PHYSICAL REVIEW C VOLUME 26, NUMBER 5 NOVEMBER 1982

Connected Chandler-Gibson equations and few-body collisions

W. N. Polyzou*
Los Alamos National Laboratory, Los Alamos, New Mexico 87545

A. G. Gibson
Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131

C. Chandler
Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131
(Received 30 November 1981)

A new connected kernel version of the Chandler-Gibson equations is derived. Both the new and the original versions of the Chandler-Gibson equations are then modified to incorporate identical particle symmetries. Both types of equations are written down in detail for two different six-body models of alpha-deuteron scattering. Various features of these equations are then compared.

[NUCLEAR REACTIONS Connected kernel equations for distinguishable and identical particles; equations for alpha-deuteron scattering.]

I. INTRODUCTION

It is typical of modern theories of nuclear reactions which are of the integral equation type (cf. Ref. 1 and references cited therein) that some iterate of the kernel of the exact equation be compact in an appropriate Banach space. For systems of very few nucleons these equations can thus be solved directly with Fredholm methods and standard numerical technology (cf. Ref. 2 and references cited therein). For systems with a larger number of particles the complexity of the equations requires some type of approximation.[3–5]

The property that an iterate of the kernel is compact is not shared by the dynamical equations recently proposed by Chandler and Gibson.[4–7] This difficulty is circumvented by constructing a sequence of approximate transition operators which converge to the exact transition operator. Each approximate transition operator is itself the solution of an "approximate" dynamical equation that does have a compact kernel. These approximate equations are obtained from the exact dynamical equation by approximating the asymptotic channel subspaces (and possibly the asymptotic Hamiltonians) in physically meaningful ways. Because these approximate equations have a compact kernel, the same standard mathematical technology can be applied to them as is applied to more conventional theories. This method of approximation is potentially very powerful, but much experience must be accumulated before it is known how to construct good approximations in practice.

The purpose of this paper is to derive connected (hence possibly compact) kernel equations within the framework of the Chandler-Gibson (CG) theory and to compare these equations with the CG equations. Such connected kernel equations would, presumably, be less demanding of the particular choice of approximations of the channel subspaces.

We review the important features of the CG theory in Secs. II and III. We then use a Weinberg-Van Winter-type coupling scheme[8] in Secs. IV and V to derive connected kernel equations. In Sec. VI we use the methods of Bencze and Redish[9] to modify these equations to incorporate identical particle symmetries. In Sec. VII we use a model of alpha-deuteron scattering as a concrete example to exhibit our connected kernel equations and the CG equations. Finally, in Sec. VIII we use the example of Sec. VII to compare the various forms of the equations.

II. *N*-BODY NOTATION

Our notation follows that of Refs. 5 and 6 except that lower case Latin letters $a, b, c, \ldots$ ($A, B, C, \ldots$

 ©1982 The American Physical Society

in Refs. 5 and 6) will denote partitions (clusterings) of the N particles into n_a, n_b, n_c, . . . clusters (fragments). The free partition will be denoted by O ($n_O = N$), and the partition containing all N particles in one cluster will be denoted by N. The symbol $\mathscr{P}'$ denotes the set of all partitions of n particles while $\mathscr{P}$ denotes the set

$$\mathscr{P} \equiv \{a \in \mathscr{P}' : n_a > 1\} \ . \tag{2.1}$$

(Note that these meanings of $\mathscr{P}$ and $\mathscr{P}'$ are the reverse of those in Refs. 10 and 11.)

Partitions are used in N-body theories to index the connectivity of N-body operators. Roughly speaking, an operator has connectivity a if and only if it commutes with the unitary group of translations of the clusters of a and vanishes as any particles in the same cluster of a are asymptotically separated. An operator is said to be a connected if it has connectivity a and connected if it has connectivity N ($n_N = 1$). A precise definition for bounded operators is given in Ref. 10.

Most N-body operators A can be expressed in terms of a cluster expansion of the form

$$A = \sum_{a \in \mathscr{P}'} [A]_a \equiv \sum_a{}' [A]_a \ , \tag{2.2}$$

where $[A]_a$ is the a-connected part of A. In this paper unrestricted sums will be considered as sums over $\mathscr{P}$.

It is useful to single out the part of A with the connections between the clusters of a turned off. This is most effectively expressed using the lattice structure on $\mathscr{P}'$.[12] We say $a \subseteq b$ if any two particles in the same cluster of a are in the same cluster of b. It is easy to show that $\subseteq$ is a partial ordering on $\mathscr{P}'$ and introduces a natural lattice structure on $\mathscr{P}'$. The operator $(A)_a$ (A_a in Refs. 10 and 11) which is the part of A with the connections between the clusters of a turned off, can be expressed in this notation as

$$(A)_a = \sum_{\substack{b \in \mathscr{P}' \\ b \subseteq a}} [A]_b \equiv \sum_{b(\subseteq a)}{}' [A]_b \ . \tag{2.3}$$

It is also useful to define the part of the operator A which has the residual connections between the clusters of a. We denote this operator by the symbol $(A)^a$ (A^a in Refs. 10 and 11). It is defined in terms of $(A)_a$ by

$$(A)^a \equiv A - (A)_a = \sum_{b(\subsetneq a)}{}' [A]_b \ . \tag{2.4}$$

We see that it contains the terms in the cluster expansion of A that connects different clusters of a.

The total Hamiltonian H_N of our N-body system acts on the Hilbert space $\mathscr{H}_N$ and has the form

$$H_N = H_O + V_N \ , \tag{2.5}$$

where H_O is the N-body kinetic energy operator, and V_N is the sum of all interparticle interactions. For any partition a we may express the total Hamiltonian as

$$H_N = (H_N)_a + (H_N)^a \equiv H_a + V_N^a \ , \tag{2.6}$$

where H_a is the part of H_N with the interactions between the clusters of a turned off and V_N^a ($\bar{V}_a$ in Refs. 4–6) is the sum of all interactions between particles in different clusters of a.

The dynamics of the N-body system is contained in the resolvent of H_N,

$$R_N(z) \equiv (z - H_N)^{-1} \ , \tag{2.7}$$

or in the dynamically equivalent (prior) transition operators

$$U_{ba}(z) \equiv V_N^a + V_N^b R_N(z) V_N^a \ , \tag{2.8}$$

where $Im z \neq 0$. These can be related to proper subsystem transition operators through the second resolvent relations

$$R_N(z) = R_a(z) + R_a(z) V_N^a R_N(z) \ , \tag{2.9}$$

with

$$R_a(z) \equiv (z - H_a)^{-1} \ . \tag{2.10}$$

For each a we let P_a denote the orthogonal projection onto the invariant subspace $\mathscr{H}_a$ of $\mathscr{H}_N$ corresponding to free motion of the n_a bound clusters of a. We note that $[P_a]_a = P_a$. These operators will carry the bound state input in the CG equations. We also define the projected transition operators

$$\begin{aligned} T_{ba}(z) &\equiv P_b U_{ba}(z) P_a \\ &= P_b R_b{}^{-1}(z) R_N(z) V_N^a P_a \ . \end{aligned} \tag{2.11}$$

It is only these projected operators that are physically relevant, and it is these that appear as the unknowns in the CG equations.

III. THE CHANDLER-GIBSON EQUATIONS

Define the operator JJ^* by

$$JJ^* \equiv \sum_a P_a \ . \tag{3.1}$$

Here J is the injection operator of the two-Hilbert space formulation used in Refs. 4–7, and JJ^* has a bounded inverse (Ref. 6, *Theorem 7*). The operators $T_{ba}(z)$ are the unique solution of the exact T equa-

tions [Ref. 6, Eq. (3.68)]

$$T_{ba}(z)=P_b V_N^a P_a+P_b V_N^b (JJ^*)^{-1} \times \sum_c R_c(z)P_c T_{ca}(z) \ . \tag{3.2}$$

Equation (3.2) is derived in Ref. 6 using two-Hilbert space methods. Since these methods are not used in this paper, an alternative derivation is given in Appendix B.

One nice feature of these equations is that the operators $R_c(z)P_c$ can be constructed from the masses, binding energies, and bound state wave functions of the clusters of c as follows:

$$R_c(z)P_c=\int \frac{|\phi(c)\vec{q}^c\rangle d\vec{q}^c\langle\phi(c)\vec{q}^c|}{z-T_c(\vec{q}^c)+\epsilon(c)} \ , \tag{3.3}$$

where $\vec{q}^c$ is a $3(n_c-1)$ dimensional vector describing the relative momentum of the clusters of a, $|\phi(c)\rangle$ is the direct product of the bound cluster wave functions, $T_c(\vec{q}^{\,c})$ is the kinetic energy of relative motion of the clusters, and $\epsilon(c)$ is the total internal energy of the clusters. This leads to an important practical advantage. To reduce these formal operator equations to integral equations one inserts a complete set of states between P_c and T_{ca}. In Eq. (3.2) this complete set can be truncated to the physically relevant states in the range of T_{ca}, while other equations require the full set of intermediate states. These other equations, therefore, include additional amplitudes that do not have a simple physical interpretation.

The approximations of Refs. 4 and 5 involve replacements of the form

$$P_a \rightarrow \Pi_a \ , \tag{3.4}$$

where Π_a is an approximation to P_a satisfying *Assumption* (II) of Refs. 4 and 5. In particular, Π_a is an orthogonal projection operator on $\mathscr{H}_N$ satisfying (i) $\Pi_a P_a=\Pi_a$, (ii) $[\Pi_a,H_a]=0$, and (iii) the operator

$$J\Pi J^* \equiv \sum_a \Pi_a \tag{3.5}$$

has a bounded inverse on $\mathscr{H}_\pi \equiv \overline{\mathscr{R}}(J\Pi J^*)$, the closure of the range of $(J\Pi J^*)$.

