PHYSICAL REVIEW C VOLUME 26, NUMBER 5 NOVEMBER 1982

Zero range scattering theory. I. Nonrelativistic three- and four-particle equations

H. Pierre Noyes
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
(Received 22 February 1982)

We prove here that by taking the zero range limit of conventional nonrelativistic three-particle theories and restricting the two-particle amplitude in this limit to have no singularities at negative energy other than bound state poles, we can derive unitary three-particle equations depending only on two-particle physical observables. Phenomenological extensions of this theory suitable for data analysis of systems with three-particle final states and two- and three-cluster reaction theories are briefly discussed. The extension of the theory to four-particle systems is sketched. Nonrelativistic and relativistic applications will be discussed in subsequent papers in this series.

[NUCLEAR REACTIONS 3, 4 particle equations, generalized to N. Restricted to 2 particle observable input, no left hand cuts. Faddeev-Yakubovsky combinatorics.]

I. INTRODUCTION

The research program reported in this series of papers had its origin in the obvious proposition that we can learn more about the dynamics of three-particle systems if we can "subtract" from the three-particle system the consequences of the scatterings and bound states of the two-particle subsystems in a way that preserves asymptotic probability flux and can be uniquely and unambiguously computed using only the probabilities that can be unambiguously determined (in the sense of the law of large numbers) from the statistics of two-particle scattering experiments. In this paper we assume in addition that all "particles" and "bound states" considered can be characterized by unique finite mass values, or failing that in the limit of small mass differences, by parameters which are unambiguously assigned by the "detectors" implied in the reference to asymptotic states.

Once launched, it became clear that this program, if successful, would also find application in elementary particle theory. Thanks to Faddeev it was already clear that the simplest route to asymptotic flux conservation ("unitarity" in the context of a nonrelativistic Hamiltonian theory) requires in the three-particle system the (unobservable for three free-particle states) separation of the wave function into "channels" referring to an "interacting pair" and a "spectator"—an "overcomplete" description. Once this is done using the Faddeev prescription a vertex which opens between two subsystems cannot again close until some novel component of the system has intervened. Hence the "self-energy diagrams" which are a major source of infinities in the quantum field theory approach to elementary particle physics cannot occur if one follows Faddeev's prescription. Further, if the theory is consistently formulated in terms of "free particle" asymptotic states of specified finite mass, one has immediately available a covariant relativistic description of the observables simply by using covariant kinematics, and does not have to enter the vexed question of how the "interaction energy" transforms from one coordinate system to another or of how that quantity is to be "localized" in the laboratory in a manner consistent with the presupposed quantum theory.[1,2] The success of Lindesay in constructing a minimal relativistic three-particle dynamical theory[3] has proved that these hopes were not illusory; consequences will be discussed in subsequent papers in this series.[4]

Despite the simplicity of the physical ideas that launched the program, it ran into technical difficulties, some of which are explored in this paper. In the nonrelativistic context the naive approach is simply to replace the "fully off-shell" scattering amplitude $t(q,\bar{q};z-\tilde{p}^2)$ of the two-body subsystems,[5] which contains the dynamical content of the Faddeev equations, by the "on-shell" (or zero range;

see Sec. II) amplitude $t(z-\tilde{p}^2)$. However, conventional two-particle on-shell scattering amplitudes generated by an exponentially bounded "potential" or extracted by some prescription from a nonrelativistic limit of an elementary particle theory based on the Wick[6]-Yukawa[7] model of finite mass hadronic quanta usually require singularities ("left-hand cuts" in the language of dispersion theory) when the energy argument $\tilde{q}^2$ of the on-shell two-particle amplitude $\tau(\tilde{q}^2+i0^+)$ is analytically continued to negative real values. Since the Faddeev formalism requires the spectator particle energy $\tilde{p}^2$ to range over all positive values, the Faddeev equations become ambiguous in the zero range limit in the presence of left-hand cuts in the model used for the on-shell amplitude.

The naive answer to this technical difficulty is to confine our zero range theory to models that have no left hand cuts. The simplest such model, the scattering length model, has a long history, starting with the proof by Thomas that such a model in the case of attractive interactions would give infinite binding to the triton.[8] Although subsequent work has shown[9] that it is possible to define "zero range" limits in this problem in such a way as to achieve finite results, our approach here differs. As has been shown by Brayshaw,[10] if one uses relativistic kinematics and argues from the requirement that the spectator in a three particle problem should not affect the scattering of the pair, and hence, must be represented by all momenta between zero and infinity when the pair are in their own zero momentum system, then simply transforming these limits to the three particle zero momentum system one finds that the spectator momentum lies between zero and $(M^2-m^2)/2M$, where M is the invariant four momentum and m the mass of the spectator. Thus by using relativistic kinematics the scattering length model becomes well defined without additional argument, and as Lindesay[3] has shown, yields precise quantitative results. Indeed as the scattering length goes to infinity he shows that the nonrelativistic Efimov accumulation of a logarithmically infinite spectrum of three particle bound states is obtained, in quantitative agreement in the appropriate region with nonrelativistic calculations based on separable potentials. For work in the nonrelativistic region, it suffices to simply use a fixed momentum cutoff by taking $M=3m$ (for equal masses) and use nonrelativistic kinematics. Calculations in this approximation for the three nucleon problem made by Orlowski will be presented in the third paper in this series.[11] But the objective of this theory was not to give a new method of meeting the scattering length problem but rather to find a general on-shell theory that could use only empirical input.

For the general case, the "no left hand cut" assumption forces us to part company both with potential theory and with the usual ideas about how particle exchanges are reflected in the analytic structure of on-shell two-particle amplitudes. One reason that this program has taken so long to reach definitive publication was that the author was extremely reluctant to take this step. However, once taken, the theory is at least well defined, and leads to interesting results in both the nonrelativistic[11] and relativistic[3,4] applications already made. We therefore beg the skeptical reader to reserve judgment on the usefulness of this step until he has seen the reasoning that has forced us to it.

Our first step in what follows is to define the zero range limit in both configuration and momentum space, and explicate why well understood physical principles force us to a restricted class of models when we take that limit, which is done in Sec. II. In Sec. III we derive three-particle equations by imposing a finite range boundary condition on the asymptotic form of the three-particle wave function in each Faddeev channel and taking the zero range limit. The resulting equations are identical to the on-shell limit of either the Faddeev or the Karlsson-Zeiger equations under our model assumption. In Sec. IV we show that the three-particle scattering amplitudes calculated from these equations satisfy three-particle unitarity on-shell (i.e., satisfy the physical requirements of asymptotic flux conservation and detailed balance). In Sec. V we show how to introduce "reduced widths" and zero range "three-body forces" into the theory with an eye to data analysis. We sketch a three-cluster multichannel reaction theory. In particular, we also show that if the two-particle subsystems contain only bound states and no scattering states, we can calculate asymptotically unitary "three-particle" amplitudes describing the elastic and rearrangement scattering of a particle and a two-particle cluster which never lead to breakup at any energy. This has obvious application to nuclear reaction theory using cluster models, and in the relativistic version of the theory[4] to be developed subsequently, to fully covariant "constituent" models. Thus we have found a practical way to implement the proposal of Fermi and Yang[12] that the pion be considered to be a "bound state" of a nucleon and an antinucleon. In Sec. VI we show that the same approach leads to well-defined four-particle equations. Assuming that the covariant 4,5, . . . particle equations follow as easily from the nonrelativistic combinatorics as do the three-particle equations, this will open up the exploration of phenomenological covariant constituent quark models, that is, asymptotic quantum chromodynamics (QCD) without gluons, and also, we hope, with confined gluons.

II. THE ZERO RANGE LIMIT

As mentioned in the Introduction, we restrict ourselves here to particles and "bound states" of finite mass. Although our focus in this paper is "nonrelativistic," our aim is to extend our treatment to all hadrons in a covariant way. Hence we will use freely general ideas that come from the broader context of the relativistic quantum mechanics of particles of finite mass. The basic mechanism for scattering is therefore taken from Wick's discussion[6] of Yukawa's meson theory of nuclear forces.[7] If two systems are brought together within some distance r where they can interact coherently during the time δt when they are so localized, special relativity requires that $r \le c\delta t$. By Heisenberg's uncertainty principle $\delta t \approx \hbar/\delta E$. Assume that the interaction is in some sense due to the presence of some particle of mass μ and (from special relativity again) rest energy μc^2. This can only happen if the uncertainty in energy $\delta E \ge \mu c^2$. Hence

$$r \le c\delta t \approx \frac{c\hbar}{E} \le \frac{c\hbar}{\mu c^2} = \frac{\hbar}{\mu c} \; . \tag{2.1}$$

Following Newton we assume that the total momentum of the system must be conserved, but this does not define the relative momentum between the two systems before and after they enter the region of dimension r; we conclude that they will "scatter" in some manner that will be connected to the way they share momentum with μ during the time interval δt. Further, if the energy is high enough, the "hadronic quantum" of mass μ will appear, sometimes, in the final state. Our nonrelativistic restriction precludes this possibility in the discussion of this paper, and hence, requires us to define the scattering process in such a way that the "range" r, and the corresponding degrees of freedom of the mass μ, never enter our equations. This is the physics behind the zero range limit which we now define in the nonrelativistic context.

One consequence of our approach is that we will, at least until the relativistic version of the theory is developed, always be able to restrict ourselves to a finite number of angular momentum states; we also ignore Coulomb effects. Since we end up with a result identical to the on-shell limit of the Faddeev or Karlsson-Zeiger[13] equations, where the consequences of angular momentum conservation have been worked out in complete detail,[5] we will restrict ourselves throughout this paper to the state of zero total angular momentum for three spinless particles scattering only in states with relative angular momentum zero, and with only one bound state in each two-particle channel; the generalization is immediate, and uninteresting except for specific application. Hence the two-particle radial scattering wave function outside the Wick-Yukawa range r will be

$$u_q(y) = e^{i\delta_q} \sin(qy + \delta_q)/q \; ,$$

and satisfies the boundary condition

$$u_q'(r)/u_q(r) = q \cot(qr + \delta_q) \; .$$

Our zero range limit then consists simply of assuming that we can take the limit $r=0$ in this equation, or that

$$\lim_{r\to 0^+} \frac{u_q'(r)}{u_q(r)} = q \cot\delta_q \; . \tag{2.2}$$

For finite r this would be the boundary condition model first proposed by Breit and Bouricius[14] and explored in detail by Feshbach and Lomon[15] and Brayshaw.[16] Our approach differs for reasons discussed below.

