{W^2}{4M^2}-\frac{t}{4M^2}\right]y^2(1-\alpha^2)+\frac{1}{2}\frac{yW}{M}(p_0(\nu)+p_0'(\nu))\left[\frac{1}{\nu}-\alpha+\frac{y}{\nu M}(1-\alpha^2)\right]$$
$$-\frac{1}{2}\frac{yW}{M}(p_0(\nu)-p_0'(\nu))\gamma\left[1-\frac{\alpha}{\nu}\right]. \tag{3.32}$$

To obtain the static limit, assume that $|\vec{p}|$ and $|\vec{p}\,'|$ are of order of the meson masses μ_1 (or μ_2), and let $M\to-\infty$. Also $W=2M+\epsilon$, where $\epsilon\simeq\mu^2/M$, so that

$$1-\frac{W^2}{4M^2}\equiv-\frac{\alpha_0{}^2}{M^2}\simeq\left[\frac{\mu}{M}\right]^2$$

and these terms can be ignored if terms of order $(\mu/M)^2$ are to be neglected (they cancel for all ν anyway). From (3.10) we have

$$\frac{1}{\nu}[p_0(\nu)+p_0'(\nu)]\simeq\left[\frac{1}{2M}\right][\vec{p}^{\,2}+\vec{p}\,'^2-2\alpha_0{}^2]. \tag{3.33}$$

Then expand the denominators, keeping terms of order M^{-1} only. The individual terms can be cast into a convenient spectral form using the techniques reviewed in Appendix C. The result is

$$V_{4s}=V_0-\frac{1}{M}\left[(1-\nu^2)+\frac{2(1+\nu^2)}{\mu_1\mu_2}\times(\vec{p}^{\,2}+\vec{p}\,'^2-2\alpha_0{}^2)\right]V_1,$$

$$V_{4-}=-V_0+\frac{1}{M}\left[2+\frac{4}{\mu_1\mu_2}\times(\vec{p}^{\,2}+\vec{p}\,'^2-2\alpha_0{}^2)\right]V_1, \tag{3.34}$$

where we assumed the potential was symmetric in μ_1 and μ_2 (which would be guaranteed by the regularization; see Sec. IV), and

$$V_0(t)=\frac{g^4}{16\pi^2M^2}\int_{(\mu_1+\mu_2)^2}^{\infty}\frac{d\xi}{\xi-t}\times\Delta^{-1/2}(\xi,\mu_1{}^2,\mu_2{}^2),$$
$$V_1(t)=\frac{g^4}{128\pi M^2}\frac{\mu_1+\mu_2}{(\mu_1+\mu_2)^2-t}, \tag{3.35}$$

where

$$\Delta(a,b,c)=a^2+b^2+c^2-2ab-2ac-2bc. \tag{3.36}$$

The total fourth order kernel is

$$\begin{aligned} V_4 &= V_{4s} + V_{4-} \\ &= \frac{1}{M}\left[1+\nu^2+\frac{2}{\mu_1\mu_2}(1-\nu^2)\right. \\ &\qquad \left.\times(\vec{p}^{\,2}+\vec{p}^{\,\prime 2}-2\alpha_0{}^2)\right]V_1(t)\,. \end{aligned} \tag{3.37}$$

This is the principal result of this section. For an unregularized theory with a single meson ($\mu_1=\mu_2=\mu$), the special cases are

$$V_4(\nu=0)=\frac{1}{M}\left[1+\frac{2}{\mu^2}(\vec{p}^{\,2}+\vec{p}^{\,\prime 2}-2\alpha_0{}^2)\right]V_1(t)\,,$$

$$V_4(\nu=1)=\frac{2}{M}V_1(t)\,. \tag{3.38}$$

Note that only the case $\nu=1$ gives an *energy independent*, local potential in the static limit. The otherwise attractive $\nu=0$ case has a potential with a strong energy dependence which goes like ϵ/μ. Since $\vec{p}^{\,2}\sim\vec{p}^{\,\prime 2}\sim\mu^2$, the $\nu=0$ potential is also probably just as large as the $\nu=1$ potential. We would have drawn somewhat different conclusions if we had ignored the p_0 and p_0' terms, instead of treating them consistently.[6] We are lead to the conclusion that the greater tendency for the $\nu=1$ potential to be energy independent is a strong argument in favor of this equation.

E. Accuracy of the expanded static limit

This section is devoted to a brief discussion of the accuracy of the expanded static limit, where terms up to $\mathcal{O}(\mu/M)$ are retained.

We will begin the discussion with the crossed box. The approximation in which the last three terms are ignored in (3.32) is sometimes referred to as the adiabatic approximation.[14] It gives

$$\xi_-{}^2 \cong y^2+\tfrac{1}{2}\left[1+\frac{y}{M}\right] \times[\mu_1{}^2(1+\gamma)+\mu_2{}^2(1-\gamma)]-\tfrac{1}{4}t(1-\gamma^2) \tag{3.39}$$

and can be directly justified if we are near threshold and neglect off mass shell effects ($p_0=p_0'=0$), or if we are using the $\nu=1$ equation. Charap and Fubini[14] then argue that while the adiabatic limit may be good, it is bad to proceed further to the static limit, which is equivalent to letting $M\to\infty$ in (3.39) and dropping the y/M term. Their point is that since the y integration runs to ∞, y/M cannot be regarded as small. However, y/M is small except when y is very large, and in this case the y^2 term in (3.39) guarantees that the integrand will be small. In short, since the integrand is uniformly convergent, the limit $M\to\infty$ can be taken either before or after the integration has been performed. (With spin, this requires that the theory first be regularized in order that the integral exist.)

In this paper we have used what we call the *expanded* static limit, in which the y/M term in (3.39), and others like it, are taken into account by expanding the denominator in a power series and retaining the first two terms. As we shall see shortly, while the y/M term is quite important at the 10% level, it can be well approximated by this expansion.

To gain insight into the error introduced by expanding the y/M term as opposed to keeping it in the denominator (3.39), we examine both the subtracted box and crossed box at $t=0$.