In practical cases $\Pi_a=0$ for several partitions a, and the nonzero Π_a operators should include a large enough segment of the space to include all important real and virtual cluster states. Assumption (i) is equivalent to requiring the range of Π_a to be a subspace of $\mathscr{H}_a$. Assumption (ii) requires that the energy variable is left as continuous in the approximation. Assumption (iii) is known to be satisfied if there is a dominant projector Π_d such that $\Pi_a\Pi_d=\Pi_a$ for all partitions a or if $\Pi_a\Pi_b$ is a compact operator for all $a\neq b$ (cf. *Theorem 4.4* of Ref. 5).

If P_π is the orthogonal projection of $\mathscr{H}_N$ onto the subspace $\mathscr{H}_\pi$, then the formula

$$P_\pi=(J\Pi J^*)^{-1}J\Pi J^* \tag{3.6}$$

is valid (cf. *Proposition 4.3* of Ref. 5). The approximate total Hamiltonian H_π of the approximate scattering system is given by

$$H_\pi \equiv P_\pi H_N P_\pi \ , \tag{3.7}$$

and $R_\pi(z)\equiv(z-H_\pi)^{-1}$. The approximate transition operators

$$T_{ba}^\pi=T_{ba}^\pi(z) \equiv \Pi_b R_b{}^{-1}(z)R_\pi(z)P_\pi V_N^a \Pi_a \tag{3.8}$$

are the unique solution of the approximate T equations [Ref. 5, Eq. (6.4)]

$$T_{ba}^\pi=I_{ba}^{(1)}+\sum_c K_{bc}^{(1)}R_c\Pi_c T_{ca}^\pi \ , \tag{3.9}$$

where

$$I_{ba}^{(1)}\equiv \Pi_b V_N^a \Pi_a \tag{3.10}$$

and

$$K_{bc}^{(1)}\equiv \Pi_b V_N^b (J\Pi J^*)^{-1}\Pi_c \ . \tag{3.11}$$

The operator $(J\Pi J^*)^{-1}$ appears in Eq. (3.11) because the asymptotic channel spaces are not orthogonal. Its presence is a major practical complication. To eliminate it we define [cf. Ref. 5, Eq. (6.6)]

$$M_{ba}^\pi=M_{ba}^\pi(z) \equiv \Pi_b R_b{}^{-1}(z)(J\Pi J^*)^{-1}R_\pi(z)P_\pi V_N^a\Pi_a \ . \tag{3.12}$$

Then $M_{ba}^\pi(z)$ is related to $T_{ba}^\pi(z)$ by

$$T_{ba}^\pi(z)=\Pi_b R_b{}^{-1}(z)\sum_c R_c(z)\Pi_c M_{ca}^\pi(z) \ , \tag{3.13}$$

and satisfies the approximate M equations [Ref. 5, Eq. (8.16)]

$$M_{ba}^\pi=I_{ba}^{(3)}+\sum_c K_{bc}^{(3)}R_c\Pi_c M_{ca}^\pi \ , \tag{3.14}$$

where

$$I_{ba}^{(3)}=I_{ba}^{(1)} \ , \tag{3.15}$$

and

$$K_{bc}^{(3)}=K_{bc}^{(3)}(z) \equiv \Pi_b[V_N^c-\bar{\delta}_{bc}R_c{}^{-1}(z)]\Pi_c \ . \tag{3.16}$$

Here $\bar{\delta}_{bc}$ is equal to 1 if $b \neq c$ and 0 if $b = c$.

For a certain class of Π_a's and V_N^a's (cf. Sec. 14 of Ref. 4 for an example) the kernels $K_{bc}^{(1)}$ and $K_{bc}^{(3)}$ are connected and (at least for $Imz \neq 0$) compact. There are other interesting choices where they are not compact. These choices include Π_a's obtained from P_a's by eliminating channels. They are typically characterized by the property $\Pi_a = [\Pi_a]_a$. Even in this case the kernel is connected if all nonvanishing Π_a's have $n_a = 2$. If, however, the kernel of Eq. (3.9) or (3.14) is not compact, then there are two possible approaches: (1) Approximate the Π_a's by $\tilde{\Pi}_a$'s in such a way that the kernel becomes compact; or (2) modify the equations in such a way that the new equations have connected (and compact) kernels.

The first alternative is equivalent to the approach recently proposed by Chandler and Gibson.[4,5] The second approach is that developed in Secs. IV–VI of this paper.

IV. CONNECTED KERNEL T EQUATIONS

In this section we derive a connected kernel version of the approximate T equations [Eq. (3.9)]. In the particular case when $\Pi_a = P_a$ for all partitions a this gives the connected kernel version of the exact T equations (3.2) previously announced by two of us.[13]

A distinctive feature of our method for connecting these equations is that modifications of Eq. (3.9) are minimal. In particular, for $n_b = 2$ the new connected kernel is identical to $K_{ba}^{(1)}$.

Let

$$\Delta_{cb} \equiv \begin{cases} 1 \text{ if } c \supseteq b \\ 0 \text{ if } c \not\supseteq b \end{cases} . \tag{4.1}$$

Then[11]

$$\Delta_{cb}^{-1} = \begin{cases} (-1)^{n_c} \prod_{i=1}^{n_c} (-1)^{n_{b_i}} (n_{b_i} - 1)! & \text{if } c \supseteq b \\ 0 & \text{if } c \not\supseteq b \end{cases} , \tag{4.2}$$

where n_{b_i} is the number of clusters of b in the ith cluster of c. Also let

$$C_a \equiv -\bar{\delta}_{Na} \Delta_{Na}^{-1} . \tag{4.3}$$

Definition (4.1) implies that

$$\Delta_{cb} \Delta_{ca} = \Delta_{c, b \cup a} , \tag{4.4}$$

where $b \cup a$ is the partition with the most clusters satisfying both $b \cup a \supseteq a$ and $b \cup a \supseteq b$.

Let $P_{c,\pi}$ denote the orthogonal projection of $\mathscr{H}_N$ onto the closure of the range of $\sum_b \Delta_{cb} \Pi_b$ $(c \neq N)$. Then $P_{c,\pi} P_\pi = P_{c,\pi}$. Let

$$H_{c,\pi} \equiv P_{c,\pi} H_c P_{c,\pi} , \tag{4.5}$$

$$R_{c,\pi} = R_{c,\pi}(z) \equiv (z - H_{c,\pi})^{-1} , \tag{4.6}$$

and

$$V_\pi^{c,\pi} \equiv H_\pi - H_{c,\pi} . \tag{4.7}$$

The combinatorial results used in this paper are given in Appendix A.

To obtain connected kernel equations for T_{ba}^π we begin with the resolvent equations

$$P_{c,\pi} R_\pi = P_{c,\pi} R_{c,\pi} + P_{c,\pi} R_{c,\pi} V_\pi^{c,\pi} R_\pi . \tag{4.8}$$

We multiply Eq. (4.8) on the left by $C_c \Delta_{cb} R_b^{-1} \Pi_b$, on the right by $V_N^a \Pi_a$, and sum the result over c $(c \neq N)$. Using $\Pi_b P_{c,\pi} = \Pi_b$ for $b \subseteq c$, *Theorem A2*, and Eq. (3.8), we obtain

$$T_{ba}^\pi = \sum_{c(\supseteq b)} C_c R_b^{-1} \Pi_b R_{c,\pi} V_N^a \Pi_a + \sum_{c(\supseteq b)} C_c R_b^{-1} \Pi_b R_{c,\pi} V_\pi^{c,\pi} R_\pi V_N^a \Pi_a . \tag{4.9}$$

Since $V_\pi^{c,\pi} = V_\pi^{c,\pi} P_\pi$, we may substitute

$$P_\pi = (J \Pi J^*)^{-1} \sum_e R_e R_e^{-1} \Pi_e$$

after $V_\pi^{c,\pi}$ in Eq. (4.9) and use Eq. (3.8) to obtain

$$T_{ba}^\pi = I_{ba}^{(2)} + \sum_c K_{bc}^{(2)} R_c \Pi_c T_{ca}^\pi . \tag{4.10}$$

In Eq. (4.10)

$$I_{ba}^{(2)} \equiv \sum_{c(\supseteq b)} C_c R_b^{-1} \Pi_b R_{c,\pi} V_N^a \Pi_a \tag{4.11}$$

and

$$K_{bc}^{(2)} \equiv \sum_{d(\supseteq b)} C_d R_b^{-1} \Pi_b R_{d,\pi} V_\pi^{d,\pi} (J \Pi J^*)^{-1} \Pi_c . \tag{4.12}$$

The projection operator Π_b has the property

$$\Pi_b = \Pi_b P_b = \Pi_b [P_b]_b .$$

The kernel $K_{bc}^{(2)}$ is therefore connected by *Theorem A5*. For $n_b = 2$ the only c satisfying $c \supseteq b$, $c \neq N$, is $c = b$. In this case Eq. (4.10) reduces to Eq. (3.9).