Conventional models require the wave function to depart from the asymptotic form inside the range r, and hence, in momentum space, introduce an "off-shell" momentum parameter k in addition to the asymptotic momentum q. It is easy to show[17] that the momentum space wave function then has the form

$$\psi_q(k) = \frac{\delta(q-k)}{qk} - \frac{\tau(\tilde{q}^2 + i0^+)[1 + (\tilde{k}^2 - \tilde{q}^2) f_{q^2}(k)]}{\tilde{k}^2 - \tilde{q}^2 - i0^+} \; , \tag{2.3}$$

where $f_{q^2}(k)$ is real and $\tau(\tilde{q}^2 \pm i0^+)$ is the on-shell two-particle scattering amplitude normalized to

$$\tau^{\pm}(\tilde{q}^2) \equiv \tau(\tilde{q}^2 + i0^{\pm}) = \frac{-ie^{\pm i\delta_q}\sin\delta_q}{\pi\mu(-q^2 \mp i0^+)^{1/2}} \; . \tag{2.4}$$

Here μ is the reduced mass $m_1 m_2/(m_1+m_2)$ and

$q^2=2\mu\tilde{q}^2$. Hence the "half off-shell" amplitude is[17]

$$\tau(k;\tilde{q}^2+i0^+)=\tau(\tilde{q}^2+i0^+)\times[1+(\tilde{k}^2-\tilde{q}^2)f_{q^2}(k)] . \tag{2.5}$$

We note that in this momentum space formalism our zero range limit is simply to take $f_{q^2}(k)=0$.

According to the dispersion theory of two-particle scatterings generated by the exchange of particles of mass greater than or equal to m_x, the on-shell amplitude can always be represented by

$$\tau^{\pm}(\tilde{q}^2)=-\int_0^\infty \frac{k^2\,|\,\tau(\tilde{k}^2)\,|^2 dk}{\tilde{k}^2-\tilde{q}^2\mp i0^+}+\frac{\Gamma^2}{\pi\mu(\tilde{q}^2+\epsilon\pm i0^+)}+\int_{m_x^2/4}^\infty \frac{\rho(\sigma^2)d\sigma^2}{\tilde{\sigma}^2+\tilde{q}^2\pm i0^+} , \tag{2.6}$$

where ϵ is the binding energy of the (single) two-particle bound state. In this dispersion relation the left-hand cut specified by $\rho(\sigma^2)$ must be consistent with the "two-particle unitarity" relation

$$\tau^+(\tilde{q}^2)-\tau^-(\tilde{q}^2)=-2(-q^2-i0^+)^{1/2}\pi\mu\,|\,\tau(\tilde{q}^2)\,|^2 . \tag{2.7}$$

The conventional Faddeev treatment requires us to know instead the "fully off-shell" amplitude $t(q,\bar{q};z-\tilde{p}^2)$ from which the half off-shell amplitude

$$t\{q,[2\mu(z-\tilde{p}^2)]^{1/2};z-\tilde{p}^2\}$$

and the on-shell amplitude $\tau(z-\tilde{p}^2)$ can be obtained. In order that the three particle amplitudes calculated from this driving term satisfy three-particle on-shell unitarity, t must satisfy full off-shell unitarity, and in order that they satisfy time reversal invariance (detailed balance),

$$t(q,\bar{q};z-\tilde{p}^2)=t(\bar{q},q;z-\tilde{p}^2) .$$

We have shown[18] that these requirements can be met knowing only the half on-shell amplitude by invoking the completeness relation or Low equation

$$t(q,\bar{q};z-\tilde{p}^2)=B(q,\bar{q})-\int_0^\infty dk\frac{k^2t^+(q;k^2)t^-(\bar{q},\hat{k}^2)}{\tilde{p}^2+\tilde{k}^2-z}-\frac{\phi_\epsilon(q)\phi_\epsilon(\bar{q})}{\tilde{p}^2-\epsilon-z} . \tag{2.8}$$

The bound state wave functions can be represented in terms of the on-shell normalization and real half off-shell functions $f_\epsilon(q)$, and easily retained, but we omit them below for simplicity in presenting the algebra. The unknown function $B(q,\bar{q})$, which must equal $B(\bar{q},q)$ in order to satisfy time reversal invariance, can be eliminated by putting either q or $\bar{q}$ on shell in Eq. (2.8) and subtracting the two equations to obtain a constraint which must be satisfied if we are to preserve time reversal invariance. After some algebra, this constraint is, in our current notation,

$$\int_{m_x^2/4}^\infty \frac{\rho(\sigma^2)d\sigma^2}{(\tilde{\sigma}^2+\tilde{q}^2)(\tilde{\sigma}^2+\tilde{\bar{q}}^2)}-f_{q^2}(\bar{q})\int_{m_x^2/4}^\infty \frac{\rho(\sigma^2)d\sigma^2}{\tilde{\sigma}^2+\tilde{q}^2}-f_{\bar{q}^2}(q)\int_{m_x^2/4}^\infty \frac{\rho(\sigma^2)d\sigma^2}{\tilde{\sigma}^2+\tilde{\bar{q}}^2}$$

$$=\int_0^\infty k^2\,|\,\tau(k^2)\,|\,dk\left[\frac{f_{k^2}(\bar{q})-f_{q^2}(\bar{q})}{\tilde{k}^2-\tilde{q}^2}+\frac{f_{k^2}(q)-f_{\bar{q}^2}(q)}{\tilde{k}^2-\tilde{\bar{q}}^2}+f_{k^2}(q)f_{k^2}(\bar{q})\right] . \tag{2.9}$$

The constraint so obtained represents the same physics as is discussed by Baranger, Giraud, Mukhopadhyay, and Sauer,[19] but because we have explicitly introduced the dispersion relation (2.6), we obtain a single, *nonsingular* condition. As such, we hope it may prove of use in the construction of "phase equivalent potentials," but this application will not be pursued here. What is important to us is that if we take the zero range limit in this equation ($f=0$), we are left with the requirement that an integral over the left-hand cut weight function vanish, which cannot be met unless $\rho=0$. Thus we are led back to the naive requirement that the $\tau(z-\tilde{p}^2)$ which we obtain in the zero range limit of the Faddeev equations have no singularities for negative arguments other than bound state poles. The physics behind this is that since we are using asymptotic states with the spectator momentum p referring to a free particle, the on-shell requirement $\tilde{q}^2 = W - \tilde{p}^2$, where W is the three-particle energy in the center of mass (c.m.) system normalized to zero at three-particle breakup threshold, forces the energy $\tilde{q}^2$ of the scattering pair to negative values. Consequently our boundary condition (2.2) becomes ambiguous if $\tau(z - \tilde{p}^2)$ has singularities in this region. One might think that by using the Karlsson-Zeiger equations,[13] which refer only to on-shell phase shifts δ_q, with $q^2 \geq 0$ in the zero range limit,[20] this difficulty might be avoided. That this is not the case will be demonstrated in the next section.

Further insight into the problem is provided by noting that if we accept the restriction $\rho = 0$, and have satisfied the constraint on f given by the vanishing of the right-hand side of Eq. (2.9), we can then construct the interaction term $B(q, \bar{q})$ which occurs in the Low equation. But then it is easy to see that $B = 0$ in the zero range limit. In other words we are restricted to solutions of the Low equation which persist in the absence of interactions, a class of Castillejo-Dalitz-Dyson solutions.[21] These are of course a useful representation of elementary particle scattering amplitudes, which tells us that by sticking to the particulate degrees of freedom which occur asymptotically, we are dealing with a nonrelativistic limit of some elementary particle phenomenology. Thus we accept this restriction as consistent with our overall approach.

At first sight we might invoke the well known boundary condition model[14–16] to avoid the left-hand cut in $\tau(z - \tilde{p}^2)$. Unfortunately, the left-hand cut for such amplitudes is replaced by an essential singularity at infinity. As was pointed out to the author by Orlowski, this prevents representation of the amplitude by the usual dispersion-theoretic formula [Eq. (2.6)] with $\rho = 0$, which is required in our treatment below. This creates a serious difficulty in applying our theory to nuclear force problems since the effective range formula and similar simple descriptions of the nucleon-nucleon S wave amplitudes predict an "interaction pole" at about -20 MeV. To avoid this catastrophe we can use[11] the modified effective range formula

$$q \cot\delta = [\alpha + \beta q^2 + \epsilon q^8][1 - q^2/q_0{}^2]^{-1} \times [1 + \lambda q^4]^{-1} , \qquad (2.10)$$

where $q_0{}^2$ is chosen to reproduce the zero in the S phases at around 250 MeV. The results will be discussed in the third paper in this series.[11]

III. DERIVATION OF THE THREE PARTICLE ZERO RANGE EQUATIONS

We have shown elsewhere that if we accept free particle wave functions and make the postulate that there are no hidden variables, we can derive the conventional Goldberger-Watson form of the n-particle scattering wave functions[22] with the important generalization that the scattering amplitudes so defined can describe any *conceivable* scattering process with N_A free, massive particles in and N_B free, massive particles out. That is, we have supplied a kinematics for this part of the S-matrix theory which is independent of the dynamics. Since the amplitudes so defined are *not* the "matrix elements of an interaction," any dynamics used to compute them must be proven to satisfy the physical requirements of flux conservation and detailed balance. We derive the dynamical equations for our special case in this section and prove that they predict unitary three-particle amplitudes in the next section.