The adiabatic limit for $\nu=1$ is

$$\begin{aligned} V_{4s}(t=0) &= \frac{g^4}{32\pi^2M^2\mu^2}\left\{\frac{1}{\eta\sqrt{1-\eta^2}}\tan^{-1}\left[\frac{\eta}{\sqrt{1-\eta^2}}\right]\right\}, \\ V_{4-}(t=0) &= \frac{-g^4}{32\pi^2M^2\mu^2}\left\{\frac{1}{1-\eta^2}+\frac{\eta}{(1-\eta^2)^{3/2}}\left[\tan^{-1}\left[\frac{\eta}{\sqrt{1-\eta^2}}\right]-\frac{\pi}{2}\right]\right\}, \end{aligned} \tag{3.40}$$

where $\eta=\mu/2M=0.074$ (for the pion-nucleon case). If (3.40) is expanded, then

$$\begin{aligned} V_{4s}(t=0) &= \frac{g^4}{32\pi^2M^2\mu^2}\left\{1+\mathcal{O}\left[\frac{\mu^2}{M^2}\right]\right\}, \\ V_{4-}(t=0) &= \frac{-g^4}{32\pi^2M^2\mu^2}\left\{1-\frac{\pi}{2}\eta+\mathcal{O}\left[\frac{\mu^2}{M^2}\right]\right\}. \end{aligned} \tag{3.41}$$

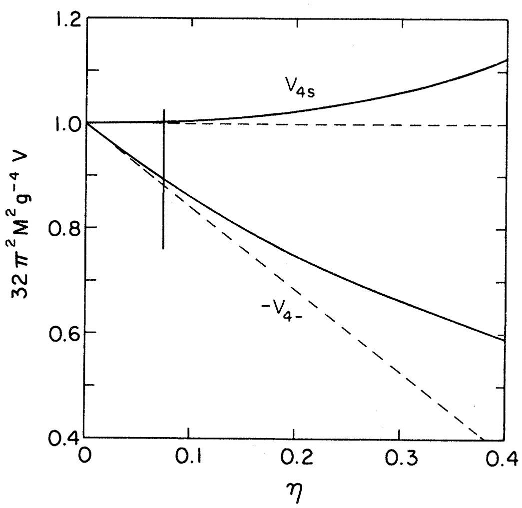

FIG. 5. Comparison of the adiabatic limit (solid line) Eq. (3.40) with the expanded static limit (dashed line) Eq. (3.41) as a function of η. The vertical solid line marks $\eta=0.074$.

These quantities are shown as a function of the small parameter η in Fig. 5. It is clear that the terms of order η^2 can be safely neglected, and the expanded static limit is accurate to better than 2% for $\eta \lesssim 0.1$. The strict static limit is less accurate; for the value of η expected in NN scattering the $\pi\eta/2$ term contributes about a 10–20% correction to V_{4-}. It can be concluded that, for small η, the static limit itself gives a reasonable estimate of the result, and that a fairly precise approximation can be obtained from the expanded static limit.

IV. N-N INTERACTIONS WITH CHIRAL SYMMETRY

In this section, the two pion exchange (TPE) kernel is examined for the family of relativistic equations introduced in the previous section. To ensure that the interaction is realistic, discussion is restricted to a class of π-N interactions derivable from chiral symmetry; the π-N scattering amplitudes which serve as input to the TPE have the correct behavior at threshold (at least). To keep the discussion as simple as possible, and directed toward the study of relativistic wave equations, which is the primary focus of this paper, all baryon and meson resonances are neglected; the world is composed of spin $\frac{1}{2}$ nucleons and pseudoscalar pions only. Furthermore, all kernels will be evaluated in the expanded static limit.

A. Chiral π-N interactions

A class of π-N interactions consistent with chiral symmetry may be derived from a Lagrangian[18,19] with a nonlinear realization of this symmetry. Two examples come to mind. These are[18]

$$L_\rho=\bar{\psi}\left[\frac{i}{2}\overleftrightarrow{\partial}-M-\frac{g}{2M\left[1+\dfrac{g^2\phi^2}{4M^2}\right]}\gamma^5\gamma^\mu\vec{\tau}\cdot\partial_\mu\vec{\phi}-\frac{g^2}{4M^2\left[1+\dfrac{g^2\phi^2}{4M^2}\right]}\gamma^\mu\vec{\tau}\cdot(\vec{\phi}\times\partial_\mu\vec{\phi})\right]\psi+L_\pi \tag{4.1}$$

and

$$L_\sigma=\bar{\psi}\left[\frac{i}{2}\overleftrightarrow{\partial}-M+\frac{1}{1+\dfrac{g^2\phi^2}{4M^2}}\left[\frac{g^2}{2M}\phi^2-ig\gamma^5\vec{\tau}\cdot\vec{\phi}\right]\right]\psi+L_\pi\ . \tag{4.2}$$

In both examples, $\overleftrightarrow{\partial}=\gamma\cdot\vec{\partial}-\gamma\cdot\overleftarrow{\partial}$, and the purely pion part is

$$L_\pi=\frac{1}{2\left[1+\dfrac{g^2\phi^2}{4M^2}\right]}\times\left[\left[1+\frac{g^2\phi^2}{4M^2}\right]^{-1}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}-\mu^2\phi^2\right]. \tag{4.3}$$

In both cases the infinitesimal chiral transformation of the pion is

$$\vec{\phi}\,'=\vec{\phi}+\frac{2M}{g}\vec{\epsilon}+\frac{g}{M}\vec{\phi}(\vec{\epsilon}\cdot\vec{\phi})-\frac{g}{2M}\vec{\epsilon}\phi^2\ , \tag{4.4}$$

while the nucleon field transforms according to

$$\psi'=\left[1+i\frac{g}{2M}\vec{\tau}\cdot(\vec{\phi}\times\vec{\epsilon})\right]\psi\ , \tag{4.5}$$

in (4.1) and

$$\psi' = (1 - i\vec{\tau}\cdot\vec{\epsilon}\gamma^5)\psi \tag{4.6}$$

in (4.2).