Equations (4.11) and (4.12) give a very concise representation of the driving term and kernel. Un-

fortunately this representation involves delicate cancellations associated with the combinatorial coefficients C_a. If V_N is a sum of only two-body forces, then we can construct more useful representations for the kernel and driving term of Eq. (4.10), and exhibit a hierarchy of equations to generate these operators. Let

$$L_b^d = L_b^d(z,\Pi) \equiv \Delta_{db}[R_b{}^{-1}\Pi_b R_{d,\pi}]_d R_O{}^{-1} , \quad (4.13)$$

and

$$V_{c,\pi}^{d,\pi} \equiv H_{c,\pi} - H_{d,\pi} . \quad (4.14)$$

Also let

$$Z_{c,\pi} \equiv \sum_e \Delta_{ce}\Pi_e , \quad (4.15)$$

and suppose that $Z_{c,\pi}$ has closed range. It is proved in Appendix C that

$$L_b^c = B_b^c + \sum_{e(\subseteq c)} K_b^c R_e L_e^c , \quad (4.16)$$

where

$$B_b^c \equiv -\sum_{d,e,f,g} \bar{\delta}_{cd}\bar{\delta}_{cf}\Delta_{cd}{}^{-1}\Delta_{db}\Delta_{ce} \times \Delta_{cf}\Delta_{dg} L_b^g R_O V_{c,\pi}^{d,\pi} R_e L_e^f , \quad (4.17)$$

and

$$K_b^c \equiv -\sum_{d,g} \bar{\delta}_{cd}\Delta_{cd}{}^{-1}\Delta_{db}\Delta_{dg} L_b^g R_O V_{c,\pi}^{d,\pi} Z_{c,\pi}{}^{-1} . \quad (4.18)$$

The sum over d in Eqs. (4.17) and (4.18) may be evaluated in essentially the same way as the sum over c was evaluated in Eqs. (C6)–(C13). We omit the details.

Equation (4.16) is a linear c-connected kernel equation for L_b^c that uses the L_b^d for $d \subseteq c$, $d \neq c$, as input. This gives a recursive scheme for evaluating the L_b^c operators starting from $L_d^d = \Pi_d R_O{}^{-1}$ and L_O^c for $n_c = N-1$ which, by Eq. (4.16), satisfies

$$L_O^c = \Pi_O V_{c,\pi}^{O,\pi} + \Pi_O V_{c,\pi}^{O,\pi} Z_{c,\pi}{}^{-1} R_O L_O^c , \quad (4.19)$$

where

$$\Pi_O V_{c,\pi}^{O,\pi} = \Pi_O (H_c P_{c,\pi} - H_O \Pi_O) . \quad (4.20)$$

Equations (4.10) and (4.16) simplify considerably if V_N is a sum of only two-body forces and $P_{c,\pi} = P_\pi$ for all partitions c. This last assumption is satisfied if there is a "dominant projector" d such that $\Pi_a \Pi_d = \Pi_a$ for all partitions a.

For example, if $\Pi_a = P_a$ for all partitions a then Π_O is a dominant projector. In this exact theory case $P_{c,\pi} = P_\pi = I_N$ for all c and the operator ${}^2K_{ba}^{(2)}$ defined in Eq. (C8) of Appendix C is zero. Equation (4.10) then becomes

$$T_{ba} = \sum_d L_b^d R_O V_N^a P_a + \sum_{d,n_d=2,\ e} L_b^d R_O V_N^d (JJ^*)^{-1} R_e T_{ea} . \quad (4.21)$$

The operator $Z_{c,\pi}$ defined in Eq. (4.15) may be replaced by

$$Y_{c,\pi} \equiv \sum_e \bar{\delta}_{ce}\Delta_{ce}\Pi_e \quad (4.22)$$

in this case and $I_N = Y_{c,\pi}{}^{-1} Y_{c,\pi}$ may be inserted into Eq. (C19) in place of $P_{c,\pi}$. The analog of Eq. (4.16) is then

$$L_b^c = D_b^c + \sum_{e(\subseteq c), e\neq c} F_b^c R_e L_e^c , \quad (4.23)$$

where

$$D_b^c \equiv \sum_{\substack{e,f,g \\ n_g = n_c + 1}} \bar{\delta}_{ce}\bar{\delta}_{cf}\Delta_{c,e\cup f\cup g} \times L_b^g R_O V_c^g Y_{c,\pi}{}^{-1} R_e L_e^f , \quad (4.24)$$

and

$$F_b^c \equiv \sum_{g(\subseteq c), n_g = n_c + 1} L_b^g R_O V_c^g Y_{c,\pi}{}^{-1} . \quad (4.25)$$

When $n_c = N-1$ and $b = 0$, Eq. (4.23) becomes

$$L_O^c = V_c + V_c R_O L_O^c , \quad (4.26)$$

which is the two-body Lippmann-Schwinger equations embedded in the N-body Hilbert space. It is Eqs. (4.21)–(4.25) which were previously announced by two of us in Ref. 13. Using the methods of Ref. 14 we have shown that these equations possibly admit spurious solutions.

Another example which has a dominant projector will be given in Sec. VII [cf. Eqs. (7.13) and (7.14)].

V. CONNECTED KERNEL M EQUATIONS

In this section we derive a connected kernel version of the approximate M equations [Eqs. (3.14)–(3.16)]. The first step is to express the kernels $K_{bc}^{(3)}$ of Eq. (3.16) as

$$K_{bc}^{(3)} = (K_{bc}^{(3)})_d + (K_{bc}^{(3)})^d . \quad (5.1)$$

In the notation of Secs. II and IV we find that

$$(K_{bc}^{(3)})_d=\Delta_{db}\Pi_b[V_d^c-\bar{\delta}_{bc}R_c{}^{-1}]\Pi_c\Delta_{dc}\ . \tag{5.2}$$

Using (5.1) rewrite Eq. (3.14) as

$$\sum_c(\delta_{bc}-(K_{bc}^{(3)})_dR_c\Pi_c)M_{ca}^{\pi} =I_{ba}^{(3)}+\sum_c(K_{bc}^{(3)})^dR_c\Pi_cM_{ca}^{\pi}\ . \tag{5.3}$$

A connected kernel equation for M_{ba}^{π} is obtained if we formally invert the left side of (5.3), multiply by C_d, and sum on d. The result is

$$M_{ba}^{\pi}=I_{ba}^{(4)}+\sum_c K_{bc}^{(4)}R_c\Pi_cM_{ca}^{\pi}\ , \tag{5.4}$$

where

$$I_{ba}^{(4)}\equiv\sum_{c,d}C_d[\delta_{bc}+(U_{bc})_d]I_{ca}^{(3)}\ , \tag{5.5}$$

and

$$K_{bc}^{(4)}\equiv\sum_{d,e}C_d[\delta_{be}+(U_{be})_d](K_{ec}^{(3)})^d\ . \tag{5.6}$$

The operators $(U_{bc})_d=(U_{bc})_d(z)$ satisfy

$$(U_{bc})_d=(K_{bc}^{(3)})_dR_c+\sum_e(K_{be}^{(3)})_dR_e(U_{ea})_d\ . \tag{5.7}$$

The theorems of the Appendix guarantee that (5.4) has a connected kernel, although (5.7) does not. This is because $(U_{ba})_d$ has no fully connected terms in its cluster expansion. For this operator it is sufficient to construct a d-connected kernel equation because the nonessential degrees of freedom associated with the relative motion of the clusters harmlessly factor out. What remains is a connected kernel equation on a smaller space with the nonessential degrees of freedom appearing as parameters. Solutions for different values of the parameters are related by known unitary transformations.

To construct a d-connected kernel equation we utilize the following relation between the various operators $(U_{ba})_d$

$$(U_{ba})_d=\sum_c[\delta_{bc}+(U_{bc})_e](K_{ca}^{(3)})_dR_a+\sum_{c,f}[\delta_{bc}+(U_{bc})_e](K_{cf}^{(3)})_d^eR_f(U_{fa})_d\ . \tag{5.8}$$

The derivation of this equation follows closely the derivation of (5.4) if we note that for $e\subseteq d$

$$(K_{ca}^{(3)})_d=(K_{ca}^{(3)})_e+(K_{ca}^{(3)})_d^e\ . \tag{5.9}$$

To obtain the desired equation multiply Eq. (5.8) by $-\bar{\delta}_{de}\Delta_{de}{}^{-1}$ and sum over e. This gives

$$(U_{ba})_d=-\sum_{c,e}\bar{\delta}_{de}\Delta_{de}{}^{-1}[\delta_{bc}+(U_{bc})_e](K_{ca}^3)_dR_a-\sum_{c,e,f}\bar{\delta}_{de}\Delta_{de}{}^{-1}[\delta_{bc}+(U_{bc})_e](K_{cf}^3)_d^eR_f(U_{fa})_d\ , \tag{5.10}$$

where we have used *Theorem A3*. The kernel of this equation is d connected by *Theorem A6*. The restriction $\bar{\delta}_{de}\Delta_{de}{}^{-1}$ requires $e\subseteq d$ and $e\neq d$, so the $(U_{bc})_e$ appearing in the kernel and driving term of Eq. (5.10) correspond to e's that are strict refinements of d. Thus this equation can be used to recursively construct $(U_{ba})_d$ on decreasing numbers of clusters of d beginning with $n_d=N-1$ for which $(U_{ba})_d\equiv 0$.

Equation (5.4) and the hierarchy (5.10) are the fundamental connected kernel equations of this section. Since these equations are derived from the approximate Chandler-Gibson M equations, the solutions should have the same physical content. As with the connected T equations, spurious solutions of these equations cannot be ruled out.[14]

VI. EQUATIONS FOR IDENTICAL PARTICLES

When some particles are identical, their interchange is a symmetry of the system that should be incorporated into the basic dynamical equations. This is done in this section for the equations developed in Secs. III–V, at least to the extent required by the example of the next section. We follow the strategy of Bencze and Redish.[9]

The permutations p of identical particles form a finite group $\mathcal{S}_N$. For simplicity of notation the unitary operators associated with the permutations are also denoted by p. Operators

$$\hat{p}=f_pp\ , \tag{6.1}$$

where $f_p=-1$ if p involves an odd number of fer-

mion permutations and $+1$ otherwise, are defined. The Young symmetrizer of $\mathscr{S}_N$ is given by

$$Y_N \equiv |\mathscr{S}_N|^{-1} \sum_{p\in\mathscr{S}_N} \hat{p} , \tag{6.2}$$

where $|\mathscr{S}_N|$ is the order of $\mathscr{S}_N$.