We start from an initial state of three free particles scattering only in s waves and project out the $J = 0$ wave function. We will see below that this approach will allow us to calculate the equations for elastic scattering, rearrangement, breakup, and coalescence as well as 3-3 scattering without additional effort. The resulting radial wave function, expressed in terms of the coordinate x_a of the spectator relative to the c.m. of the scattering pair whose relative coordinate is y_a, is (with $\bar{\delta}_{ac} = 1 - \delta_{ac}$)

$$U(x_a, y_a) = \frac{\sin p_a^{(0)} x_a \sin q_a^{(0)} y_a}{p_a^{(0)} q_a^{(0)}} \delta_{ab} - \int_0^\infty p_a{}^2 dp_a \int_0^\infty q_a{}^2 dq_a \frac{M_{ab}(p_a, p_b^{(0)}; z) \sin p_a x_a \sin q_a y_a}{(\tilde{p}_a{}^2 + \tilde{q}_a{}^2 - W - i0^+) p_a q_a}$$

$$- \sum_c \bar{\delta}_{ac} \int_0^\infty p_c{}^2 dp_c \int_0^\infty q_c{}^2 dq_c \left[\frac{M_{cb}(p_c, p_b^{(0)}; z)}{\tilde{p}_c{}^2 + \tilde{q}_c{}^2 - W - i0^+} \frac{1}{2} \int_{-1}^{1} \frac{\sin p_{ca}(\xi) x_a \sin q_{ca}(\xi) y_a}{p_{ca}(\xi) q_{ca}(\xi)} d\xi \right] , \qquad (3.1)$$

where the on-shell condition

$$\tilde{p}_a^{(0)2}+\tilde{q}_a^{(0)2}=W ;$$

$$\tilde{p}_a^{(0)2}=(m_a+m_b+m_c)\frac{p_a^{(0)2}}{2m_a(m_b+m_c)} \tag{3.2}$$

is implied, and our restriction to the asymptotic form is expressed by requiring the amplitudes M to depend only on the spectator momenta p and not on the internal momenta of the scattering pairs q.

The integral over ξ arises from the projection onto the $J=0$ state and the kinematic channel relations[5]

$$\vec{p}_{ca}=-\frac{m_a}{m_a+m_{a\mp}}\vec{p}_c\mp\vec{q}_c;\ \ c=a\pm ,$$

$$\vec{q}_{ca}=\pm\frac{m_aM\vec{p}_c}{(m_a+m_{a\mp})(m_{a-}+m_{a+})}-\frac{m_{a\pm}\vec{q}_c\xi}{(m_{a+}+m_{a-})};\ \ \xi=\frac{\vec{p}_c\cdot\vec{q}_c}{p_cq_c} , \tag{3.3}$$

$$M=m_1+m_2+m_3 ,$$

and the convenient identity

$$\vec{p}_c\cdot\vec{x}_{ca}(\vec{x}_a,\vec{y}_a)+\vec{q}_c\cdot\vec{y}_{ca}(\vec{x}_a,\vec{y}_a)=\vec{p}_{ca}(\vec{p}_c,\vec{q}_c)\cdot\vec{x}_a+\vec{q}_{ca}(\vec{p}_c,\vec{q}_c)\cdot\vec{y}_a , \tag{3.4}$$

which defines the corresponding relations in coordinate space.

If we now replace the dependence on x_a by a dependence on p_a by defining

$$U_{p_a}(y_a)=\frac{2}{\pi}\int_0^\infty dx_a\frac{\sin p_ax_a}{p_a}U(x_a,y_a) \tag{3.5}$$

using

$$\frac{2}{\pi}\int_0^\infty dx_a\frac{\sin p_ax_a\,\sin p_a^{(0)}x_a}{p_ap_a^{(0)}}=\frac{\delta(p_a-p_a^{(0)})}{p_ap_a^{(0)}} , \tag{3.6}$$

our two-particle boundary condition Eq. (2.1) in the three-particle space becomes

$$\lim_{y_a\to0^+}\left[\frac{U'_{p_a}(y_a)}{U_{p_a}(y_a)}=k_a\cot(k_ay_a+\delta_{k_a})\right] , \tag{3.7}$$

where, because of our asymptotic condition, k_a is the on-shell value of q_a defined by

$$k_a=[2\mu_a(W+i0^+-\tilde{p}_a^{\,2})]^{1/2} . \tag{3.8}$$

Using, in addition, the fact that for $y_a>0$

$$\int_0^\infty\frac{dq_aq_a\sin q_ay_a}{\tilde{p}_a^{\,2}+\tilde{q}_a^{\,2}-W-i0^+}=\pi\mu_ae^{ik_ay_a} , \tag{3.9}$$

we find in this way that

$$k_a\cot(k_ar+\delta_{k_a})\left[\frac{\delta(p_a-p_b^{(0)})}{p_ap_b^{(0)}}\delta_{ab}\frac{\sin k_ar}{k_a}-\pi\mu_aM_{ab}(p_a,p_b^{(0)};z)e^{ik_ar}-\sum_{c=a\pm}\bar{\delta}_{ac}\int_0^\infty p_c^{\,2}dp_c\int_0^\infty q_c^{\,2}dq_c\frac{M_{cb}(p_c,p_b^{(0)};z)}{\tilde{p}_c^{\,2}+\tilde{q}_c^{\,2}-W-i0^+}\frac{1}{2}\int_{-1}^1d\xi\frac{\delta(p_a-p_{ca}(\xi))}{p_ap_{ca}(\xi)}\frac{\sin q_{ca}(\xi)r}{q_{ca}(\xi)}\right]$$
$$=\frac{\delta(p_a-p_b^{(0)})}{p_ap_b^0}\delta_{ab}\cos k_ar-ik_a\pi_\mu M_{ab}(p_a,p_b^{(0)};z)e^{ik_ar}-\sum_{c=a\pm}\bar{\delta}_{ac}\int_0^\infty p_c^{\,2}dp_c\int_0^\infty q_c^{\,2}dq_c\frac{M_{cb}(p_c,p_b^{(0)};z)}{\tilde{p}^2+\tilde{q}^2-W-i0^+}\frac{1}{2}\int_{-1}^1d\xi\frac{\delta(p_a-p_{ca}(\xi))}{p_ap_{ca}(\xi)}\cos q_{ca}(\xi)r , \tag{3.10}$$

which, by solving for M_{ab} and taking the $r=0$ limit, gives us immediately that

$$M_{ab}(p_a,p_b^{(0)};z)-\tau_a(z-\tilde{p}_a^{\,2})\delta_{ab}\frac{\delta(p_a-p_b^{(0)})}{p_ap_b^{(0)}}=-\tau_a(z-\tilde{p}_a^{\,2})\sum_{c=a\pm}\int_0^\infty p_c^{\,2}dp_c\int_0^\infty q_c^{\,2}dq_c\frac{M_{cb}(p_c,p_b^{(0)};z)}{\tilde{p}_c^{\,2}+\tilde{q}_c^{\,2}-z}\frac{1}{2}\int_{-1}^1d\xi\frac{\delta(p_a-p_{ca}(\xi))}{p_ap_{ca}(\xi)} . \tag{3.11}$$

This is precisely the zero range limit of the Faddeev equations in the usual theory, as can be seen by performing the integral over the δ function and comparing with Bollé and Osborn (BO), Eq. (3.7) for the $J=0=l=0=\lambda$ case. Hence we have derived our basic equation directly from free particle wave functions and the usual scattering boundary condition by simply imposing the two-particle boundary condition at $y_a=0^+$. We emphasize that nowhere in the theory have we used the concept of interaction, having replaced it by the two-particle on-shell scattering amplitude as observed, appropriately extended to negative energies.

If we had applied the boundary condition on y_b rather than y_a, we would have found a different equation for M_{ab} proportional to τ_b rather than τ_a. To show that these define the same function, it is convenient to define a more symmetric amplitude $Z_{ab}(p_a,p_b;z)$ by

$$M_{ab}(p_a,p_b;z)-\tau_a(z-\tilde{p}_a{}^2)\delta_{ab}\frac{\delta(p_a-p_b)}{p_ap_b}=\tau_a(z-\tilde{p}_a{}^2)Z_{ab}(p_a,p_b;z)\tau_b(z-\tilde{p}_b{}^2) \tag{3.12}$$

and iterate the equations once to obtain

$$\begin{aligned}Z_{ab}(p_a,p_b;z)+\bar{G}^0_{ab}(p_a,p_b;z)&=-\sum_c\int_0^\infty p_c{}^2dp_c\bar{G}^0_{ac}(p_a,p_c;z)\,\tau_c(z-\tilde{p}_c{}^2)Z_{cb}(p_c,p_b;z)\\&=-\sum_c\int p_c{}^2dp_cZ_{ac}(p_a,p_c;z)\tau_c(z-\tilde{p}_c{}^2)\bar{G}^0_{cb}(p_c,p_b;z)\,,\end{aligned} \tag{3.13}$$

where

$$\bar{G}^0_{ab}(p_a,p_b;z)=\frac{\bar{\delta}_{ab}}{2p_ap_b}\ln\frac{\left[\dfrac{p_a{}^2}{2\mu_b}+\dfrac{p_b{}^2}{2\mu_a}+\dfrac{p_ap_b}{m_c}-z\right]}{\left[\dfrac{p_a{}^2}{2\mu_b}+\dfrac{p_b{}^2}{2\mu_a}-\dfrac{p_ap_b}{m_c}-z\right]}\,. \tag{3.14}$$

Using the symmetry thus established it is easy to prove by iteration that these two equations do indeed define the same function, establishing at this level the consistency of our boundary condition approach at zero range. The existence of these two forms is critical for our proof of the unitarity of M_{ab}. They obviously immediately establish the time reversal invariance.