Note that L_ρ involves a $\gamma^5\gamma^\mu$ πNN coupling, with a ρ-type $NN2\pi$ contact term, while L_σ involves a γ^5 coupling with a σ-type $NN2\pi$ contact term. The S wave πN scattering lengths corresponding to these two Lagrangians, while constructed from different mechanisms, are equivalent to order g^2. For the σ-type Lagrangian, the nucleon pole terms at threshold give a large contribution,

$$\mathcal{M}_{\pi N}^{(\sigma\,\mathrm{pole})} = \frac{g^2}{M}\left[1 - \frac{\mu^2}{4M^2}\right]^{-1} \times \left\{\delta_{ij} - \frac{\mu}{M}\tfrac{1}{2}[\tau_j,\tau_i]\right\}, \tag{4.7}$$

where i and j are the isospin of the initial and final pion, respectively. The σ contact term cancels nearly all of the δij term

$$\mathcal{M}_{\pi N}^{(\sigma)} = -\frac{g^2}{M}\delta_{ij} . \tag{4.8}$$

For the ρ-type Lagrangian, the nucleon pole terms are much smaller

$$\mathcal{M}_{\pi N}^{(\rho\,\mathrm{pole})} = \frac{g^2}{M}\left[1 - \frac{\mu^2}{4M^2}\right]^{-1} \times \left\{\frac{\mu^2}{4M^2}\delta_{ij} - \frac{\mu^3}{8M^3}\tfrac{1}{2}[\tau_j,\tau_i]\right\} . \tag{4.9}$$

In this case, the ρ contact term supplies most of the interaction

$$\mathcal{M}_{\pi N}^{(\rho)} = -\frac{g^2\mu}{2M^2}\tfrac{1}{2}[\tau_j,\tau_i] , \tag{4.10}$$

and there is no delicate cancellation. In both cases the sum of the pole term plus the contact term is identical, and the scattering lengths are in qualitative agreement with the data

$$\begin{aligned}\mu a^{(+)} &= \tfrac{1}{3}\mu(a^{(1/2)} + 2a^{(3/2)}) \\ &= -\frac{g^2}{4\pi}\frac{\mu^3}{4M^3}\left[1 + \frac{\mu}{M}\right]^{-1}\left[1 - \frac{\mu^2}{4M^2}\right]^{-1} \\ &= -0.010 \ \text{(theoretical)} \\ &= -0.002 \pm 0.004 \ \text{(experimental)} ,\end{aligned} \tag{4.11}$$

$$\begin{aligned}\mu a^{(-)} &= \tfrac{1}{3}\mu(a^{(1/2)} - a^{(3/2)}) \\ &= \frac{g^2}{4\pi}\frac{\mu^2}{2M^2}\left[1 + \frac{\mu}{M}\right]^{-1}\left[1 - \frac{\mu^2}{4M^2}\right]^{-1} \\ &= 0.139 \ \text{(theoretical)} \\ &= 0.086 \pm 0.003 \ \text{(experimental)} .\end{aligned} \tag{4.12}$$

It is possible to transform L_ρ into L_σ, so that one could consider an arbitrary linear combination of γ^5 and $\gamma^5\gamma^\mu$ coupling, with a corresponding mixture of σ- and ρ-type contact terms. This generalized σ-ρ Lagrangian would give the same scattering lengths, and would be invariant under chiral symmetry. We will not pursue this further here.

In the remainder of this section, we will work with the σ-type Lagrangian. We have not investigated the consequences of using the ρ type, or a σ-ρ mixture, but it seems likely that the general conclusions arrived at in this section would apply to these cases as well.

B. The fourth order kernel

The full fourth order kernel for a chirally invariant theory with γ^5 interaction and a σ-type contact term is constructed from the diagrams shown in Fig. 1. In the figure particle 1 has been placed on shell, corresponding to the $\nu=1$ equation, but we will consider the case of arbitrary ν in this section just as we did in Sec. III above. Diagram (a) is the full box and (b) is the subtraction, so that (a)$-$(b) gives the subtracted box V_{4s} as discussed in Sec. III. The crossed box is shown in (c), and the three remaining terms arise from the σ contact interaction.

The full box and crossed box diagrams are

$$M_{4\pm} = \frac{ig^4}{(2\pi)^4}\int \frac{d^4k' N(\pm)}{D_1 D_2 A_+ A_-(\pm)} , \tag{4.13}$$

where the quantities are identical to those given in Eq. (3.12), and the numerator functions are

$$N(\pm) = (3 \mp 2\vec{\tau}_1\cdot\vec{\tau}_2)[M - \tfrac{1}{2}\not{P} - \not{Q} - \not{k}']_1 \times [M - \tfrac{1}{2}\not{P} + \not{Q} + \not{k}']_2 , \tag{4.14}$$

where $\vec{\tau}_i$ is the isospin operator for particle i, and $\not{Q} = Q\cdot\gamma$, etc. Integrating over k', using the same Feynman parameters introduced in Sec. III, and subtracting the box just as done in Sec. III, gives

$$V_{4s}=(3-2\vec{\tau}_1\cdot\vec{\tau}_2)\frac{g^4}{64\pi^2}\int_{-1}^{1}d\gamma\int_0^\infty x\,dx\left[\int_{-\left[\frac{1+x}{\nu x}\right]}^{-(1+\nu)^{-1}}+\int_{(1-\nu)^{-1}}^{\infty}\right]d\alpha\left\{\frac{\theta_1(+)\theta_2(-)}{\eta_+{}^4}+\frac{\gamma_1\cdot\gamma_2}{2\eta_+{}^2[1+x(1+\nu\alpha)]^2}\right\},$$

$$V_{4-}=(3+2\vec{\tau}_1\cdot\vec{\tau}_2)\frac{g^4}{64\pi^2}\int_{-1}^{1}d\gamma\int_0^\infty x\,dx\int_{-1}^{1}d\alpha\left\{\frac{-\theta_1'(+)\theta_2'(-)}{\eta_-{}^4}+\frac{\gamma_1\cdot\gamma_2}{2\eta_-{}^2(1+x)^2}\right\}, \tag{4.15}$$

where $\eta_\pm$ were given in Eqs. (3.18) and (3.23), and

$$\theta_i(\pm)=\left[M-\tfrac{1}{2}\not{P}\mp\left[\frac{\not{Q}+\gamma\not{\Delta}-\alpha x\frac{1}{2}\not{P}}{1+x(1+\nu\alpha)}\right]\right]_i,$$
$$\theta_i'(\pm)=\left[M\mp\not{Q}-\left[\frac{-\alpha x\not{Q}+\gamma\not{\Delta}+\frac{1}{2}\not{P}}{1+x}\right]\right]_i. \tag{4.16}$$