For any partition a there is a subgroup $\mathscr{S}_a$ of permutations that leave the partition invariant. The corresponding Young symmetrizer is given by

$$Y_a \equiv |\mathscr{S}_a|^{-1} \sum_{p\in\mathscr{S}_a} \hat{p} , \tag{6.3}$$

where $|\mathscr{S}_a|$ is the order of $\mathscr{S}_a$.

Since interchange of identical particles, even if the partition is changed, does not change any physical properties of the system, quantities labeled by a and a', with $a'=pa$ for some $p\in\mathscr{S}_N$, are physically equivalent. It is therefore useful to divide the set $\mathscr{P}$ by partitions into equivalence classes

$$[a]\equiv\{a'\in\mathscr{P} \mid a'=pa \text{ for some } p\in\mathscr{S}_N\} . \tag{6.4}$$

The number N_a of elements of $[a]$ is given by

$$N_a = |\mathscr{S}_N||\mathscr{S}_a|^{-1} . \tag{6.5}$$

These equivalence classes are to be treated as the physical entities. To facilitate this a canonical member a^o is designated for each equivalence class $[a]$.

The various operators appearing in the dynamical equations of the previous sections are labeled by partitions. For example, the projection operators Π_a are labeled by one partition index. We assume that they are chosen to have the property

$$\hat{p}\Pi_a = \Pi_{pa}\hat{p} , \tag{6.6}$$

where p is any member of $\mathscr{S}_N$. Another example is provided by the transition operators T^π_{ba} and M^π_{ba}, which are labeled by two partition indices. They also have the property

$$\hat{p}T^\pi_{ba} = T^\pi_{pb,pa}\hat{p} , \tag{6.7}$$

$$\hat{p}M^\pi_{ba} = M^\pi_{pb,pa}\hat{p} , \tag{6.8}$$

where p is any member of $\mathscr{S}_N$. Operators with the properties of Eqs. (6.6)–(6.8) are called "label transforming" by Bencze and Redish.[9]

Because the operators T^π_{ba} and M^π_{ba} are label transforming, their correctly symmetrized forms are[9]

$$\hat{T}^\pi_{b^oa^o} = (N_{b^o}N_{a^o})^{-1/2} \sum_{\substack{b\in[b^o]\\ a\in[a^o]}} Y_{b^o}\hat{p}_{b^ob}T^\pi_{ba}\hat{p}_{aa^o}Y_{a^o} \tag{6.9}$$

and

$$\hat{M}^\pi_{b^oa^o} = (N_{b^o}N_{a^o})^{-1/2} \sum_{\substack{b\in[b^o]\\ a\in[a^o]}} Y_{b^o}\hat{p}_{b^ob}M^\pi_{ba}\hat{p}_{aa^o}Y_{a^o} . \tag{6.10}$$

In Eqs. (6.9) and (6.10) the operators $\hat{p}_{b^ob}$ and $\hat{p}_{aa^o}$ denote the operators $\hat{p}$ associated with the permutations that change b into b^o and a^o into a, respectively.

The dynamical equations that the operators T^π_{ba} and M^π_{ba} satisfy are of the general form studied by Bencze and Redish.[9] Both the inhomogeneous terms and the kernel operators are label transforming. It follows that the symmetrized forms of the basic dynamical equations have the following form:

$$\hat{T}^\pi_{b^oa^o} = \hat{I}^{(j)}_{b^oa^o} + \sum_{c^o} \hat{K}^{(j)}_{b^oc^o}R_{c^o}\hat{T}^\pi_{c^oa^o} \tag{6.11}$$

for $j=1,2$; and

$$\hat{M}^\pi_{b^oa^o} = \hat{I}^{(j)}_{b^oa^o} + \sum_{c^o} \hat{K}^{(j)}_{b^oc^o}R_{c^o}\hat{M}^\pi_{c^oa^o} \tag{6.12}$$

for $j=3,4$. The operators $\hat{I}^{(j)}_{b^oa^o}$ are given by[9]

$$\hat{I}^{(j)}_{b^oa^o} \equiv (N_{b^o}/N_{a^o})^{1/2} \sum_{a\in[a^o]} I^{(j)}_{b^oa}\hat{p}_{aa^o}Y_{a^o} , \tag{6.13}$$

where the $I^{(j)}_{b^oa}$ are defined in Eqs. (3.10), (3.15), (4.11), and (5.5). The operators $\hat{K}^{(j)}_{b^oa^o}$ are given by[9]

$$\hat{K}^{(j)}_{b^oa^o} \equiv (N_{b^o}/N_{a_o})^{1/2} \sum_{a\in[a^o]} K^{(j)}_{b^oa}\hat{p}_{aa^o}Y_{a^o} , \tag{6.14}$$

where $K^{(j)}_{b^oa}$ are defined in Eqs. (3.11), (3.16), (4.12), and (5.6).

The machinery developed by Bencze and Redish[9] can be used to obtain simpler expressions for $\hat{I}^{(1)}_{b^oa^o}$, $\hat{I}^{(3)}_{b^0a^o}$, $\hat{K}^{(1)}_{b^oa^o}$, and $\hat{K}^{(3)}_{b^oa^o}$ than those of Eqs. (6.13) and (6.14). The appropriate formulas are the following:

$$\hat{I}^{(1)}_{b^oa^o} = \hat{I}^{(3)}_{b^oa^o} = (N_{b^o}N_{a^o})^{1/2}\Pi_{b^o}Y_N V^{a^o}_N \Pi_{a^o} , \tag{6.15}$$

$$\hat{K}^{(1)}_{b^oa^o} = (N_{b^o}N_{a^o})^{1/2}\Pi_{b^o}V^{b^o}_N(J\Pi J^*)^{-1}Y_N\Pi_{a^o} , \tag{6.16}$$

and

$$\hat{K}^{(3)}_{b^oa^o} = \Pi_{b^o}(\delta_{b^oa^o}Y_{a^o} - (N_{b^o}N_{a^o})^{1/2}Y_N)\Pi_{a^o}R_{a^o}{}^{-1} + \hat{I}^{(1)}_{b^oa^o} . \tag{6.17}$$

In Eq. (6.16) the operator $J\Pi J^*$ can be replaced by

$$Y_N J\Pi J^* Y_N = Y_N \sum_{c^o} N_{c^o}\Pi_{c^o}Y_N . \tag{6.18}$$

Similarly, the operators $\hat{I}^{(2)}_{b^oa^o}$, $\hat{I}^{(4)}_{b^oa^o}$, and $\hat{K}^{(2)}_{b^oa^o}$ associated with the connected kernel equations can be simplified. The appropriate formulas are the following:

$$\hat{I}^{(2)}_{b^o a^o} = (N_{b^o} N_{a^o})^{1/2} \sum_{c(\supseteq \underset{d}{b^o})} C_c \Delta_{cd} L^d_{b^o} R_O Y_N V^{a^o}_N \Pi_{a^o} , \tag{6.19}$$

$$\hat{I}^{(4)}_{b^o a^o} = (N_{b^o} N_{a^o})^{1/2} \sum_{c,d} C_d [\delta_{b^o c} + (U_{b^o c})_d] \Pi_c Y_N V^{a^o}_N \Pi_{a^o} , \tag{6.20}$$

and

$$\hat{K}^{(2)}_{b^o a^o} = (N_{b^o} N_{a^o})^{1/2} \sum_{c(\supseteq \underset{d}{b^o})} C_c \Delta_{cd} L^d_{b^o} R_O V^{c,\pi}_{\pi} (J \Pi J^*)^{-1} Y_N \Pi_{a^o} . \tag{6.21}$$

The operator $\hat{K}^{(4)}_{b^o a^o}$ does not seem to have any generally useful representation that is simpler than the basic definition (6.14).

Equations (6.11) and (6.12) involve only correctly symmetrized quantities and treat equivalence classes (via canonical partitions) as single entities. It would therefore appear that they represent a complete solution to the problem of incorporating the symmetry under interchange of identical particles. This is the case for the unconnected equations ($j=1,3$).

It is not the case for the connected equations ($j=2,4$). There are also the subsidiary equations for the operators L^d_b and $(U_{ba})_d$. Unfortunately, the symmetry group of these operators is $\mathcal{S}_d$, leading to a complicated situation when different partitions d appear in the same equation. For this reason we have not pursued the symmetrization of the subsidiary equations and have left it as a problem for future research. Fortunately, for some systems the symmetrization of the subsidiary equations is not necessary. An example of such a system is discussed in the next section.

VII. EXAMPLE: ALPHA-DEUTERON SCATTERING

In this section we exhibit the four forms of the approximate CG equations in the context of alpha-deuteron scattering. We first give the connected and unconnected T equations [Eqs. (3.9)–(3.11) and (4.10)–(4.12)] for a simple model of the process. In this model only the elastic and the deuteron-breakup channels are included, and exchange effects arising from the Pauli principle are ignored. We then write out the connected and unconnected M equations [Eqs. (3.14)–(3.16) and (5.4)–(5.6)] for a more complicated model of the process. This more complicated model includes the ^{3}H-^{3}He channel in addition to those channels of the simple model. Exchange effects are also included. For simplicity we neglect the electromagnetic interaction and the nucleon spin. A comparison of the two models of alpha-deuteron scattering and the four forms of the equations will be given in Sec. VIII.