It is instructive at this point to ask what would have happened if we had not gone to the zero range limit, but applied the boundary condition at finite r. As we might expect, the driving term is unaltered, but the kernel acquires the additional factor

$$\left[1+\frac{\cos q_{ac}r\,\sin(k_ar+\delta)}{\sin\delta}-1-\frac{k_a\,\sin q_{ac}r\,\cos(k_ar+\delta)}{\sin\delta}\right], \tag{3.15}$$

where we have grouped the terms to make the $r=0$ limit transparent. Thus the equation remains well defined, but the added term removes the symmetry we needed above to prove the time-reversal invariance of the amplitude. Hence, additional work is needed before we can make a consistent finite range boundary condition model out of this approach. That such a program works has been amply demonstrated by Brayshaw using a different approach in both the nonrelativistic[16] and the relativistic[23] cases, so we do not pursue this question further in this paper.

The factorization of the on-shell amplitude M_{ab} occurs naturally in the Karlsson-Zeiger equation,[13] thanks to Eq. (2.4), even in the conventional theory for the off-shell amplitudes since they start from the half-off shell t matrices, with the trivial difference that we have factored out $\tau(\tilde{q}^2)$ while they factor out the Jost function. From the fact[13] that the amplitudes they define are identical to the Faddeev amplitudes for all physically observable (i.e., three-particle on-shell) three-particle processes, it follows immediately that even the conventional theory can always rigorously be cast into the form of Eq. (3.12) no matter what theory is used to compute

$M_{ab}(p_a,q_a;p_b,q_b;z)$,

although of course Z_{ab} will have another significance and will not be given by Eq. (3.13). As has been pointed out previously[24] this fact could have considerable importance for data analysis, since the rapid variation of the two-particle amplitudes (e.g., resonances) has been factored out leaving only the more smoothly varying function Z_{ab} to be determined from theory or phenomenological fits to the data. This has an advantage over "isobar models" for three-particle final states in that the interference terms between overlapping resonances have an unambiguously defined phase relation, and hence, can yield information difficult to obtain from simpler models. We intend to exploit this fact in subsequent papers in this series.

Having reached this point it should be obvious why we did not have to include the bound state terms in our treatment explicitly. These terms are contained, so far as the primary singularities go, simply by noting that if $\tau(z-\tilde{p}^2)$ contains the correct bound state poles, these primary singularities are correctly given by Eq. (3.11), and for any explicit representation for τ, the elastic scattering, rearrangement, breakup, and coalescence amplitudes can be read out of this equation immediately simply by specifying the appropriate arguments in Z_{ab}.

The route by which Eq. (3.13) for Z_{ab} was originally derived was to take the zero range limit of the Karlsson-Zeiger equations.[20] At first sight this gives a very different result since the Z_{ab} so defined turns out to be

$$Z_{ab}(p_a,p_b;z)+\bar{G}^0_{ab}(p_a,p_b;z)$$
$$=-\sum_{c=a\pm}\bar{\delta}_{ac}\tfrac{1}{2}\int_{-1}^{1}d\xi\int_0^\infty p_c^2dp_c\frac{\Gamma_c^{\,2}Z_{cb}(p_c,p_b;z)}{(\tilde{q}_{ac}^{(2)2}+\epsilon_c)(\tilde{p}_c^{\,2}-\epsilon_c-z)}$$
$$-\sum_{c=a\pm}\bar{\delta}_{ac}\tfrac{1}{2}\int_{-1}^{1}d\xi\int_0^\infty p_c^2dp_c\int_0^\infty q_c^2dq_c\frac{\tau_c^+(q_c^2)\psi^-_{\tilde{q}_c^2}(q_{ac}^{(2)})Z_{cb}(p_c,p_c;z)}{\tilde{p}_c^{\,2}+\tilde{q}_c^{\,2}-z}, \tag{3.16}$$

where

$$\psi^\pm_{\tilde{k}^2}(q)=\frac{\delta(q-k)}{qk}-\frac{\tau^\pm(\tilde{k}^2)}{\tilde{q}^2-\tilde{k}^2\pm i0^+}$$
$$=e^{\pm i\delta_k}\left[\cos\delta_k\frac{\delta(k-q)}{kq}+\frac{2\mathscr{P}}{\pi}\frac{\sin\delta_k}{q^2-k^2}\right] \tag{3.17}$$

and

$$q_{ac}^{(2)2}=p_a^{\,2}+\left[\frac{m_a}{m_a+m_{a\mp}}\right]^2p_{a\pm}^{\,2}+\frac{2p_ap_{a\pm}}{m_a+m_{a\mp}}\xi\,. \tag{3.18}$$

This is puzzling, since we are presumably dealing with the same theory as before, yet the kernel now depends exclusively on binding energies, reduced widths, and two-particle phase shifts in the region where they are physically observed or observable. Hence, once this form is adopted we would seem to have avoided the problem posed by the left-hand cuts in the Faddeev form.

However, if we use the dispersion relation Eq. (2.6) retaining the left-hand cut and perform a sequence of not completely transparent, though simple, algebraic steps,[25] we find that the $t_c(z-\tilde{p}_c^{\,2})$ in Eq. (3.12) becomes replaced by

$$\tau^{KZ}(z-\tilde{p}_c^{\,2};q_{ac}^{(2)})=-\int_0^\infty\frac{k^2|\tau(k^2)|^2dk}{\tilde{p}_c^{\,2}+\tilde{k}^2-z}-\frac{\Gamma^2}{\pi\mu(\tilde{p}_c^{\,2}-\epsilon_c-z)}+\int_{m_x^2/4}^\infty\frac{\rho(\sigma^2)d\sigma^2}{\tilde{\sigma}^2+\tilde{q}_{ac}^{(2)2}}, \tag{3.19}$$

which indeed has no singularities and leaves the resulting equation well defined for $\rho\neq 0$. However, as was pointed out by Karlsson,[26] if we use the equation in the second form $q_{ac}^{(2)}$ becomes replaced by $q_{cb}^{(1)}$ and the two different forms of the equation no longer define the same function, thus once again destroying the time reversal invariance of the theory. The models with no left hand cut are time reversal invariant thanks to the fact[13] that

$$\tilde{p}_a^{\,2}+\tilde{q}_{ac}^{(2)2}=\tilde{p}_b^{\,2}+\tilde{q}_{cb}^{(1)2}=p_a^{\,2}/2\mu_b+p_b^{\,2}/2\mu_a+p_ap_b\xi/m_c\,.$$

At first glance this approach gets us little more than the bald restriction that we found necessary in Sec. II, namely that the theory is consistent only if $\tau(z-\tilde{p}^2)$ has no left-hand cut. But if we think a little about Eq. (3.15), and realize that the kernel does indeed depend only on the phase shifts in the physical region, regardless of how they are computed, we see that we have, in a practical sense, reached our goal. As already remarked, if our theory depends in a critical way on the behavior of the phase shifts at relativistic energies, neither our approach nor the conventional approach makes any sense in the first place. So all we need ensure is that our fitting procedure in using Eqs. (2.9) and (2.10) gives just as good a fit to the phase shifts in the physical region as any conventional model *with* a left-hand cut, and our zero-range equations will be just as good a representation of the zero-range dynamics of the conventional model as for our own choice of amplitude. Of course it would be more satisfactory from a mathematical point of view if we could define our zero range limit in the presence of a left-hand cut and then compute the off-shell effects as a perturbation. It was hoped that the so-called Kowalski-Noyes representation[27] would provide such a theory, but this suffers from the left-hand cut problem once the nonseparable term is dropped even before the zero-range limit is taken, as has been pointed out by Oryu.[28] Nevertheless, we claim that this would be more of a mathematical nicety than a practical matter, and that either Eqs. (3.10), (3.12), or (3.15) can be used for our zero-range theory provided only that *some* reasonable fit in the region of interest can be made by using Eqs. (2.9) and (2.10). Of course for the *rigorous* proof of unitarity in the next section, we will have to rely on Eq. (3.10). That, quantitatively, this model is not a good first approximation for the three-nucleon system has interesting implications, as will be discussed in our third paper in this series.[11]

IV. UNITARITY OF THE THREE-PARTICLE ZERO-RANGE EQUATIONS

As was discussed at the beginning of the last section, any theory such as ours which has no "interactions" but relies instead on observed asymptotic two particle scattering amplitudes with appropriate analytic continuations in the three particle space requires a separate proof of unitarity based explicitly on the dynamical equations used to compute the three-particle amplitudes. Fortunately, as was shown by Freedman, Lovelace, and Namyslowski[29] and discovered independently by Kowalski,[30] if the Faddeev equations hold in both orders, the algebraic form of the equations guarantees three-particle on-shell unitarity provided only the two-particle amplitudes themselves are unitary. As was pointed out some time ago,[31] the energy conserving δ function in the three-particle unitarity relation ensures that for the on-shell Faddeev equations only the two particle unitarity in the physical region is required for the proof. In the reference given[31] the "on-shell Faddeev equations" used $t(\tilde{q}^2)$ rather than $t(z-\tilde{p}^2)$, and do not even define three-particle bound states properly, as was pointed out at the time by Kok.[32] As we now know, this criticism was correct, and as can be seen from the discussion in the last two sections, the arbitrary prescription used in Ref. 31 does not even guarantee time-reversal invariance. However, the formal algebraic proof of unitarity given there is still valid for Eq. (3.11) and its time-reversed partner. Since this statement has been questioned, in particular by Alt,[33] we provide here an explicit version of the Freedman, Lovelace, and Namyslowski (FLN) proof[29] using the integral equations rather than a symbolic representation of them.