The two triangle diagrams, Figs. 1(d) and (e), are

$$M_{\Delta\pm}=\frac{-3ig^4}{M}\int\frac{d^4k}{(2\pi)^4}\frac{[M-\frac{1}{2}\not{P}\mp\not{k}]_{1,2}}{D_1(\mu_1{}^2)D_2(\mu_2{}^2)A_\pm}, \tag{4.17}$$

where a factor of 3 for isospin has been included, the projection operator is on line 1 for $M_{\Delta+}$ and line 2 for $M_{\Delta-}$, and

$$D_1(\mu_1{}^2)=\mu_1{}^2-(p-k)^2,$$
$$D_2(\mu_2{}^2)=\mu_2{}^2-(p'-k)^2. \tag{4.18}$$
$$A_\pm=M^2-(\tfrac{1}{2}P\pm k)^2,$$

Using the identity

$$(D_1D_2A)^{-1}=2\int_{-1}^{1}d\gamma\int_0^1(1-y)dy([D_1\tfrac{1}{2}(1+\gamma)+D_2(1-\gamma)](1-y)+Ay)^{-3} \tag{4.19}$$

and integrating over k gives:

$$M_{\Delta\pm}=\frac{3g^4}{32M\pi^2}\int_{-1}^{1}d\gamma\int_0^1(1-y)dy\left[M-\frac{(1-y)}{2}(\not{P}\mp\not{p}(1+\gamma)\mp\not{p}'(1-\gamma))\right]_{1,2}\eta_\Delta{}^{-2}(\pm), \tag{4.20}$$

where

$$\eta_\Delta{}^2(\pm)=M^2y^2+\tfrac{1}{2}(1-y)[\mu_1{}^2(1+\gamma)+\mu_2{}^2(1-\gamma)]-\tfrac{1}{4}t(1-\gamma^2)(1-y)^2$$
$$\pm\tfrac{1}{2}W(p_0(\nu)(1+\gamma)+p_0'(\nu)(1-\gamma))\left[1\pm\frac{1}{\nu}\right]y(1-y). \tag{4.21}$$

We will discuss the bubble diagram, Fig. 1(f), in the next section.

C. Regularization and the bubble diagram

We now wish to examine the expanded static limit. However, none of the diagrams examined so far has a static limit, because the terms involving η^{-2} do not converge when we take the $M\to\infty$ limit. In order for a static limit to exist in this theory with spin, the pion exchange must be further regularized to make it more convergent at large momenta. One convenient way to do this, which also takes into account the composite nature of nucleons, is to insert form factors at the πNN vertices. If a simple monopole form for the form factor is used, this is equivalent to replacing the pion propagator by

$$\frac{1}{\mu^2-t}\to\frac{1}{\mu^2-t}\frac{(\Lambda^2-\mu^2)^2}{(\Lambda^2-t)^2}, \tag{4.22}$$

where the normalization has been chosen to give the same residue at $t=\mu^2$. If we now assume Λ^2 is not too much larger than μ^2, and that it remains fixed as M is increased, the static limit will exist.

The regularized potential (4.22) can be readily calculated if we use the following decomposition

$$\frac{(\Lambda^2-\mu^2)^2}{(\mu^2-t)(\Lambda^2-t)^2}=\frac{1}{\mu^2-t}-\frac{1}{\Lambda^2-t}+(\Lambda^2-\mu^2)\frac{d}{d\Lambda^2}\left[\frac{1}{\Lambda^2-t}\right]. \tag{4.23}$$

If the unregularized potential $V(\mu_1{}^2,\mu_2{}^2)$ is known, then

$$V_{\mathrm{Reg}}=V(\mu^2,\mu^2)-V(\mu^2,\Lambda^2)-V(\Lambda^2,\mu^2)+V(\Lambda^2,\Lambda^2)+(\Lambda^2-\mu^2)\frac{d}{dx}$$
$$\times[V(\mu^2,x)+V(x,\mu^2)-V(\Lambda^2,x)-V(x,\Lambda^2)]\,|_{x=\Lambda^2}+(\Lambda^2-\mu^2)^2\frac{d}{dx}\frac{d}{dy}V(x,y)\,|_{\substack{x=\Lambda^2\\ y=\Lambda^2}}=\mathscr{R}V(\mu_1{}^2,\mu_2{}^2)\,. \tag{4.24}$$

In what follows, we will think of (4.24) as an operation which can be performed on any unregularized potential $V(\mu_1{}^2,\mu_2{}^2)$.

The unregularized potentials we will examine all have the form

$$V(\mu_1{}^2,\mu_2{}^2)=\int_{-1}^{1}d\gamma\,N[B^2+C(\mu_1{}^2(1+\gamma)+\mu_2{}^2(1-\gamma))]^{-n}\,. \tag{4.25}$$

The operation (4.24) when applied (4.25) leaves an integrand which is symmetric in γ, so that any terms in N which are odd in γ (and do not depend on μ_1,μ_2) can be neglected. Furthermore, the regularization increases the rate at which the potential decreases as B^2 increases. If the unregularized potential goes like B^{-2n}, as shown above, then the regularized version will go like $B^{-(2n+6)}$.