We introduce the following notation. We label the nucleons by integers and the various partitions as follows:

$$\begin{aligned} &d=(12)(3456), \quad N=(123456) , \\ &1=(1)(23456), \quad t=(123)(456) , \\ &2=(2)(13456), \quad O=(1)(2)(3)(4)(5)(6) , \\ &\alpha=(1)(2)(3456) . \end{aligned} \tag{7.1}$$

The Jacobi momenta and reduced masses are

$$\begin{aligned} &\vec{p}_d = \tfrac{1}{2}(\vec{k}_1 - \vec{k}_2) , \\ &\vec{q}_d = \tfrac{1}{6}(2\vec{k}_{(3456)} - 4\vec{k}_1 - 4\vec{k}_2) , \\ &\vec{p}_1 = \tfrac{1}{5}(4\vec{k}_2 - \vec{k}_{(3456)}) , \\ &\vec{q}_1 = \tfrac{1}{6}(5\vec{k}_1 - \vec{k}_2 - \vec{k}_{(3456)}) , \\ &\vec{p}_2 = \tfrac{1}{5}(\vec{k}_{(3456)} - 4\vec{k}_1) , \\ &\vec{q}_2 = \tfrac{1}{6}(5\vec{k}_2 - \vec{k}_1 - \vec{k}_{(3456)}) , \\ &\vec{q}_t = \tfrac{1}{2}(\vec{k}_{(123)} - \vec{k}_{(456)}) , \end{aligned} \tag{7.2}$$

$$\begin{aligned} &\mu_d = \tfrac{1}{2}m, \quad \mu_1 = \mu_2 = \tfrac{4}{5}m, \quad \mu_t = \tfrac{3}{2}m , \\ &M_\alpha = \tfrac{4}{3}m, \quad M_1 = M_2 = \tfrac{5}{6}m , \end{aligned} \tag{7.3}$$

where m is the mass of a single nucleon and $\vec{k}_a$ is the total momentum of the cluster a. We let $\epsilon(\alpha)$, $\epsilon(d)$, and $\epsilon(t)$ denote the alpha, deuteron, and triton binding energies measured from the breakup threshold. The corresponding bound state vectors are $|\phi(\alpha)\rangle$, $|\phi(d)\rangle$, and $|\phi(t)\rangle$, respectively. The vectors $|\phi(\alpha)\vec{q}_d\vec{p}_d\rangle$, $|\phi(\alpha)\phi(d)\vec{q}_d\rangle$, and

$|\phi(t)\phi(t')\vec{q}_t\rangle$ are normalized as

$$\begin{aligned}
&\langle\phi(\alpha)\vec{q}_d\vec{p}_d\,|\,\phi(\alpha)\vec{q}'_d\vec{p}'_d\rangle=\delta(\vec{q}_d-\vec{q}'_d)\delta(\vec{p}_d-\vec{p}'_d)\ ,\\
&\langle\phi(\alpha)\phi(d)\vec{q}_d\,|\,\phi(\alpha)\phi(d)\vec{q}'_d\rangle=\delta(\vec{q}_d-\vec{q}'_d)\ ,\\
&\langle\phi(t)\phi(t')\vec{q}_t\,|\,\phi(t)\phi(t')\vec{q}'_t\rangle=\delta(\vec{q}_t-\vec{q}'_t)\ ,
\end{aligned}\tag{7.4}$$

where $\phi(t)$ and $\phi(t')$ represent the three particle bound states consisting of particles (123) and (456), respectively. In what follows we will also need the operators

$$\begin{aligned}
&\Pi_\alpha=\int|\phi(\alpha)\vec{q}_d\vec{p}_d\rangle d\vec{q}_d d\vec{p}_d\langle\phi(\alpha)\vec{q}_d\vec{p}_d|\ ,\\
&\Pi_d=\int|\phi(\alpha)\phi(d)\vec{q}_d\rangle d\vec{q}_d\langle\phi(\alpha)\phi(d)\vec{q}_\alpha|\ ,\\
&\Pi_t=\int|\phi(t)\phi(t')\vec{q}_t\rangle d\vec{q}_t\langle\phi(t)\phi(t')\vec{q}_t|\ ,
\end{aligned}\tag{7.5}$$

$$\begin{aligned}
&R_\alpha\Pi_\alpha=\int\frac{|\phi(\alpha)\vec{q}_d\vec{p}_d\rangle d\vec{q}_d d\vec{p}_d\langle\phi(\alpha)\vec{q}_d\vec{p}_d|}{z-q_d^{\,2}/2M_d-p_d^{\,2}/2\mu_d+\epsilon(\alpha)}\ ,\\
&R_d\Pi_d=\int\frac{|\phi(\alpha)\phi(d)\vec{q}_d\rangle d\vec{q}_d\langle\phi(\alpha)\phi(d)\vec{q}_d|}{z-q_d^{\,2}/2M_d+\epsilon(\alpha)+\epsilon(d)}\ ,\\
&R_t\Pi_t=\int\frac{|\phi(t)\phi(t')\vec{q}_t\rangle d\vec{q}_t\langle\phi(t)\phi(t')\vec{q}_t|}{z-q_t^{\,2}/2M_t+2\epsilon(t)}\ ,
\end{aligned}\tag{7.6}$$

$$\begin{aligned}
&V_N=\sum_{i<j}V_{ij}\ ,\\
&V_d=V_{12}+V_\alpha\ ,\\
&V_1=V_{23}+V_{24}+V_{25}+V_{26}+V_\alpha\ ,\\
&V_2=V_{13}+V_{14}+V_{15}+V_{16}+V_\alpha\ ,\\
&V_\alpha=V_{34}+V_{35}+V_{36}+V_{45}+V_{46}+V_{56}\ ,\\
&V_t=V_{12}+V_{13}+V_{23}+V_{45}+V_{46}+V_{56}\ ,\\
&V_N^d=V_N-V_d\ ,\\
&V_N{}^1=V_N-V_1\ ,\\
&V_N{}^2=V_N-V_2\ ,\\
&V_N^t=V_N-V_t\ .
\end{aligned}\tag{7.7}$$

Our simple model of alpha-deuteron scattering is characterized by

$$\Pi_a=\delta_{a\alpha}\Pi_\alpha+\delta_{ad}\Pi_d\ .\tag{7.8}$$

In this approximation $J\Pi J^*=\Pi_\alpha+\Pi_d$, and $(J\Pi J^*)^{-1}P_\pi$ and P_π can be computed in the closed forms

$$(J\Pi J^*)^{-1}P_\pi=\Pi_\alpha-\tfrac{1}{2}\Pi_d\ ,\tag{7.9}$$

$$P_\pi=\Pi_\alpha\ .\tag{7.10}$$

We assume that the initial change is fixed at d.

The approximate T equations (3.9)–(3.11) in this model are

$$\begin{aligned}
T^\pi_{dd}&=\Pi_d V_N^d\Pi_d+\Pi_d V_N^d\\
&\quad\times(\Pi_\alpha-\tfrac{1}{2}\Pi_d)(R_d\Pi_d T^\pi_{dd}+R_\alpha\Pi_\alpha T^\pi_{\alpha d})\ ,\\
T^\pi_{\alpha d}&=\Pi_\alpha V_N^d\Pi_d+\Pi_\alpha V_N^\alpha\\
&\quad\times(\Pi_\alpha-\tfrac{1}{2}\Pi_d)(R_d\Pi_d T^\pi_{dd}+R_\alpha\Pi_\alpha T^\pi_{\alpha d})\ ,
\end{aligned}\tag{7.11}$$

The connected form of the approximate T equations are Eqs. (4.10)–(4.12). If we use the notation

$$\tau^\pi_a=\tau^\pi_a(z)\equiv L^a_\alpha R_O R_\alpha{}^{-1}\tag{7.12}$$

for $a=d$, 1, or 2, these equations are

$$T^{\pi}_{dd}=\Pi_d V^d_N\Pi_d+\Pi_d V^d_N\Pi_d(\Pi_\alpha-\tfrac{1}{2}\Pi_d)(R_d\Pi_d T^{\pi}_{dd}+R_\alpha\Pi_\alpha T^{\pi}_{\alpha d}) ,$$
$$T^{\pi}_{\alpha d}=\Pi_\alpha V^d_N\Pi_d+(\tau^{\pi}_d+\tau^{\pi}_1+\tau^{\pi}_2)R_\alpha\Pi_\alpha V^d_N\Pi_d \tag{7.13}$$
$$+(\tau^{\pi}_d R_\alpha\Pi_\alpha V^d_N+\tau^{\pi}_1 R_\alpha\Pi_\alpha {V_N}^1+\tau^{\pi}_1 R_\alpha\Pi_\alpha {V_N}^2)\Pi_\alpha(\Pi_\alpha-\tfrac{1}{2}\Pi_d)(R_\alpha\Pi_\alpha T^{\pi}_{\alpha d}+R_d\Pi_d T^{\pi}_{dd}) .$$

Although the L^a_α, and hence τ^{π}_a, can be computed from Eqs. (4.16)–(4.18), in this special case τ^{π}_a are the solutions of the Lippman-Schwinger-type equations

$$\tau^{\pi}_a=\Pi_\alpha V^\alpha_a\Pi_\alpha+\Pi_\alpha V^\alpha_a\Pi_\alpha R_\alpha\tau^{\pi}_a , \tag{7.14}$$

$a=d$, 1, or 2. It is clear that the kernel of Eq. (7.13) is connected, and the kernel of Eq. (7.14) is a connected.

There is no advantage to using the M equations when $(J\Pi J^*)^{-1}P_\pi$ can be computed exactly as in this model. Therefore we do not write them down.

Next we consider a more realistic model of this system. This model includes the effects of exchanging one or both nucleons from the deuteron with the nucleons in the alpha particle. It also includes the ^{3}H-^{3}He channel. In this case the nonvanishing Π_a's are Π_d, Π_α, Π_t, and all Π_a's related to these by exchange of identical particles. To make a rigorous connection with an approximate time dependent theory one must show that the $J\Pi J^*$ associated with this approximation has closed range. Although this has not been proved for this example, we will assume that the range is closed.

In this model $(J\Pi J^*)^{-1}P_\pi$ does not have a simple closed form. Therefore the T equations which require this inverse will not be written down. Instead, we formulate this model with both the unconnected and connected forms of the M equations. The symmetrized form of these equations is Eqs. (6.12)–(6.14) with $j=3$ and 4, respectively. The resulting equations have the three coupled canonical channels d, α, and t. (We dispense with the superscripts o.) We only consider the case where the incoming state is in the d channel.