Since the wave functions we consider are of the form

$$U_i(x,y)=\frac{\sin p_i x}{p_i}\frac{\sin k_i y}{k_i}-\int_0^\infty p^2dp\int_0^\infty q^2dp\frac{T(p,p_i;z)}{\tilde{p}^2+\tilde{q}^2-W-i\eta}\frac{\sin py}{p}\frac{\sin qy}{q}\,, \tag{4.1}$$

where k is the on-shell value of q given by Eq. (3.8), orthogonality of the wave functions, or flux conservation, requires that

$$\frac{4}{\pi^2}\int_0^\infty dx\int_0^\infty dy\,U_1(x,y)U_2^*(x,y)-\frac{\delta(p_1-p_2)}{p_1p_2}\frac{\delta(q_1-q_2)}{q_1q_2}$$
$$=-\frac{T^*(p_2,p_1;W)}{\tilde{p}_2{}^2+\tilde{k}_2{}^2-W+i\eta}-\frac{T(p_2,p_1;W)}{\tilde{p}_1{}^2+\tilde{k}_1{}^2-W-i\eta}+\int_0^\infty p^2dp\int_0^\infty q^2dq\frac{T(p_2,p;W)T^*(p,p_i;W)}{(\tilde{p}^2+\tilde{q}^2-W)^2+\eta^2}=0\,. \tag{4.2}$$

Hence our three particle unitarity condition is

$$T(p_2,p_1;W)-T^*(p_2,p_1;W)=-\int_0^\infty p^2dp\int_0^\infty q^2dq\,T(p_1,p)\times\left[\frac{1}{\tilde{p}^2+\tilde{q}^2-W-i\eta}-\frac{1}{\tilde{p}^2+\tilde{q}^2+W+i\eta}\right]T^*(p,p_2)\,. \tag{4.3}$$

In evaluating this expression we must take care to remember that, according to our general on-shell expression Eq. (3.12), T has poles at

$$p_{\epsilon_a}=[2n_a(W+\epsilon_a)]^{1/2};\quad n_a=\frac{m_a(m_b+m_c)}{m_a+m_b+m_c}\,, \tag{4.4}$$

which will contribute terms in addition to those coming from the physical three particle states given by

$$2\pi i\delta(\tilde{p}^2+\tilde{q}^2-W)=2i\eta/[(\tilde{p}^2+\tilde{q}^2-W)^2+\eta^2]\,,$$

when $\eta\to0^+$. Defining

$$\lim_{p_a\to p_{\epsilon_a}}(\tilde{p}_a{}^2-\epsilon_a-W)T(p_a,p_b;W)=T^{\epsilon_a}(p_b;W) \tag{4.5}$$

we find that we must evaluate

$$\int_0^\infty p^2dp\int_0^\infty q^2dq\frac{2i\eta}{[(\tilde{p}^2-\epsilon-W)^2+\eta^2]}\,\frac{1}{[(\tilde{p}^2+q^2-W)^2+\eta^2]}=2\pi i\int_0^\infty\frac{q^2dq}{(\tilde{q}^2+\epsilon)^2}=2\pi i\frac{\pi\mu}{\sqrt{\mu\epsilon}}\,. \tag{4.6}$$

Hence our final finite condition is

$$T(p_2,p_1;W)-T^*(p_2,p_1;W)=-2i\pi\mu\left\{\int_0^{\sqrt{2nW}}p^2[2\mu(W-\tilde{p}^2)]^{1/2}\times T(p_1,p_2;W)T^*(p_1,p_2;W)+\pi\sum_a\frac{T^{\epsilon_a}(p_1)T^{\epsilon_a}(p_2)}{2\mu\epsilon_a}\right\} \tag{4.7}$$

from which the unitarity conditions for elastic scattering, rearrangement, breakup, and coalescence can easily be extracted by using Eq. (4.5).

The key to including the bound states in the FLN unitarity proof is to note that the bound state poles in τ, $-\Gamma_a{}^2/\pi\mu(\tilde{p}_a{}^2-\epsilon_a-W-i0^+)$, when multiplied by the spectator wave function $\delta(p_1-p_2)/p_2p_2$, satisfy the three-body unitary condition Eq. (4.3) thanks to Eq. (4.6) provided only that

$$\Gamma_a{}^2=2(2\mu_a\epsilon_a)^{1/2}\,, \tag{4.8}$$

which is indeed the correct normalization of our zero-range bound state wave function, provided it is interpreted to represent exactly two particles. Consequently, when we replace $-2i\pi\mu k\,|\tau|^2$ by $\tau-\tau^*$ in the diagonal term given below, the correct bound state terms appearing in Eq. (4.7) are indeed preserved. In Eq. (4.3), taking proper account of the Faddeev channels, we therefore have that

$$-\sum_{cc'}\int_0^\infty p_c{}^2dp_c\int_0^\infty q_c{}^2dq_cM_{ac}(p_a,p_c;W)\left[\frac{1}{\tilde p_c{}^2+\tilde q_c{}^2-W-i\eta}-\frac{1}{\tilde p_c{}^2+\tilde q_c{}^2-W+i\eta}\right]$$
$$\times\tfrac{1}{2}\int_{-1}^{1}\frac{\delta(p_c-p_{c'c}(\xi))}{p_cp_{c'c}(\xi)}d\xi\,M^*_{c'b}(p_c,p_b;W)$$
$$=\sum_c\int p_c{}^2dp_c\int q_c{}^2dq_c\left[\delta_{ac}\frac{\delta(p_a-p_c)}{p_ap_c}-\sum_{c'}\int_0^\infty p_{c'}{}^2dp_{c'}\int_0^\infty q_{c'}{}^2dp_{c'}\frac{M_{ac}(p_a,p_{c'};W)}{\tilde p_{c'}{}^2+\tilde q_{c'}{}^2-W-i\eta}\right.$$
$$\times\bar\delta_{c'c}\tfrac{1}{2}\int_{-1}^{1}d\xi'\frac{\delta(p_c-p_{c'c})}{q_cq_{c'c}}$$
$$\left.\times\frac{\delta(\tilde p_c{}^2+\tilde q_c{}^2-W)}{k_c}\right]$$
$$\times[\tau_c(\tilde k_c{}^2)-\tau_c^*(\tilde k_c{}^2)]\left[\delta_{cb}\frac{\delta(p_c-p_b)}{p_cp_b}\right.$$
$$-\sum_{c''}\bar\delta_{cc''}\int_0^\infty p_{c''}{}^2dp_{c''}\int_0^\infty q_{c''}{}^2dq_{c''}\tfrac{1}{2}\int_{-1}^{1}d\xi''\frac{\delta(p_c-p_{c''c})}{p_cp_{c''c}}$$
$$\left.\times\frac{M^*_{c''b}(p_{c''},p_b;W)}{\tilde p_{c''}{}^2+\tilde q_{c''}{}^2-W+i\eta}\right]$$
$$-\sum_{cc'}\bar\delta_{cc'}\int_0^\infty p_c{}^2dp_c\int_0^\infty q_c{}^2dq_cM_{ac}(p_a,p_c;W)$$
$$\times\left[\frac{1}{\tilde p_c{}^2+\tilde q_c{}^2-W-i\eta}-\frac{1}{\tilde p_c{}^2+\tilde q_c{}^2-W+i\eta}\right]$$
$$\times\tfrac{1}{2}\int_{-1}^{1}d\xi\frac{\delta(p_c-p_{c'c})}{p_cp_{c'c}}M^*_{c'b}(p_c,p_b;W)\,, \tag{4.9}$$

where we have invoked the channel independence of $\tilde p^2+\tilde q^2$ and have used the two forms of Eq. (3.11) in appropriate order in the replacement on the right-hand side. It is now simply a matter of algebra to see that the δ_{ac} and δ_{cb} give us simply [with a second invocation of Eq. (3.11)]

$$M_{ab}(p_a,p_b;W)-M^*_{ab}(p_a,p_b;W)$$

and that the rest of the terms cancel identically, completing the proof. We trust that the channel form of Eq. (4.7), which follows immediately from the left-hand side of Eq. (4.9) by the same steps as before, need not be written out explicitly. Further, since the angular momentum kinematics for any finite number of states is identical to that in the conventional Faddeev theory, we claim that this generalization is immediate, and that specific formulas are best left to subsequent papers which apply this theory to specific problems.

V. EXTENSION TO CONSTITUENT MODELS AND THREE-PARTICLE FORCES

Since the theory as so far developed completely ignores the effects to be expected from the short-range degrees of freedom predicted by the Wick-Yukawa mechanism, except insofar as they are reflected in the asymptotic two-particle scattering states, we cannot anticipate the equations of the last two sections to be in agreement with experiment. However they do provide, we claim, an unambiguous description of these asymptotic effects in the low energy region. What we must now provide is a practical, and hopefully reasonably unambiguous, way to supplement these equations with parameters that will not destroy the unitarity we have finally achieved, and which can be fitted to experiment. This simply cannot be done in terms of current nonrelativistic theories. As is well known, there is an infinite number of nonrelativistic (nonlocal) "potential models" which will give identical fits, even in a mathematical sense, to the two-particle scattering data. One of the hopes in the early days of the study of the three-body problem was that this ambiguity could be removed, or at least diminished, by comparing different two-particle models with three-particle experiments. However, as was pointed out in 1972, and indeed was a major motivation for this program,[34] in the three particle system the same Wick-Yukawa mechanism which is supposed to generate the two-particle scatterings will necessarily give rise to three-particle forces in the three-particle systems which are, phenomenologically speaking, unpredictable. Hence two-body off-shell effects can be traded off against three-body forces while preserving a fit to three-body experiments and the ambiguity remains, a fact demonstrated in specific contexts by Brayshaw.[16] If instead one tries to compute the interaction to be used in the nonrelativistic Hamiltonian theory ordinarily employed in nuclear physics from elementary particle theory, there was no consensus as of 1960 as to how this is to be done.[35] In this author's opinion that situation has not basically changed, and in fact has been further compounded in recent times by the controversy over "big quark bags" and "little quark bags." So we must find our own route.