The bubble diagram, which is finite only if regularized, can now be calculated. Using the form (4.22), the diagram can be written

$$M_\sigma=(\Lambda^2-\mu^2)^4\frac{d}{d\Lambda_1{}^2}\frac{d}{d\Lambda_2{}^2}M'_\sigma\Bigg|_{\substack{\Lambda_1{}^2=\Lambda^2\\ \Lambda_2{}^2=\Lambda^2}}, \tag{4.26}$$

where including a factor of 3 for isospin, and the factor of $\frac{1}{2}$ required for such bubbles

$$M'_\sigma=\frac{3ig^4}{2M^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{D_1(\mu^2)D_1(\Lambda_1{}^2)D_2(\mu^2)D_2(\Lambda_2{}^2)}\,. \tag{4.27}$$

Introducing Feynman parameters and integrating over k gives

$$M'_\sigma=\frac{-3g^4}{32\pi^2M^2}\frac{1}{(\Lambda_1{}^2-\mu^2)(\Lambda_2{}^2-\mu^2)}\int_{\mu^2}^{\Lambda_1{}^2}dm_1\int_{\mu^2}^{\Lambda_2{}^2}dm_2\int_0^1dy\frac{1}{[m_1y+m_2(1-y)-ty(1-y)]^2}\,. \tag{4.28}$$

Performing the operation of Eq. (4.26), and using the techniques developed in Appendix C, M_σ may be cast into a convenient spectral form:

$$M_\sigma=V_\sigma=-6\mathscr{R}U_0(\mu_1{}^2,\mu_2{}^2)\,, \tag{4.29}$$

where

$$U_0(\mu_1{}^2,\mu_2{}^2)=\frac{g^4}{64\pi^2M^2}\int_{(\mu_1+\mu_2)^2}^{\infty}\frac{d\omega\,\Delta^{1/2}(\omega,\mu_1{}^2,\mu_2{}^2)}{\omega(\omega-t)}\,. \tag{4.30}$$

As will be seen shortly, (4.29) has the same form as some of the static potentials to be obtained below, but for the bubble diagram it is an exact result.

D. Static limits of the fourth order regularized kernel

Once the potentials in Sec. IV B have been regularized, the static limit can be taken just as was done in Sec. III. Now we must assume that the regularization mass $\Lambda \ll M$, as discussed above. The results to order Λ/M will be retained, as was done in Sec. III.

The presence of spin introduces one extra complication into the process of estimating the size of the leading terms. It is necessary to take account of off-diagonal matrices which couple positive and negative energy subspaces, or couple large and small components within the positive energy subspace.

To estimate the size of potentials which couple positive and negative energy subspaces, recall the nonrelativistic limit of the coupled equations for the $\nu=1$ case,[20] which can be written

$$\left[\frac{\nabla^2}{M}+\epsilon\right]\psi^+=\left[V^{++}+\frac{|V^{+-}|^2}{2M}\right]\psi^+ . \tag{4.31}$$

If $V^{+-}\simeq V^{++}$, which is the case here, and we assume that $V^{++}\simeq\Lambda^2/M$, then

$$\frac{|V^{+-}|^2}{2M}\simeq V^{++}\left[\frac{\Lambda^2}{M^2}\right] \quad (V^{+-}\simeq V^{++}) \tag{4.32}$$

and can be neglected. The V^{+-} potentials need only be considered in cases when $V^{+-}\gg V^{++}$. For future reference, note that if

$$V^{+-}\simeq(M/\Lambda)V^{++} ,$$

then

$$\frac{|V^{+-}|^2}{2M}\simeq V^{++} \quad \left[V^{+-}\simeq\frac{M}{\Lambda}V^{++}\right] \tag{4.33}$$

and, in this case, the coupling between positive and negative energy channels should be taken into account.

Since the negative energy channels can be ignored, the positive energy projection operators can be used to eliminate all reference to most remaining off-diagonal matrix elements. For example, from the relations

$$[M-E_{p'}\gamma^0+\vec{p}\,'\cdot\vec{\gamma}]_1 u_1(\vec{p}\,')=0 , \tag{4.34}$$

$$\bar{u}_1(\vec{p})[M-E_p\gamma^0+\vec{p}\cdot\vec{\gamma}]_1=0 ,$$

and similar relations for particle 2, it is easy to show that

$$\begin{aligned}
\Lambda_1(p)&\equiv[M-(\tfrac{1}{2}\not{P}+\not{p})]_1\\
&=\gamma_1^0[E_p-\tfrac{1}{2}W-p_0(\nu)]\\
&\to(1-\nu)\frac{\vec{p}\,'^2-\alpha_0^2}{2M}+\mathcal{O}\left[\frac{\Lambda^2}{M^2}\right] ,\\
\Lambda_2(p)&\equiv[M-(\tfrac{1}{2}\not{P}-\not{p})]_1\\
&=\gamma_1^0[E_p-\tfrac{1}{2}W+p_0(\nu)]\\
&\to(1+\nu)\frac{\vec{p}\,^2-\alpha_0^2}{2M}+\mathcal{O}\left[\frac{\Lambda^2}{M^2}\right] ,
\end{aligned} \tag{4.35}$$

and also

$$P_2=P_1=2M+\mathcal{O}\left[\frac{\Lambda^2}{M^2}\right] . \tag{4.36}$$

Using these simplifications, the static limit of the fourth order kernel is

$$\begin{aligned}
V_{4s}&=(3-2\vec{\tau}_1\cdot\vec{\tau}_2)\\
&\quad\times\left\{(-2+\gamma_1\cdot\gamma_2)U_0+(1-\nu^2)\frac{U_1}{M}\right.\\
&\quad\left.+(\vec{p}\,^2+\vec{p}\,'^2-2\alpha_0^2)(1-\nu^2)\frac{U_2}{M}\right\} ,\\
V_{4-}&=(3+2\vec{\tau}_1\cdot\vec{\tau}_2)\left\{(-2+\gamma_1\cdot\gamma_2)U_0+\frac{2}{M}U_1\right\} ,\\
V_{4\Delta}&=12U_0-\frac{12}{M}U_1 ,\\
V_\sigma&=-6U_0 ,
\end{aligned} \tag{4.37}$$

where U_0 was already defined in Eq. (4.30), $V_{4\Delta}=M_{\Delta+}+M_{\Delta-}$, and

$$\begin{aligned}
U_1&=\frac{g^4}{64\pi^2M^2}\pi/4\\
&\quad\times\int_{(\mu_1+\mu_2)^2}^{\infty}\frac{d\omega}{\sqrt{\omega}\,(\omega-t)}[\omega-(\mu_1^2+\mu_2^2)] ,\\
U_2&=\frac{g^4}{64\pi^2M^2}\pi\int_{(\mu_1+\mu_2)^2}^{\infty}\frac{d\omega}{\sqrt{\omega}\,(\omega-t)} .
\end{aligned} \tag{4.38}$$

The regularization operation is understood to apply to all potentials.