In this model the operators $\hat{I}^{(3)}_{ba}$ and $\hat{K}^{(3)}_{ba}$ of the unconnected M equations (6.12) are

$$\hat{I}^{(3)}_{dd}=N_d\Pi_d Y_N V^d_N\Pi_d ,$$
$$\hat{I}^{(3)}_{\alpha d}=(N_\alpha N_d)^{1/2}\Pi_\alpha Y_N V^d_N\Pi_d ,$$
$$\hat{I}^{(3)}_{td}=(N_t N_d)^{1/2}\Pi_t Y_N V^d_N\Pi_d ,$$
$$\hat{K}^{(3)}_{dd}=N_d\Pi_d Y_N(V^d_N-{R_d}^{-1})\Pi_d+Y_d{R_d}^{-1}\Pi_d ,$$
$$\hat{K}^{(3)}_{d\alpha}=(N_d N_\alpha)^{1/2}\Pi_d Y_N(V^\alpha_N-{R_\alpha}^{-1})\Pi_\alpha ,$$
$$\hat{K}^{(3)}_{dt}=(N_d N_t)^{1/2}\Pi_d Y_N(V^t_N-{R_t}^{-1})\Pi_t ,$$
$$\hat{K}^{(3)}_{\alpha d}=(N_\alpha N_d)^{1/2}\Pi_\alpha Y_N(V^d_N-{R_d}^{-1})\Pi_d , \tag{7.15}$$
$$\hat{K}^{(3)}_{\alpha\alpha}=N_\alpha\Pi_\alpha Y_N(V^\alpha_N-{R_\alpha}^{-1})\Pi_\alpha+Y_\alpha{R_\alpha}^{-1}\Pi_\alpha ,$$
$$\hat{K}^{(3)}_{\alpha t}=(N_\alpha N_t)^{1/2}\Pi_\alpha Y_N(V^t_N-{R_t}^{-1})\Pi_t ,$$
$$\hat{K}^{(3)}_{td}=(N_t N_d)^{1/2}\Pi_t Y_N(V^d_N-{R_d}^{-1})\Pi_d ,$$
$$\hat{K}^{(3)}_{t\alpha}=(N_t N_\alpha)^{1/2}\Pi_t Y_N(V^\alpha_N-{R_\alpha}^{-1})\Pi_\alpha ,$$
$$\hat{K}^{(3)}_{tt}=N_t\Pi_t Y_N(V^t_N-{R_t}^{-1})\Pi_t+Y_t{R_t}^{-1}\Pi_t .$$

The numerical factors are easily computed to be

$$N_1=N_2=6 ,$$
$$N_d=N_\alpha=15 , \tag{7.16}$$
$$N_t=10 .$$

The operators $\hat{I}^{(4)}_{ba}$ and $\hat{K}^{(4)}_{ba}$ of the connected M equations (6.12) are

$$\hat{I}^{(4)}_{dd}=N_d\Pi_d[I_N+(U_{dd})_d+(U_{d\alpha})_d]Y_N V^d_N\Pi_d ,$$
$$\hat{I}^{(4)}_{\alpha d}=(N_\alpha N_d)^{1/2}\Pi_\alpha[I_N+(U_{\alpha\alpha})_d+(U_{\alpha\alpha})_1+(U_{\alpha\alpha})_2+(U_{\alpha d})_d]Y_N V^d_N\Pi_d ,$$
$$\hat{I}^{(4)}_{td}=(N_t N_d)^{1/2}\Pi_t Y_N V^d_N\Pi_d ,$$
$$\hat{K}^{(4)}_{dd}=\Pi_d[I_N+(U_{dd})_d+(U_{d\alpha})_d](V^d_N+\sum_{a\in[d]}\bar{\delta}_{ad}\hat{p}_{ad}Y_d{R_N}^{-1})Y_d\Pi_d ,$$
$$\hat{K}^{(4)}_{d\alpha}=(N_d/N_\alpha)^{1/2}\Pi_d[I_N+(U_{dd})_d+(U_{d\alpha})_d](V^d_N+\sum_{a\in[\alpha]}\bar{\delta}_{a\alpha}\hat{p}_{a\alpha}Y_\alpha{R_N}^{-1})Y_\alpha\Pi_\alpha ,$$
$$\hat{K}^{(4)}_{dt}=(N_d N_t)^{1/2}\Pi_d[I_N+(U_{dd})_d+(U_{d\alpha})_\alpha](Y_N V^t_N\Pi_t-Y_N{R_t}^{-1}\Pi_t) , \tag{7.17}$$
$$\hat{K}^{(4)}_{\alpha d}=(N_\alpha/N_d)^{1/2}\Pi_\alpha\{[I_N+(U_{\alpha\alpha})_d+(U_{\alpha d})_d][V^d_N-\sum_{a\in[d]}\bar{\delta}_{ad}\hat{p}_{ad}Y_d{R_N}^{-1}]$$
$$-N_\alpha[(U_{\alpha\alpha})_1+(U_{\alpha\alpha})_2]Y_N{R_N}^{-1}\}Y_d\Pi_d ,$$

$$\hat{K}^{(4)}_{\alpha\alpha}=\Pi_\alpha\{[(U_{\alpha\alpha})_d+(U_{\alpha d})_d](V^d_N-\sum_{a\in[\alpha]}\bar{\delta}_{a\alpha}\hat{p}_{a\alpha}Y_\alpha R_N{}^{-1})$$
$$+(U_{\alpha\alpha})_1(V_N{}^1\sum_{a\in[\alpha]}\Delta_{1a}\hat{p}_{a\alpha}-\sum_{a\in[\alpha]}\bar{\Delta}_{1a}\hat{p}_{a\alpha}Y_\alpha R_N{}^{-1})$$
$$+(U_{\alpha\alpha})_2(V_N{}^2\sum_{a\in[\alpha]}\Delta_{2a}\hat{p}_{a\alpha}-\sum_{a\in[\alpha]}\bar{\Delta}_{2a}\hat{p}_{a\alpha}Y_\alpha R_N{}^{-1})\}Y_\alpha\Pi_\alpha\ ,$$
$$\hat{K}^{(4)}_{\alpha t}=-(N_\alpha N_t)^{1/2}\Pi_\alpha[I_N+(U_{\alpha\alpha})_d+(U_{\alpha\alpha})_1+(U_{\alpha\alpha})_2+(U_{\alpha d})_d]Y_N R_N{}^{-1}\Pi_t\ ,$$
$$\hat{K}^{(4)}_{td}=-(N_t N_d)^{1/2}\Pi_t Y_N R_N{}^{-1}\Pi_d\ ,$$
$$\hat{K}^{(4)}_{t\alpha}=-(N_t N_\alpha)^{1/2}\Pi_t Y_N R_N{}^{-1}\Pi_\alpha\ ,$$
$$\hat{K}^{(4)}_{tt}=\Pi_t(-\sum_{a\in[t]}\bar{\delta}_{at}\hat{p}_{at}R_t{}^{-1}Y_t+N_t Y_N V^t_N)\Pi_t\ .$$

The operators $(U_{ab})_c=(U_{ab})_c(z)$ which are needed as input to Eqs. (7.17) are the solutions of Eqs. (5.7). For this particular model these equations are

$$(U_{\alpha\alpha})_1=\Pi_\alpha V^\alpha_1 R_\alpha\Pi_\alpha+\Pi_\alpha V^\alpha_1 R_\alpha\Pi_\alpha(U_{\alpha\alpha})_1\ ,$$
$$(U_{\alpha\alpha})_2=\Pi_\alpha V^\alpha_2 R_\alpha\Pi_\alpha+\Pi_\alpha V^\alpha_2 R_\alpha\Pi_\alpha(U_{\alpha\alpha})_2\ ,$$
$$(U_{\alpha\alpha})_d=\Pi_\alpha[(I_N-\Pi_d)V^\alpha_d R_\alpha+\Pi_d]\Pi_\alpha$$
$$+\Pi_\alpha[(I_N-\Pi_d)V^\alpha_d R_\alpha+\Pi_d]\Pi_\alpha(U_{\alpha\alpha})_d\ . \tag{7.18}$$

The remaining nonvanishing $(U_{ab})_d$ operators can be computed from the operator $(U_{\alpha\alpha})_d$ by quadrature

$$(U_{\alpha d})_d=-\Pi_\alpha[I_N+(U_{\alpha\alpha})_d]\Pi_d\ ,$$
$$(U_{d\alpha})_d=\Pi_d(V^\alpha_d R_\alpha-I_N)[I_N+(U_{\alpha\alpha})_d]\Pi_d\ ,$$
$$(U_{dd})_d=-\Pi_d(V^\alpha_d R_\alpha-I_N)[I_N+(U_{\alpha\alpha})_d]\Pi_\alpha\Pi_d\ . \tag{7.19}$$

To obtain transition operators with the appropriate symmetry we use a symmetrized form of Eq. (3.13)

$$\hat{T}^\pi_{ba}=\Pi_b R_b{}^{-1}Y_N N_b^{1/2}$$
$$\times(N_d^{1/2}R_d\Pi_d\hat{M}^\pi_{da}+N_\alpha^{1/2}R_\alpha\Pi_\alpha\hat{M}^\pi_{\alpha a}$$
$$+N_t^{1/2}R_t\Pi_t\hat{M}^\pi_{ta})\ . \tag{7.20}$$

VIII. DISCUSSION

In this paper we have considered the new dynamical equation [Eq. (3.2)] for N-body scattering that has recently been proposed by Chandler and Gibson.[6] More precisely, we have studied approximate versions [Eqs. (3.9)–(3.11) and (3.14)–(3.16)] of this equation. As the kernel of these equations may be noncompact so that standard numerical technology may be inapplicable, we have derived connected kernel versions [Eqs. (4.10)–(4.12) and (5.4)–(5.6)] of these equations. These connected kernel equations preserve most of the important features of the original equations (cf. Refs. 4–6), but the kernels are more complicated and the possibility of spurious solutions is raised [cf. the remark following Eq. (4.26)]. The advantage of the connected kernel equations is that standard compact kernel methods can be used. We then went on to write down (in Sec. VI) the properly symmetrized form of the equations, both connected and unconnected, for systems containing identical particles.