One way to introduce fitting parameters into the theory with a well-defined significance is to define

$$\tau_a(z-\tilde{p}^2)=-\frac{N_a{}^2}{\pi\mu_a(\tilde{p}_a{}^2-\epsilon_a-z)}+\hat{\tau}_a(z-\tilde{p}^2)\ . \tag{5.1}$$

This allows us, thanks to Eq. (3.12), to explicitly separate out the primary singularities, and hence the physical amplitudes, following the detailed treatment of Osborn and Bollé.[5] Calling their amplitudes $\mathscr{K}_{ab}$, $\mathscr{G}_{ab}$ and $\mathscr{B}_{0b}$, K_{ab}, G_{ab}, and B_{0b} in our zero-range limit we then find by comparing our Eqs. (3.12) and (5.1) with the OB Eqs. (4.7) and (4.8) that $K_{ab}=N_aZ_{ab}N_b$, $G_{ab}=\hat{\tau}_aZ_{ab}N_b$, and that the physical breakup amplitude is given by [see OB Eq. (1.2)]

$$B_{0b}(p_a,q_a;\vec{\mathrm{p}}_b^{(0)};z)=\sum_a\left[\hat{\tau}_a(\tilde{q}_a{}^2)Z_{ab}(p_a,p_b^{(0)};z)-\frac{N_a{}^2Z_{ab}N_b}{\tilde{q}_a{}^2+\epsilon_a}\right]\ . \tag{5.2}$$

From this identification we can immediately take over their formulas for differential and total cross sections. Provided

$$N_a{}^2=\Gamma_a{}^2=2(2\mu_a\epsilon_a)^{1/2}$$

we see that the "bound-state wave function" in configuration space is

$$\Gamma_a\exp\{-(2\mu_a\epsilon_a)^{1/2}y\}/y$$

and indeed has the correct normalization for a bound state with precisely two particles in a zero-range theory.

However, we claim that we have the freedom to replace Γ_a by a parameter N_a to be fitted to experiment. Why can we do this? In a conventional theory where the bound state has short-range structure, whether due to a potential or to some more complicated degrees of freedom which are not excited asymptotically in the reaction under consideration, N_a is indeed different from Γ_a, and has to be determined either from some microscopic theory, or, for example, by extrapolating physically observed cross sections to the bound-state pole; in that context $N_a{}^2$ is called the "reduced width." It is then a task for both theory and data analysis to prove that $N_a{}^2$ so determined is indeed a unique constant independent of the particular reaction channels used to make the determination. Thus, by introducing this freedom into our own theory, we are not departing from standard practice.

When it comes to physical interpretation, we can say that $(1-f_a{}^2)=(1-N_a{}^2/\Gamma_a{}^2)$ is the fraction of the bound state which, at the level of analysis under consideration, can be thought of as "elementary" and $f_a{}^2$ as the fraction of the bound state which is indeed composite and can contribute to the reaction. This idea was discussed long ago by Weinberg[36] and has been exploited in discussions of n-d scattering both by Aaron, Amado, and Yam[37] using Amado's "nonrelativistic field theory" and by Barton and Phillips[38] in a dispersion theoretic approach. A more satisfactory way of looking at the situation has been suggested by Lindesay.[39] Consider an "elementary" state e_a which is simply a particle of mass $(m_a+m_b-\epsilon_a)/c^2$ that never comes apart and a "composite" zero-range state of the same mass and quantum numbers called ϵ_a, with the

$$\Gamma_a{}^2=2(2\epsilon_a\mu_a)^{1/2}$$

normalization. These two sectors of the theory refer to different particle number and as pure states are separately unitary. However, for detectors that respond only to the mass and quantum numbers, they cannot be experimentally distinguished. If we now form physical states as an incoherent mixture of the two pure states with weights $(1-f_a^2)$ for e_a and f_a^2 for ϵ_a and multiply them by the appropriate spectator wave function, the consequences will be the same as for our *ad hoc* replacement of Γ_a by N_a.

Our next step is to consider the effect of a "three-body force" of zero range, which adds a direct 3-3 scattering channel, which we label by "0" in addition to the channels $a,b,c \in 1,2,3$ already considered. For this channel we express the wave function in terms of the hyper-radius $R^2=\tilde{x}^2+\tilde{y}^2$ and require that

$$U'(R)/U(R)=W^{1/2}\cot\delta_W$$

as $R\to 0$. The conjugate "momentum" is

$$(\tilde{p}^2+\tilde{q}^2)^{1/2}=W^{1/2}$$

on shell, and unitarity is preserved by taking $\tau(W)$ proportional to

$$e^{i\delta_W}\sin\delta_W/W^{1/2}\ .$$

The Faddeev equations are unchanged in form if we allow the sums to run over all four channels, but it is more convenient to eliminate the fourth channel and obtain modified equations for M_{ab} in terms of the original three channels. Explicitly,

$$\begin{aligned}
M_{00}&=\tau_0\left[1-\sum_{a=1}^{3} G_0 M_{a0}\right]\\
&=\left[1-\sum_{a=1}^{3} M_{0a}G_0\right]\tau_0\ ,\\
M_{0a}&=-\tau_0\sum_{a=1}^{3} G_0 M_{ca}\ ,\\
M_{a0}&=-\sum_{c=1}^{3} M_{ac}G_0\tau_0\ ,\\
M_{ab}&=\tau_a\left[\delta_{ab}-G_0M_{0b}-\sum_{c=1}^{3}\bar{\delta}_{ac}G_0M_{cb}\right]\\
&=\tau_a\left[\delta_{ab}+\sum_{c=1}^{3}(G_0\tau_0G_0-\bar{\delta}_{ac}G_0)M_{cb}\right]\\
&=\left[\delta_{ab}+\sum_{c=1}^{3}M_{ac}(G_0\tau_0G_0-G_0\bar{\delta}_{cb})\right]\tau_b\ .
\end{aligned} \tag{5.3}$$

When there is a single three-particle bound state at $W=-\epsilon_0$, we can guarantee fitting this binding with our model simply by taking

$$\tau_0=-N_0^2/(W+\epsilon_0)\ .$$

Solving the homogeneous equation then allows us to compute the "reduced widths" of the breakup of the bound state into the 2+1 channels. An analysis of low-energy *n-d* scattering using this approach is in progress in collaboration with Orlowski. We anticipate good results, since Barton and Phillips[38] have already shown that the on-shell terms in a model similar to ours already give reasonable predictions once the sensitive doublet scattering length is fitted; our freedom in the choice of N_0 will accomplish this.

To extend our approach to general data fitting of systems with three-particle final states, this zero-range three-body force will probably not be sufficient, since it only provides a single parameter, δ_W, at each energy in each total angular momentum state of the three-particle system. Returning to Eq. (3.13) and introducing a matrix notation, since

$$\begin{aligned}
&(1+\bar{G}_0t)Z=-\bar{G}_0=Z(1+t\bar{G}_0)\ ,\\
&(1+Zt)(1+\bar{G}_0t)Z=Z\\
&\qquad=Z(1+t\bar{G}_0)(1+tZ)\ ,
\end{aligned} \tag{5.4}$$

we see that $(1+Zt)$ is the left inverse operator for the first form of the equation and $(1+tZ)$ is the right inverse operator for the second form. Thus, once we have solved the equation for Z, we have the inverse operators, without further effort, directly expressed in terms of Z. Further, since [Eq. (3.12)] $M=t+tZt$, we also can invert the Faddeev equations directly. Hence we can define a new amplitude M' by adding a driving term to the original equation and obtain the solution to the modified equation by quadrature. Explicitly, since

$$\begin{aligned}
&(1+t\bar{G}_0)M=t=M(1+\bar{G}_0t)\ ,\\
&(1-M\bar{G}_0)(1+t\bar{G}_0)M=M\\
&\qquad=M(1+\bar{G}_0t)(1-\bar{G}_0M)\ ,
\end{aligned} \tag{5.5}$$

we take

$$\begin{aligned}
&(1+t\bar{G}_0)M'=t+tXM\ ,\\
&t+MXt=M'(1+\bar{G}_0t)\ ,\\
&M'=M+MXM=M'\ ,
\end{aligned} \tag{5.6}$$

with X arbitrary. The form of the driving term has been chosen asymmetrically in the two equations to guarantee that both define the same M' and hence ensure time reversal invariance. But X must also be chosen so as to guarantee unitarity

$$M'=M'^*=-M(G_0-G_0^*)M'^* , \tag{5.7}$$

$$\begin{aligned} MXM-M^*X^*M^* &= -MXM(G_0-G_0^*)M^* \\ &\quad -M(G_0-G_0^*)M^*X^*M^* \\ &\quad -MXM(G_0-G_0^*)M^*X^*M^* \\ &= MX(M-M^*)+(M-M^*)X^*M^* \\ &\quad +MX(M-M^*)X^*M^* , \end{aligned}$$

or

$$M(X-X^*)M^*=MX(M-M^*)X^*M^* . \tag{5.8}$$

Since M has been shown above to be invertible, our unitarity condition on X thus reduces to

$$X-X^*=X(M-M^*)X^* . \tag{5.9}$$

We can therefore always construct an appropriate parametrization for X.

We have now reached our goal of providing a general method for analyzing three-particle states which explicitly takes account of the known two-particle scattering phase shifts and bound states in the two-particle subsystems. From those we compute, once and for all, Z and hence M. We then introduce in X parameters for those amplitudes not well-fitted by the zero-range model consistent with Eq. (5.9). This allows us to compute M' by quadrature, and by computing observables from M' fit those parameters to experimental three particle observables by a conventional least squares search. In this way we will find an explicit description of whatever in the three-particle system goes beyond the physics already contained in the two-particle scatterings. Applications of this approach will form part of subsequent papers in this series.

To extend our theory to a model for the scattering of more complicated composite structures restricted to initial and final states containing no more than three clusters, we must extend our notation as follows. For concreteness we can take the case of a system with A nucleons, Z protons, and $N=A-Z$ neutrons, but the approach itself is more general. Consider first a state a_i containing Z_{a_i} protons and N_{a_i} neutrons. In general this system will have several levels, so we immediately extend the notation to states to a_i,l_i each of which has a mass $m(a_i,l_i)$. Each of these is considered to be a "pure state" of that mass when acting as a spectator. The residual system will have $\bar{N}_{a_i}=N-N_{a_i}$ neutrons and $\bar{Z}_{a_i}=Z-Z_{a_i}$ protons. This in turn can be decomposed into states $b_{j,l}(a_i,l_i)$ containing N_{b_j} neutrons and Z_{b_j} protons connected to states $c_{k,l}(b_j)$ containing

$$N_{c_k}=\bar{N}_{a_i}-N_{b_j}$$

neutrons and

$$Z_{c_k}=\bar{Z}_{a_i}-Z_{b_j}$$

protons. Out of this complicated description we now select those clusters which we consider significant in any particular physical process we wish to study and write the zero-range equations for each partition of three clusters considered in isolation using Eq. (5.1) with $N_a{}^2=\Gamma_a{}^2$ as the driving term, and a τ_a which satisfies two-particle on-shell unitarity in this restricted environment. As we have already shown, each such system satisfies unitarity in the two- and three-cluster space so generated and can be considered a pure state. Our last step is then to form incoherent mixtures of these pure states to construct the physical states of the reaction theory.