The results (4.37) again display a remarkable cancellation of the leading term U_0. Furthermore, the nonlocal energy dependent term involving U_2 cancels when $\nu=\pm1$. The full potential for the $\nu=\pm1$ case is

$$\begin{aligned}
V_4&=-6[1-\gamma_1\cdot\gamma_2]U_0\\
&\quad-(3-2\vec{\tau}_1\cdot\vec{\tau}_2)\frac{2}{M}U_1 .
\end{aligned} \tag{4.39}$$

While the $\gamma_1\cdot\gamma_2$ term has been carried along so far, it should be noted that

$$\gamma_1\cdot\gamma_2\simeq1+\mathcal{O}\left[\frac{\Lambda^2}{M^2}\right]$$

(on positive energy subspace) and

$$\gamma_1\cdot\gamma_2\simeq\mathcal{O}\left[\frac{\Lambda}{M}\right] \tag{4.40}$$

(on negative energy subspace if $|\nu|=1$), so that actually the U_0 term can be completely neglected. The remaining term can be viewed as composed of an (attractive) σ-like term and a (repulsive) ρ-like term, of equal effective mass but coupling in the ratio of 3 to 2.

E. Cancellation of the negative energy terms

It is instructive to estimate the size of the quadratic term[20] generated by OPE. To the same order of accuracy as (4.39), this is obtained from the leading contribution to V^{+-}, which is (for $\nu=1$):

$$V^{+-}\simeq\frac{-g^2(\vec{\tau}_1\cdot\vec{\tau}_2)\vec{\sigma}_1\cdot\vec{q}}{2M(\mu^2+\vec{q}^{\,2})}\left[\frac{\Lambda^2-\mu^2}{\Lambda^2+\vec{q}^{\,2}}\right] . \tag{4.41}$$

The quadratic term is therefore

$$V_Q\cong(\vec{\tau}_1\cdot\vec{\tau}_2)^2\frac{g^4}{8M^3}\mathscr{R}\int\frac{d^3k}{(2\pi)^3}\frac{\vec{\sigma}_1\cdot(\vec{p}\,'-\vec{k})\sigma_1\cdot(\vec{k}-\vec{p})}{[\mu_1{}^2+(\vec{p}\,'-\vec{k})^2][\mu_2{}^2+(\vec{p}-\vec{k})^2]} . \tag{4.42}$$

Introducing Feynman parameters into this expression, and shifting the momentum $\vec{k}$ gives a form identical to that for U_1 described in Appendix C. The final result is

$$V_Q=(3-2\vec{\tau}_1\cdot\vec{\tau}_2)\frac{2}{M}U_1 . \tag{4.43}$$

Note that this repulsive contribution is canceled by the attractive contribution from (4.39), removing all terms, to order (Λ/M), except the original V^{++} from OPE. This simple model therefore suggests that the TPE contributions known to be of major importance in the nuclear force must come predominately from diagrams containing an intermediate Δ or other higher states.

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APPENDIX A

In this appendix we show that the subtraction term for the box diagram takes on the form (3.20). Begin by substituting the identity (3.14) into (3.16). This gives

$$I_+=\tfrac{3}{2}\int_{-1}^{1}d\gamma\int_0^\infty x\,dx\int_{-(1+\nu)^{-1}}^{(1-\nu)^{-1}}d\alpha \times[\Delta(\nu)\alpha x+D]^{-4} , \tag{A1}$$

where

$$D=\tfrac{1}{2}(A_++A_-)x +\tfrac{1}{2}D_1(1+\gamma)+\tfrac{1}{2}D_2(1-\gamma) ,$$

and $A_\pm$, $\Delta(\nu)$, and $D_{1,2}$ were all defined in Sec. III. Multiplying (A1) by suitable constants, and integrating over d^4k' would give (3.17), our result for the full box.

To obtain the subtraction term, we recall that the propagator (3.1) does not involve $\Delta(\nu)$ in the denominator. Using the more elementary identity from which (3.16) was derived, gives

$$\begin{aligned}I_s&=\int_{-1}^{1}d\gamma\int_0^\infty\frac{dx}{2}2\pi i\delta_+[\Delta(\nu)]D^{-3}\\&=\int_{-1}^{1}d\gamma\int_0^\infty\frac{dx}{2}\frac{2\pi i\delta[m_+-k_0]}{\nu(m_++m_-)D^3} ,\end{aligned} \tag{A2}$$

where we wrote

$$\Delta(\nu)=[(m_-+k_0)(m_+-k_0)-i\epsilon]\nu . \tag{A3}$$

In order to convert (A2) into a form identical to (A1) except for different limits on the α integration, we need an identity like

$$\frac{2\pi i\delta[m_+-k_0]}{\nu(m_++m_-)D^3}=3x\int_{\alpha_1}^{\alpha_2}d\alpha[\Delta(\nu)\alpha x+D]^{-4} . \tag{A4}$$

This is one special case of a general class of identities. If $\nu>0$, $x>0$, and

$$M_2+(m_--m_+)(1+x)>0 , \tag{A5}$$

and $N(k_0)$ is a polynomial of order $m\le 2n-2$, then

$$\begin{aligned}J&=\int_{-\left[\frac{1+x}{\nu x}\right]}^{\infty}d\alpha\int_{-\infty}^{\infty}\frac{dk_0N(k_0)}{[\Delta(\nu)\alpha x+D(k_0)]^{1/2}}\\&=\frac{2\pi iN(m_+)}{(n-1)\nu x(m_++m_-)[D(k_0=m_+)]^3} ,\end{aligned} \tag{A6}$$

where $\Delta(\nu)$ is defined in (A3) and we will write

$$D(k_0)=M_1{}^2+M_2k_0-k_0{}^2(1+x)-i\epsilon(1+x)\ . \tag{A7}$$

If $\nu<0$, a similar identity exists, but the α integration runs from $-\infty$ to $-(1+x)/\nu x$.

Equation (A6) is sufficiently general to include both the scalar and spinor cases (once the spinor case has been regularized). The condition (A5) holds in these cases.