Finally, in order to compare the connected and the unconnected equations in a more concrete context, we have formulated two models of alpha-deuteron scattering in Sec. VII. Each model corresponds to an approximate time-dependent system governed by the Hamiltonian H_π defined in Eq. (3.7). The projection operator P_π is determined by specifying the nonzero projection operators Π_a. These projection operators and the potentials given in Eq. (7.7) therefore specify the dynamical content of the model.

The first, and simpler, model of Sec. VII includes only the d-α and n-p-α channels. Exchange effects and the ^{3}H-^{3}He channel are ignored. This model is one of the few in which $(J\Pi J^*)^{-1}$ has a simple closed form [cf. Eq. (7.9)]. Consequently, there is no advantage in using the M equations and we have exhibited only the T equations, unconnected [Eqs. (7.11)] and connected [Eqs. (7.13)].

The input to Eqs. (7.11) involves only the alpha and deuteron bound state wave functions (i.e., Π_α and Π_d) and the nucleon-nucleon interactions (i.e., V^α_N and V^d_N). This input involves exactly the quan-

tities of interest in direct interaction theories. The solution method of Chandler and Gibson[4,5] for unconnected equations requires the additional approximation of the operator Π_α by an operator $\tilde{\Pi}_\alpha$ which will give a compact kernel of the resulting Eq. (7.11). The solution of these equations is approximate transition operators $\tilde{T}^\pi_{ad}$, $a=\alpha$ and d. Chandler and Gibson have proved theoretically[5] that this can be done in such a way that the $\tilde{T}^\pi_{ad}$ operators converge to the T^π_{ad} operators both on and off shell as the $\tilde{\Pi}_a$ operators converge to the Π_a operators. The practicality of this method has yet to be established, however.

On the other hand, Eqs. (7.13) require as additional input the (unphysical) matrix elements of the off-shell subsystem transition operators $\tau^\pi_a(z)$ which are the solutions of the three Eqs. (7.14). In practice one calculates only approximations $\tilde{\tau}^\pi_a$ to these operators which are then put into the kernel of Eqs. (7.13) to calculate approximations $\tilde{T}^\pi_{ad}$ to the operators T^π_{ad}. Although we are unaware of any rigorous proofs, experience with equations of the Faddeev-type suggests that the $\tilde{T}^\pi_{ad}$ converge to the T^π_{ad} as the $\tilde{\tau}^\pi_a$ converge to the τ^π_a.

The second model in Sec. VII is more sophisticated in that exchange effects and the ^{3}H-^{3}He channel are included. The exchange terms are an important ingredient of phenomenological models that is not included in the first model. The ^{3}H-^{3}He channel is believed to be important because of certain resonating group calculations.[15]

In this second model the operator $(J\Pi J^*)^{-1}$ does not have a simple closed form, considerably complicating the T equations. We have therefore considered only the unconnected [Eqs. (6.12) and (7.15)] and connected [Eqs. (6.12) and (7.17)] M equations for this model. A close inspection of the operators of either Eqs. (7.15) or (7.17) shows that the equations of this more realistic model are not substantially more complicated than the corresponding equations for the simpler model. The primary complications involve the intricate coordinate transformations associated with the symmetrizers Y_a, and the presence of the additional ^{3}H-^{3}He channel with its resulting increase in the number of kernel operators $\hat{K}^{(j)}_{ba}$ from four to nine.

The input to the unconnected kernel equations [Eqs. (6.12) and (7.15)] are the operators Π_a and the potentials V^a_N, $a=d$, α, and t. The kernel is not connected and does not become so upon iteration. The solution strategy[4,5] therefore requires an initial approximation $\tilde{\Pi}_\alpha$ to Π_α which will connect these kernels.

The connected kernel equations [Eqs. (6.12) and (7.17)] require as additional input the set of six off-shell operators $(U_{ab})_c(z)$ which are to be obtained (approximately) from Eqs. (7.18) and (7.19).

In both models, therefore, the input of the connected kernel equations is considerably more complicated than that of the unconnected kernel equations. In addition, the structure of the kernel itself is more complicated. The advantage of these equations is that they can be solved by well-established techniques. If future work shows that the initial approximations $\tilde{\Pi}_a$ of Π_a can be made in a computationally practical way, then the simpler structure of the unconnected CG equations may well shift the advantage to them.

ACKNOWLEDGMENTS

We would like to thank Gy. Bencze for several comments about the subject of this paper. W.N.P. gratefully acknowledges the Medium Energy Physics Group at Los Alamos Scientific Laboratory for support during this project. C.C. and A.G.G. gratefully acknowledge support of the National Science Foundation under Grant No. PHY-8103062.

APPENDIX A

In this appendix we give some of the main combinatorial results used in this paper.

Theorem A1.

$$\sum_{c\in\mathscr{P}'} \Delta_{ac}{}^{-1}\Delta_{cb} = \sum_{c\in\mathscr{P}'} \Delta_{ac}\Delta_{cb}{}^{-1} = \delta_{ab} \ . \tag{A1}$$

Proof. The result follows directly from Eqs. (4.1) and (4.2).

Theorem A2.

$$\text{(a)} \sum_a C_a = 1 \ , \tag{A2}$$

$$\text{(b)} \sum_{a(\supseteq b)} C_a = \sum_a C_a \Delta_{ab} = \bar{\delta}_{Nb} \ . \tag{A3}$$

Proof. Equation (A3) follows from Eq. (A1) and Eq. (4.3). Equation (A2) is obtained from Eq. (A3) by taking $b=0$.

Theorem A3.

$$-\sum_{b\in\mathscr{P}'} \bar{\delta}_{ab}\Delta_{ab}{}^{-1} = \bar{\delta}_{aO} \ . \tag{A4}$$

Proof.

$$-\sum_{b\in\mathscr{P}'} \bar{\delta}_{ab}\Delta_{ab}{}^{-1} = \sum_{b\in\mathscr{P}'} (\delta_{ab}-1)\Delta_{ab}{}^{-1}\Delta_{bO}$$
$$= 1-\delta_{aO} = \bar{\delta}_{aO} \ . \tag{A5}$$

Theorem A4. The operator

$$\sum_a C_a(A)_a(B)^a$$

is connected.

Proof. This is *Theorem 7* of Ref. 10.

Theorem A5. The operator

$$\sum_{a(\supseteq b)} C_a[D]_b(A)_a(B)^a$$

is connected.

Proof. Since

$$([D]_b(A)_a)_a=\Delta_{ab}[D]_b(A)_a\ , \tag{A6}$$

the result follows from *Theorem A4.*

Theorem A6. The operator

$$C\equiv-\sum_{b\in\mathscr{P}'}\bar{\delta}_{ab}\Delta_{ab}{}^{-1}(A)_b(B)^b_a$$

is a connected.

Proof.

$$\begin{aligned}C&=-\sum_{b,c,d\in\mathscr{P}'}\Delta_{ab}{}^{-1}\Delta_{bc}\Delta_{ad}(1-\Delta_{bd})[A]_c[B]_d\\&=\sum_{c,d\in\mathscr{P}'}\Delta_{ad}(\delta_{a,c\cup d}-\delta_{ac})[A]_c[B]_d\\&=\sum_{c,d\in\mathscr{P}'}\Delta_{ad}(\delta_{a,c\cup d}-\delta_{ac})[[A]_c[B]_d]_a\ ,\end{aligned}\tag{A7}$$

and the last expression in (A7) is a connected.

APPENDIX B

In this appendix we derive equation (3.2). Equations (2.8) and (2.11) imply that

$$T_{ba}(z)=P_bV_N^aP_a+P_bV_N^bR_N(z)V_N^aP_a\ . \tag{B1}$$

We insert

$$I=\left[\sum_c P_c\right]^{-1}\sum_d P_dR_d(z)P_dR_d{}^{-1}(z) \tag{B2}$$

between V_N^b and $R_N(z)$ in Eq. (B1) to obtain

$$T_{ba}(z)=P_bV_N^aP_a+P_bV_N^b\left[\sum_c P_c\right]^{-1}\sum_d P_dR_d(z)(P_dR_d{}^{-1}(z)R_N(z)V_N^aP_a)\ . \tag{B3}$$

Equation (3.2) now follows immediately by substituting Eqs. (2.11) and (3.1) into Eq. (B3).