Clearly the articulation of this program raises formidable combinatorial problems, whose solution will not be attempted in this paper. As is well known,[40] the inclusion of spin and the exclusion principle in dynamical equations of the Faddeev-Yakubovsky type is an unsolved problem which must also be faced in order to convert this suggestion into a practical approach to nuclear reaction

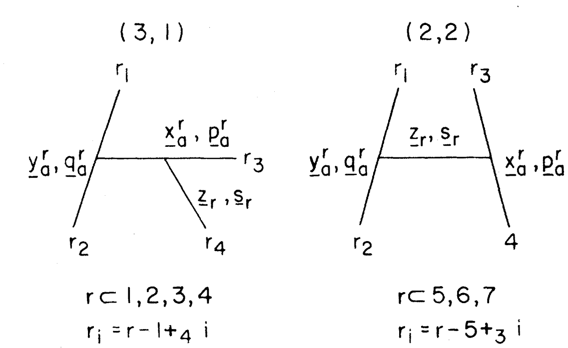

FIG. 1. Geometrical definition of the four particle coordinates used in this study for the (3,1) and (2,2) configurations. Algebraic details are given in Table I.

TABLE I. Four particle coordinates. If the four particle c.m. energy normalized to zero at four particle breakup threshold is called E, then the on shell condition $E=\tilde{p}^2+\tilde{q}^2+\tilde{s}^2$ is configuration and channel invariant. The on shell momentum for the distinguished pair is defined as $k_a^r=[2\mu_a^r(E-\tilde{p}_a^{r2}-\tilde{s}_r^2)]^{1/2}$. The spacial coordinates corresponding to $\vec{p}_a^r$, $\vec{q}_a^r$, and $\vec{s}_r$ are $\vec{x}_a^r$, $\vec{y}_a^r$, and $\vec{z}_r$, respectively. In order to express the four particle wave function in terms of a single set of coordinates we will need to know the geometrical connections $\vec{p}_a^r(\vec{p}_{a'}^{r'},\vec{q}_{a'}^{r'},\vec{s}_{r'})\equiv\vec{p}_{aa'}^{rr'}$, $\vec{x}_a^r(\vec{x}_{a'}^{r'},\vec{y}_{a'}^{r'},\vec{z}_{r'})\equiv\vec{x}_{aa'}^{rr'}$, etc. which are readily obtained from Fig. 1. If we take out the dependence on the orientation of the configurations in space, which can be done by an appropriate application of rotation matrices analogous to that done with care for the three particle case in OB, these transformations will depend on the three direction cosines $(\hat{p}_a^r\cdot\hat{q}_a^r)$, $(\hat{p}_a^r\cdot\hat{s}_r)$, $(\hat{q}_a^r\cdot\hat{s}_r)$ which we symbolize collectively by Ω. The reduction of the plane wave basis $\exp i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y}+\vec{s}\cdot\vec{z})$ to the scalar form used in the text is greatly facilitated by the identity $\vec{p}_a^r\cdot\vec{x}_{aa'}^{rr'}+\vec{q}_a^r\cdot y_{aa'}^{rr'}+\vec{s}_r\cdot\vec{z}_{rr'}=p_{a'a}^{r'r}\cdot\vec{x}_a^r+\vec{q}_{a'a}^{r'r}\cdot\vec{y}_a^r+\vec{s}_{r'r}\cdot\vec{z}_r$.

$$M=\sum_{i=1}^{4}m_i=\sum_{i=1}^{4}m_{r_i}$$

$r\subset 1,2,3,4$		$r\subset 5,6,7$
$M_r=M-m_{r_4};\ M\nu_r=M_r m_{r_4}$		$M\nu_r=(m_{r_1}+m_{r_2})(m_{r_3}+m_{r_4})$
$r_i=r-1+_4 i$		$r_i=r-5+_3 i$
	$s_r^2=2\nu_r\tilde{s}_r^2$	
	$a,b,c\subset r;\ \mu_a^r=m_{r_1}m_{r_2}/(m_{r_1}+m_{r_2})$	
	$(q_a^r)^2=2\mu_a^r(\tilde{q}_a^r)^2$	
	$a=(r_1,r_2);\ b=(r_2,r_3);\ c=(r_3,r_1)$	
$M_r n_a^r=m_{r_3}(m_{r_1}+m_{r_2})$;	$(p_a^r)^2=2n_a^r(\tilde{p}_a^r)^2$	$\bar{a}=(r_3,r_4);\ \bar{b}=(r_1,r_4);$
		$\bar{c}=(r_2,r_4)$
$M_r n_b^r=m_{r_2}(m_{r_2}+m_{r_1})$;	$(p_b^r)^2=2n_b^r(\tilde{p}_b^r)^2$	$\bar{\mu}_a^r=m_{r_3}m_{r_4}/(m_{r_3}+m_{r_4})$, etc.
$M_r n_c^r=m_{r_1}(m_{r_2}+m_{r_3})$;	$(p_c^r)^2=2n_c^r(\tilde{p}_c^r)^2$	$(\tilde{p}_a^r)^2=2\bar{\mu}_a^r(p_a^r)^2$, etc.

theory. But we thought it worthwhile to point out the possibility at this early stage in the hope that others may be attracted by the challenge. The combinatorics are no more difficult than in conventional approaches, and the dynamics are considerably simpler. In particular, if we close the three-cluster channels by taking $\hat{\tau}=0$, we obtain a dynamical theory for two-cluster multichannel problems in which only the binding energies and reduced widths of the composite systems enter, the dynamics arising solely from cluster exchanges. By using relativistic kinematics and an appropriate definition of mesons as massive quanta,[4] we can include the mesons on an equal footing with the nucleons, and construct a theory that correctly describes meson as well as nucleon exchange in the dynamics. This theory becomes considerably more powerful and more fundamental (e.g., when applied to the triton) when we make the extension to four-particle systems sketched in the next section.

VI. FOUR PARTICLE EQUATIONS

We have seen in Sec. III that zero range three particle equations can be derived simply by imposing the zero range boundary condition on each pair starting from the asymptotic form of the three particle wave function. Our approach here is similar and again requires no references to interaction.

In order to extend our treatment to the four particle case we define the (3,1) configurations with $r=1$, 2, 3, 4 and the (2,2) configurations with $r=5$, 6, 7 geometrically in Fig. 1 and algebraically in Table I. We see that, analogous to the treatment by Yakubovsky,[41] we must consider 18 initial and 18 final configurations and construct our theory in terms of the amplitudes F_{ab}^{rt} where the symbols are only defined when $a\subset r$ and $b\subset t$. Starting from a state of four free particles, we project out the state in which all angular momenta are zero and obtain the radial wave function

$$U(x_a^r,y_a^r,z_r)=\frac{\sin p_a^{r(0)}x_a^r}{p_a^{r(0)}}\frac{\sin q_a^{r(0)}y_a^r}{q_a^{r(0)}}\frac{\sin s_r^{(0)}z_r}{s_r^{(0)}}$$
$$-\sum_{r'=1}^{7}\sum_{a'\subset r'}\int_0^\infty p_{a'}^{r'2}dp_{a'}^{r'}\int_0^\infty q_{a'}^{r'2}dq_{a'}^{r'}\int_0^\infty s_{r'}^{2}ds_{r'}\frac{F_{a'b}^{r't}(p_{a'}^{r'},q_{a'}^{r'},s_{r'};E)}{\widetilde{p}_{a'}^{r'2}+\widetilde{q}_{a'}^{r'2}+\widetilde{s}_{r'}^{2}-E-i0^+}$$
$$\times\int d\Omega\frac{\sin p_{a'a}^{r'r}(\Omega)x_a^r}{p_{a'a}^{r'r}(\Omega)}\frac{\sin q_{a'a}^{r'r}(\Omega)y_a^r}{q_{a'a}^{r'r}(\Omega)}\frac{\sin s_{r'r}(\Omega)a_r}{s_{r'r}(\Omega)}\ . \tag{6.1}$$

In order to apply our zero range boundary condition to this wave function, we must first reduce the spatial dependence to the coordinate y_a^r of the distinguished pair, which can be done by Fourier transformation yielding

$$U_{p_a^r s_r}(y_a^r)=\frac{\sin q_a^{r(0)}(y)}{q_a^{r(0)}}\frac{\delta(p_a^r-p_a^{r(0)})}{p_a^r p_a^{r(0)}}\frac{\delta(s_r-s_t^{(0)})}{s_r s_t^{(0)}}\delta_{ab}\delta_{rt}-\pi\mu_r^a F_{ab}^{rt}(p_a,k_a^r,s_r;E)e^{ik_r^a y_r^a}$$
$$-\sum_{r'=1}^{7}\sum_{a'\subset r'}\bar{\delta}_{aa'}\int p_{a'}^{r'2}dp_{a'}^{r'}\int q_{a'}^{r'2}dq_{a'}^{r'}\int s_{r'}^{2}ds_{r'}\frac{F_{a'b}^{r't}}{\widetilde{p}^2+\widetilde{q}^2+\widetilde{s}^2-E}$$
$$\times\int d\Omega\frac{\sin q_{a'a}^{r'r}(\Omega)y_a^r}{q_{a'a}^{r'r}(\Omega)}\frac{\delta(p_a^r-p_{a'r}^{r'r}(\Omega))}{p_a^r p_{a'a}^{r'r}(\Omega)}\frac{\delta(s_r-s_{r'r}(\Omega))}{s_r s_{r'r}(\Omega)}\ ,$$
$$\bar{\delta}_{aa'}=1-\delta_{aa'}\ , \tag{6.2}$$

where we have kept only the asymptotic form of the amplitude corresponding to the distinguished pair, consistent with our zero range assumption, and used the on shell value for q_a^r defined in Table I. Applying our zero range boundary condition $U'/U=k_a^r\cot\delta_a$ in the limit $y_a^r\to0^+$ we find that

$$F_{ab}^{rt}=\tau_a(E-\widetilde{p}_a^{r2}-\widetilde{s}_r^{2})\left[\delta\delta-\sum_{r'=1}^{7}\sum_{a'\subset r'}\bar{\delta}_{aa'}\int R_{aa'}^{rr'}F_{a'b}^{r't}\right]\ . \tag{6.3}$$