To prove the identity, note that the coefficient of the $i\epsilon$ is $1+x+\nu\alpha x$, which is always positive as long as $\alpha>-(1+x)/\nu x$. Hence, giving this limit a small positive part, δ, we may interchange orders of integration, giving

$$J=\lim_{\delta\to 0_+}\int_{-\infty}^{\infty}dk_0\int_{-\left[\frac{1+x}{\nu x}\right]+\delta}^{\infty}d\alpha\, N(k_0)[\Delta(\nu)\alpha x+D(k_0)]^{-n}$$

$$=\lim_{\delta\to 0_+}\int_{-\infty}^{\infty}dk_0\frac{N(k_0)}{(n-1)x\Delta(\nu)}\left[\Delta(\nu)\left[-\frac{(1+x)}{\nu}+\delta x\right]+D(k_0)\right]^{-(n-1)} .$$

Now examine the singularities in the denominator. Apart from the simple poles at $m_\pm$ from Δ there are $(n-1)$ order poles from the term in square brackets. For very small δ, these are located at

$$k_0=\begin{cases}\dfrac{m_1m_2(1+x)-M_1{}^2+i\epsilon}{M_2+(m_--m_+)(1+x)}\\[2ex] \dfrac{1}{\delta\nu x}[M_2+(m_--m_+)(1+x)-i\epsilon]\end{cases} .$$

Note that (A5) ensures that these $(n-1)$th order poles remain in the upper, or lower, half plane for all values of the parameters. Closing the contour in the lower half plane proves the theorem, since the only pole which gives a finite result is at $k_0=m_+$; the other singularity coming from the $(n-1)$th order pole in D gives a vanishingly small contribution when $\delta\to 0$. It is also possible to prove (A6) by integrating over k_0 first, but this is more difficult.

APPENDIX B

In this appendix, the singularities of the subtracted box and crossed box are discussed, and it is shown how the spurious singularity in the crossed box is cancelled in higher order.

The singularities of the subtracted box are determined by the zeros of (3.18), which will be examined for $\nu\geq 0$ only. The discussion will be limited to cases where $W\geq 0$, $\vec{p}^{\,2}\geq 0$, and $\vec{p}^{\,\prime 2}\geq 0$, in which case the minimum values of p_0, p_0', and $-t$ all occur when $\vec{p}^{\,2}=\vec{p}^{\,\prime 2}=0$, and if $\mu_1=\mu_2=\mu$ the nearest singularity occurs when

$$d(x)=M^2\alpha^2x^2+[1+x(1+\nu\alpha)]\mu^2+(M^2-\tfrac{1}{4}W^2)[x^2(1+\nu\alpha)^2-x^2\alpha^2]+\frac{xW}{\nu}E_0=0\ ,$$

where

$$E_0=\frac{1}{\nu}(M^2\nu^2+\tfrac{1}{4}W^2(1-\nu^2))^{1/2}-\frac{W}{2\nu}$$

is the value of p_0 when $\vec{p}=0$. If this expression is viewed as a quadratic in x, then the coefficient of the x^2 term, $C(\alpha)$, can be shown to be positive definite in the region of α integration given in (3.21). Hence, since $d(0)=\mu^2$, the value of x at which d is a minimum, $x_{\min}$, must be positive before d can have a zero, and this gives the two requirements

$$d_{\min}(\alpha)=\mu^2-[\mu^2(1+\nu\alpha)+WE_0/\nu]\times(4C(\alpha))^{-1}=0\ , \tag{B1a}$$

$$-[\mu^2(1+\nu\alpha)+WE_0/\nu]>0\ . \tag{B1b}$$

The equation (B1a) is again a quadratic equation in α, which, for energies which satisfy (B1b), can be shown to have its minimum in the region of

$$-1/(1+\nu)\leq\alpha\leq 1/(1-\nu)\ .$$

Hence the singularity will first appear at the end points of the integration

$$d_{\min}\left[-\frac{1}{1+\nu}\right]=0\ ,$$

$$d_{\min}\left[\frac{1}{1-\nu}\right]=0\ .$$

As it turns out, the first condition is the most critical, and the energy W at which it is satisfied was given in Eq. (3.24).

To study the singularities of the crossed box, it is convenient to use the crossed variable u defined in (3.26). In terms of this variable, the expression for the crossed box is the same is given in (3.22), except with $\eta_-{}^2$ of Eq. (3.23) replaced by

$$\begin{aligned}\eta_-{}^2 = {} & M^2\alpha^2x^2+\tfrac{1}{2}(1+x)[\mu_1{}^2(1+\gamma)+\mu_2{}^2(1-\gamma)] \\ & +(M^2-\tfrac{1}{4}u)x^2(1-\alpha^2)-\tfrac{1}{4}t(1-\gamma^2) \\ & +\tfrac{1}{2}xW(p_0(\nu)+p_0'(\nu))(1/\nu-\alpha) \\ & -\tfrac{1}{2}xW(p_0(\nu)-p_0'(\nu))\gamma(1-\alpha/\nu)\ . \end{aligned} \quad \text{(B2)}$$

It can be shown that the last three terms are never negative, so that the singularity can happen only if the third time is negative. And, in the region where x is large and $\alpha x \to 0$, the leading behavior of (B2) becomes

$$(M^2-\tfrac{1}{4}u)x^2$$

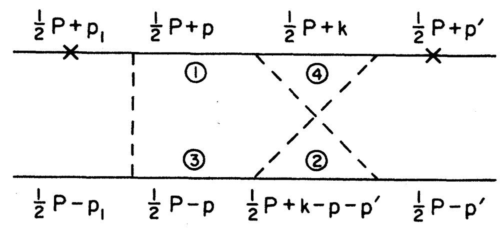

FIG. 6. The sixth order diagram which illustrates the origin and cancellation of the spurious singularities in the crossed box.

so (B2) is guaranteed to have a singularity whenever $u > 4M^2$. The location of this singularity is discussed in Sec. III.