APPENDIX C

In this appendix we derive Eqs. (4.16)–(4.18). Since for $b\subseteq d\subseteq c$,

$$\begin{aligned}\Pi_bR_{c,\pi}=&\Pi_bR_{d,\pi}+\Pi_bR_{d,\pi}\\&\times(H_{c,\pi}-H_{d,\pi})R_{c,\pi}P_{c,\pi}\end{aligned}\tag{C1}$$

and $H_{c,\pi}-H_{d,\pi}$ has connectivity outside of d, we reach the conclusion that

$$[R_b{}^{-1}\Pi_bR_{d,\pi}]_d=[R_b{}^{-1}\Pi_bR_{c,\pi}]_d\ .$$

Using Eqs. (2.3) and (4.13) we may then write

$$\begin{aligned}R_b{}^{-1}\Pi_bR_{c,\pi}&=\sum_{(b\subseteq)d(\subseteq c)}[R_b{}^{-1}\Pi_bR_{c,\pi}]_d\\&=\sum_{d(\subseteq c)}L_b^dR_O\ .\end{aligned}\tag{C2}$$

Substituting Eq. (C2) into Eqs. (4.11) and (4.12) gives

$$I_{ba}^{(2)}=\sum_{c(\supseteq b)\atop d}C_c\Delta_{cd}L_b^dR_OV_N^a\Pi_a \tag{C3}$$

and

$$K_{ba}^{(2)}=\sum_{c(\supseteq b)\atop d}C_c\Delta_{cd}L_b^dR_OV_\pi^{c,\pi}(J\Pi J^*)^{-1}\Pi_a\ . \tag{C4}$$

Using *Theorem A2* we may evaluate the sum over c in Eq. (C3) to obtain

$$I_{ba}^{(2)}=\sum_d L_b^dR_OV_N^a\Pi_a\ . \tag{C5}$$

The operator $K_{ba}^{(2)}$ in Eq. (C4) may be written as

$$K_{ba}^{(2)}={}^1K_{ba}^{(2)}+{}^2K_{ba}^{(2)}\ , \tag{C6}$$

where

$${}^1K_{ba}^{(2)}\equiv\sum_{c,d}C_c\Delta_{cd}\Delta_{cb}L_b^dR_OV_N^c(J\Pi J^*)^{-1}\Pi_a \tag{C7}$$

and

$${}^2K_{ba}^{(2)}\equiv\sum_{c,d}C_c\Delta_{cd}\Delta_{cb}L_b^dR_OH_c\times(P_\pi-P_{c,\pi})(J\Pi J^*)^{-1}\Pi_a\ . \tag{C8}$$

Using the cluster expansions

$$V_N^c=\sum_e(1-\Delta_{ce})[V_N]_e\ , \tag{C9}$$

$$H_c=\sum_f \Delta_{cf}[H_N]_f \ , \tag{C10}$$

and

$$P_\pi - P_{c,\pi}=\sum_g (1-\Delta_{cg})[P_\pi]_g \ , \tag{C11}$$

we may evaluate the sum over c in Eqs. (C7) and (C8). In particular, using Eq. (4.4) and *Theorem A2,* we obtain

$${}^1K_{ba}^{(2)}=\sum_{d,e}\delta_{N,d\cup e}L_b^d R_O[V_N]_e(J\Pi J^*)^{-1}\Pi_a \tag{C12}$$

and

$${}^2K_{ba}^{(2)}=\sum_{d,f,g}\delta_{N,d\cup f\cup g}\bar{\delta}_{N,d\cup f} \times L_b^d R_O[H_N]_f[P_\pi]_g(J\Pi J^*)^{-1}\Pi_a \ . \tag{C13}$$

If V_N is a sum of only two-body forces then $[V_N]_e\neq 0$ only for $n_e=N-1$. The condition $d\cup e=N$ in Eq. (C12) then requires $n_d=2$. Thus for two-body forces Eq. (C12) may be further simplified to

$${}^1K_{ba}^{(2)}=\sum_{d,n_d=2} L_b^d R_O V_N^d (J\Pi J^*)^{-1}\Pi_a \ . \tag{C14}$$

Now

$$\begin{aligned}\Pi_d R_{d,\pi}&=R_d\Pi_d+R_d\Pi_d(P_{d,\pi}H_d P_{d,\pi}-H_d)\\ &\quad\times R_{d,\pi}P_{d,\pi}\\ &=R_d\Pi_d \ .\end{aligned} \tag{C15}$$

The operators L_d^d thus are given by

$$L_d^d=\Pi_d R_0{}^{-1} \ . \tag{C16}$$

The remaining operators L_b^d, $b\neq d$, can be constructed recursively on the number of clusters of d decreasing from $n_d=N-1$. To obtain these recursive equations we begin with the resolvent equations

$$P_{d,\pi}R_{c,\pi}=P_{d,\pi}R_{d,\pi}+P_{d,\pi}R_{d,\pi}V_{c,\pi}^{d,\pi}R_{c,\pi} \ , \tag{C17}$$

where $d\subseteq c$, $c\neq N$, and $V_{c,\pi}^{d,\pi}$ is defined by Eq. (4.14). We multiply Eq. (C17) on the left by

$$-\bar{\delta}_{cd}\Delta_{cd}{}^{-1}\Delta_{db}R_b{}^{-1}\Pi_b \ ,$$

on the right by $R_O{}^{-1}$, and sum the result over d, $b\subseteq d\supseteq c$. By *Theorem A3* this gives for $b\neq c$

$$\begin{aligned}R_b{}^{-1}\Pi_b R_{c,\pi}R_O{}^{-1}&=-\sum_{d(\supseteq b)}\bar{\delta}_{cd}\Delta_{cd}{}^{-1}R_b{}^{-1}\Pi_b R_{d,\pi}R_O{}^{-1}\\ &\quad-\sum_{d(\supseteq b)}\bar{\delta}_{cd}\Delta_{cd}{}^{-1}R_b{}^{-1}\Pi_b R_{d,\pi}\\ &\quad\times V_{c,\pi}^{d,\pi}R_{c,\pi}R_O{}^{-1} \ .\end{aligned} \tag{C18}$$

Since

$$R_b{}^{-1}\Pi_b R_{d,\pi}=(R_b{}^{-1}\Pi_b R_{d,\pi})_d \ ,$$

the second term on the right side of Eq. (C18) is c connected by *Theorem A6.* Because the first term on the right side of Eq. (C18) has no c connected pieces for $b\subseteq d\subseteq c$, $d\neq c$, it necessarily cancels the terms on the left side of Eq. (C18) with connectivities $d\subseteq c$, $d\neq c$. What remains is

$$L_b^c=-\sum_{d(\supseteq b)}\bar{\delta}_{cd}\Delta_{cd}{}^{-1}R_b{}^{-1}\Pi_b R_{d,\pi}V_{c,\pi}^{d,\pi}R_{c,\pi}R_O{}^{-1} \ . \tag{C19}$$

We assume that $Z_{c,\pi}$ defined in Eq. (4.15) has closed range. Then [cf. Eq. (3.6)]

$$P_{c,\pi}=Z_{c,\pi}{}^{-1}Z_{c,\pi} \ . \tag{C20}$$

Since $V_{c,\pi}^{d,\pi}=V_{c,\pi}^{d,\pi}P_{c,\pi}$, we may write Eq. (C19) as

$$\begin{aligned}L_b^c=-\sum_{\substack{d(\supseteq b)\\ e(\subseteq c)}}&\bar{\delta}_{cd}\Delta_{cd}{}^{-1}R_b{}^{-1}\Pi_b R_{d,\pi}V_{c,\pi}^{d,\pi}\\ &\times Z_{c,\pi}{}^{-1}R_e R_e{}^{-1}\Pi_e R_{c,\pi}R_O{}^{-1} \ .\end{aligned} \tag{C21}$$

Now we substitute the cluster expansion (C2) into Eq. (C21) to obtain Eqs. (4.16)–(4.18).

---

*Present address: Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, MA 02139.

[1]K. L. Kowalski, in *Few Body Systems and Nuclear Forces II, Proceedings, Graz, 1978,* edited by H. Zingl, M. Haftel, and H. Zankel, Lecture Notes in Physics (Springer, Berlin, 1978), Vol. 87, pp. 393–413; F. S. Levin, Nucl. Phys. A353, 143c (1981).

[2]P. Doleschall, Nucl. Phys. A201, 264 (1973); A. W. Thomas, in *Few Body Systems and Nuclear Forces II,*

Proceedings, Graz, 1978, edited by H. Zingl, M. Haftel, and H. Zankel, Lecture Notes in Physics (Springer, Berlin, 1978), Vol. 87, pp. 247–266; J. A. Tjon, *ibid.*, p. 320–332.

[3]R. Goldflam and K. L. Kowalski, Phys. Rev. C 22, 2341 (1980); W. Polyzou and E. F. Redish, Ann. Phys. (N.Y.) 119, 1 (1979); P. C. Tandy, E. F. Redish, and D. Bolle, Phys. Rev. C 16, 1924 (1977).

[4]C. Chandler and A. G. Gibson, in *Mathematical Methods and Applications of Scattering Theory,* edited by J. A. DeSanto, A. W. Saenz, and W. W. Zachary, Lecture Notes in Physics (Springer, New York, 1980), Vol. 13, pp. 134–148.

[5]C. Chandler and A. G. Gibson, *N*-Body Quantum Scattering Theory in Two Hilbert Spaces. III. Approximation Solvability, University of New Mexico, report, 1981.

[6]C. Chandler and A. G. Gibson, J. Math. Phys. 18, 2336 (1977).

[7]C. Chandler and A. G. Gibson, J. Math. Phys. 19, 1610 (1978).

[8]S. Weinberg, Phys. Rev. 133, B232 (1964); C. van Winter, K. Dan. Vidensk. Selsk., Mat.-Fys. Medd. 2, No. 8 (1964).

[9]Gy. Bencze and E. F. Redish, J. Math. Phys. 19, 1909 (1978).

[10]W. N. Polyzou, J. Math. Phys. 21, 506 (1980).

[11]K. L. Kowalski, W. N. Polyzou, and E. F. Redish, Partition Combinatories and Multi-Particle Scattering Theory, University of Maryland, report, 1980.

[12]G. Birkhoff, *Lattice Theory,* 3rd ed. (American Mathematical Society, Providence, 1967); G. Szasz, *Introduction to Lattice Theory* (Academic, New York, 1963).

[13]C. Chandler and A. G. Gibson, in *Atomic Scattering Theory,* edited by J. Nuttall (University of Western Ontario Press, London Ontario, 1978), pp. 189–193.

[14]C. Chandler, Nucl. Phys. A301, 1 (1978).

[15]G. R. Plattner, in *Few Body Systems and Nuclear Forces II, Proceedings, Graz, 1978,* edited by H. Zingl, M. Haftel, and H. Zankel, Lecture Notes in Physics (Springer, Berlin, 1978), Vol. 87, pp. 376–392.