To understand the origin of this singularity, and how it is canceled, examine the diagram in Fig. 6, where the momenta are labeled, and we limit discussion to $\nu=1$. Consider the integration in the complex p_0 plane, and look at the contributions from the first three propagators numbered in the diagram. We have

$$I=\sum\int dp_0\{[E_p{}^2-(\tfrac{1}{2}W+p_0)^2-i\epsilon][E_{k-p-p'}{}^2-(\tfrac{1}{2}W+k_0-p_0-p_0')^2-i\epsilon][E_p{}^2-(\tfrac{1}{2}W-p_0)^2-i\epsilon]\}^{-1}\ . \quad \text{(B3)}$$

For small $\vec{p}$, the positive energy poles from propagators 1 (first term in brackets) and 3 (third term in brackets) are on opposite sides of the p_0 axis and the dominant term comes from the positive energy pole in propagator 1, which lies in the lower half plane. This is expected; the fact that this pole dominates Fig. 6 (at least at low three momentum) is one of the reasons why three dimensional equations have a one body limit.[20] However, when p is very large, the poles move in such a way that this is no longer true. For large p, the *negative* energy pole from propagator 2 (second term in brackets), which also lies in the lower half plane, can overlap the positive energy pole from 1. They lie on top of each other when

$$E_p-\tfrac{1}{2}W=E_{k-p-p'}+\tfrac{1}{2}W+k_0-p_0'\ .$$

Substituting the value of p_0' corresponding to particle 1 on the mass shell, and taking $\vec{p}+\vec{p}\,'=0$ (to push the singularity to as small a $\vec{p}$ as possible) gives

$$2(E_P-W)=E_k-\tfrac{1}{2}W+k_0\ .$$

When the k_0 integral is performed, the smallest value of k_0 comes from the positive energy pole of propagator 4, which gives

$$2(E_p-W)=2E_k-W\ .$$

The smallest value of $\vec{p}$ at which this condition is satisfied occurs when $\vec{k}=0$, giving finally

$$E_p=\tfrac{1}{2}W+M\ ,$$

which gives the result (3.27) for $\nu=1$.

Therefore, because the *positive* energy pole of propagator 1 and the *negative* energy pole of propagator 2 overlap, neither pole by itself is free of spurious singularities, even though their sum is perfectly regular. Unfortunately, the positive energy pole of propagator 1 gives rise to the fourth order crossed box kernel, while the negative energy pole of propagator 2 contributes to the irreducible sixth order kernel. When both of these contributions are taken into account, one has the full result of Fig. 6 (for example), where there are no spurious singularities, showing that the spurious singularities in the fourth order kernel are canceled exactly by compensating singularities in high order kernels.

Since the singularities cancel, it is fully correct to take the principal value of each kernel. This would give a Hermitian kernel at every stage. Unfortunately, this procedure is difficult to implement numerically. It would be more desirable to find some other subtraction procedure so that no spurious singularities of any type occur in any order. A procedure for this has not been worked out.

The above analysis also shows that such singularities would not occur if negative energy poles were neglected. While such terms must be retained to maintain covariance, at least we know from this that these singularities should not influence results at small momentum, and are absent in the extended static limit.

APPENDIX C

In this appendix, the spectral forms for various static kernels will be presented.

The identities are of two possible forms

$$\int_{-1}^{1} d\gamma \int_0^\infty \frac{z^n dz}{\eta_0^{2m}(z)} = \int_{(\mu_1+\mu_2)^2}^\infty d\omega \frac{I_{n,m}(\omega)}{\omega - t} ,$$
$$\int_{-1}^{1} d\gamma [\mu_1^2(1+\gamma)+\mu_2^2(1-\gamma)] \int_0^\infty \frac{z^n dz}{\eta_0^{2m}(z)} = \int_{(\mu_1+\mu_2)^2}^\infty d\omega \frac{J_{n,m}(\omega)}{(\omega - t)} , \tag{C1}$$

where

$$\eta_0^2(z) = z^2 + \tfrac{1}{2}[\mu_1^2(1+\gamma)+\mu_2^2(1-\gamma)] - \tfrac{1}{4}t(1-\gamma^2) .$$

The spectral functions I and J for the cases encountered in this paper are given in Table I. They can be obtained by factoring $(1-\gamma^2)/4$ out of the η_0^2 term in the denominator, eliminating z in favor of

$$\omega = [4z^2 + 2(\mu_1^2(1+\gamma)+\mu_2^2(1-\gamma))](1-\gamma^2)^{-1} ,$$

interchanging ω and γ integrations, and doing the γ integration in the complex γ plane.

In addition to the integrals (C1), we encounter integrals from the box of the form

$$\int_{-1}^{1} d\gamma \int_0^\infty dz\, z^n \left\{ \int_{-\infty}^{\frac{-z}{1+\nu}} + \int_{\frac{z}{1-\nu}}^{\infty} \right\} \frac{dy}{\eta_0^{2m}(y)} .$$

These can always be reduced to the form (C1) by integrating over z by parts and scaling the new integral over z by

$$\frac{z}{1+\nu} \to z' \text{ or } \frac{z}{1-\nu} \to z' .$$

TABLE I. The spectral functions which complete the identities of Eq. (C1).

n	m	$I_{n,m}(\omega)$
0	1	$\pi\omega^{-1/2}$
1	1	$\frac{1}{\omega}\Delta^{1/2}(\omega,\mu_1^2,\mu_2^2)$
2	1	$(\pi/8)\omega^{-3/2}\Delta(\omega,\mu_1^2,\mu_2^2)$
0	2	$4\pi\left[\frac{\mu_1+\mu_2}{\mu_1\mu_2}\right]\delta[\omega-(\mu_1+\mu_2)^2]$
1	2	$2\Delta^{-1/2}(\omega,\mu_1^2,\mu_2^2)$
2	2	$(\pi/2)\omega^{-1/2}$
4	2	$(3\pi/16)\omega^{-3/2}\Delta(\omega,\mu_1^2,\mu_2^2)$
n	m	$J_{n,m}(\omega)$
2	2	$(\pi/2)\omega^{-3/2}[\omega(\mu_1^2+\mu_2^2)-(\mu_1^2-\mu_2^2)^2]$
2	3	$\frac{\pi}{2}(\mu_1+\mu_2)\delta[\omega-(\mu_1+\mu_2)^2]$

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