Since this equation still contains disconnected scattering processes when $r=r'$, we move these to the left hand side of the equation and obtain

$$\sum_c\left[\delta_{ac}\delta+\bar{\delta}_{ac}\tau_a\int R_{ac}^{r}\right]F_{cb}^{rt}=\tau_a\left[\delta_{rt}\delta_{ab}\delta\delta-\sum_{r'}\bar{\delta}_{rr'}\sum_{a'\subset r'}\bar{\delta}_{aa'}\int R_{aa'}^{rr'}F_{a'b}^{r't}\right]\ . \tag{6.4}$$

By examination of this equation on the left for the (3,1) configurations, we find that this is simply the zero range Faddeev equation $M=t(1-\int\bar{R}M)$ clothed with the momentum conserving

$$\delta(s_r-s_r^{(0)})/s_r s_r^{(0)}$$

of the four particle spectator and with the energy W replaced by $E-\widetilde{s}_r^{\,2}$. But, as proved above, this equation also holds in the time reversed form $M=(1-\int M\bar{R})t$ obtained by applying the boundary condition to the first scattering rather than the last. Hence

$$\left[1-\int M\bar{R}\right]\left[1+t\int\bar{R}\right]M=M$$

providing an inversion of the operator on the left in Eq. (6.4) which when applied makes the driving term in the equation for F_{ab}^{rt} into

$$M_{ab}^{r}(E-\widetilde{s}_r^{\,2})\delta(s_r-s_r^{(0)})/s_r s_r^{(0)}\ .$$

For the (2,2) configurations, the only terms which couple are $F_{a\bar{a}}^{r}$ and $F_{\bar{a}a}^{r}$, establishing immediately that

$$F_{aa}^{r}=t_a(E-\widetilde{p}_a^{r2}-\widetilde{s}_r^{\,2})\delta(p_a^r-p_a^{r(0)})\delta(s_r-s_r^{(0)})/p_a^r p_a^{r(0)}s_r s_r^{(0)}\ .$$

The coupled terms appear to present a problem since neither component of either pair scatters from the other, the spectator momentum factors out, and we anticipate a factored form. The factored solution is immediate in the Schrödinger equation in configuration space, but in the integral equation we get contributions in the iterations to any finite order in the multiple scattering series. Blankenbecler[42] has pointed out to the author that the same problem occurs in the conventional theory; it is mentioned by Mitra, Gillespie, Sugar, and Panchapankesan.[43] However, if we iterate the two pair equation once we find that

$$M^r_{ab}(p^r_a,p^{r(0)}_a;E-s_r{}^2)=\delta_{ab}t_a(E-\tilde{p}^{r2}_a-\tilde{s}_r{}^2)\delta(p^r_a-p^{r(0)}_a)/p^r_a p^{r(0)}_a$$
$$-\frac{t_a(E-\tilde{p}^{r2}_a-\tilde{s}_r{}^{(0)2})t_b(E-\tilde{p}^{r2}_b-\tilde{s}^{(0)2}_r)}{\tilde{p}^{r2}_a+\tilde{p}^{r(0)2}_b+\tilde{s}^{(0)2}_r-E}-\int \bar{R}M, \quad r\subset 5,6,7\,. \tag{6.5}$$

Here we have used the fact that in this configuration $p^r_b=q^r_a$, as can be seen immediately from Fig. 1. But

$$E=\tilde{p}^{r(0)2}_a+\tilde{p}^{r(0)2}_b+\tilde{s}^{(0)2}_r\,,$$

showing that there is an on shell singularity in the first iterate. Hence we can multiply Eq. (6.5) through by this singularity and remove the unwanted multiple scattering term $\int \bar{R}M$. In configuration space this singularity does lead to the factored form

$$t_a t_b e^{ik^r_a y^r_a} e^{ik^r_b y^r_b}$$

as expected. Further, we see that for these configurations we also have Eq. (3.12) with

$$Z_{ab}=-\delta_{\bar{a}\bar{b}}(\tilde{p}^{r2}_a-\tilde{p}^{r(0)2}_a-i0^+)^{-1}\,.$$

Thus we have the Faddeev form for the equations and the algebraic inversion proceeds just as in the (3,1) case.

In the three particle equation we can see explicitly that the factorization of t allows the reduction of the equation to one variable with a geometrical kernel involving an integration over the angle $\cos^{-1}\xi$ between $\hat{p}_a$ and $\hat{q}_a$. All that happens for higher angular momentum states is that we acquire additional rotation matrices as functions of this angle and additional indices which are given explicitly in BO. The reduction occurs because of the δ function for the spectator which puts the two body scatterings in the three particle space. In the four body case we have an extra integration in momentum, which makes the factored form of the three particle equations into a convolution in the four particle case. Hence we obtain by inverting the left-hand side of Eq. (6.4) the two variable equations for the zero range four particle problem

$$^{(4)}M^{rt}_{ab}(p^r_a,s_r;p^t_b,s_b;E)-{}^{(3)}M^r_{ab}(p^r_a,p^t_b;E-\tilde{s}_r{}^2)\delta_{rt}\delta(s^r_a-s^{r(0)}_s)/s_r s^{(0)}_r$$
$$=-\sum_{a''\subset r}\sum_{r'}\sum_{a'\subset r'}\int_0^\infty dp^r_{a''}\,{}^{(3)}M^r_{aa''}(p^r_a,p^r_{a''};E-\tilde{s}_r{}^2)\int_0^\infty dp^{r'}_{a'}\,{}^{(4)}\bar{R}^{rr'}_{a''a'}(p^r_{a''},p^{r'}_{a'};E)\,{}^{(4)}M^{r't}_{a'b}(p^{r'}_{a'},p^t_b;E)\,, \tag{6.6}$$

where

$$^{(4)}\bar{R}^{rr'}_{aa'}(p^r_a,p^{r'}_{a'};E)$$
$$=\bar{\delta}_{rr'}\bar{\delta}_{aa''}\int d\Omega\int_0^\infty dq^{r'}_{a'}\int_0^\infty ds_{r'}(\tilde{p}^{r'2}_{a'}+\tilde{q}^{r'2}_{a'}+\tilde{s}^2_{r'}-E-i0^+)^{-1}$$
$$\times p^{r'2}_{a''}p^{r'2}_{a'}q^{r'2}_{a'}s_{r'}{}^2\delta(p^r_a-p^{r'r}_{a'a}(\Omega))\delta(s_r-s_{r'r}(\Omega))/p^r_a p^{r'r}_{a'a}(\Omega)s_r s_{r'r}(\Omega) \tag{6.7}$$

and we have replaced the F^{rt}_{ab} which refer explicitly to the four particle case by $^{(4)}M^{rt}_{ab}$ with an eye to generalization to the N particle case. Just as in the three particle case, we could obtain an alternative equation by applying our boundary condition to the first scattering rather than the last; that is, we also have the equation

$$^{(4)}M=\left[1-\int {}^{(4)}M\,{}^{(4)}\bar{R}\right]{}^{(3)}M\,.$$

We also have the generalization of Eq. (2), namely

$$^{(4)}M^{rt}_{ab}={}^{(3)}M^r_{ab}\delta_{rt}+{}^{(3)}M^r_{ab}\,{}^{(4)}Z^{rt}_{ab}\,{}^{(3)}M^t_{ab}\,. \tag{6.8}$$

Hence by one iteration of Eq. (6.6) we can obtain an integral equation for the smooth function $^{(4)}Z$ in which the primary singularities have been factored out. Thus knowing Z we can immediately recover all the physical four particle cross sections in a manner strictly analogous to the three particle case discussed in OB. Vanzani showed[44] that the form of our four particle equations is identical to the form of one set of such equations he has developed in the conventional theory,[45] except that the off-shell behavior in his equations requires a convolution over ${}^{(3)}M$ which prevents the factorization we have found in the zero range theory.

Since our theory does not rest on a Hamiltonian model for the interactions, we are required[1,2,34] to prove that the resulting equations are unitary. In the three particle case the unitarity condition

$$M_{ab}-M_{ab}^{*}=-\sum_{cd}M_{ac}(R_0-R_0^{*})M_{db}^{*}$$

follows immediately from the form of the Faddeev equations and the two particle on shell unitarity condition

$$t_a-t_a^{*}=-t_a(R_0-R_0^{*})t_a^{*}\ ,$$

as has been proved in detail in Sec. IV.

We claim that the generalization to the N-particle case is now transparent. We write our N-particle equation in configuration space using the full Faddeev-Yakubovsky combinatorial decomposition and reduce this to a $N-2$ variable equation in the distinguished coordinate. Applying our zero range boundary condition as before, the two particle amplitude factors out. Transferring the appropriate configurations to the left hand side we obtain spectator problems in reduced spaces which can be inverted in the same way that we demonstrated explicitly for the (3,1) and (2,2) configurations above. The driving term on the right now has $N-2$ δ functions rather than 2 before the inversion, and one δ function after the inversion. Hence, just as before, we can obtain $N-2$ variable equations driven by the appropriate analytic continuation of the $N-1$ particle amplitudes. The integral equations that provide these continuations have no singularities other than bound state poles provided only the two particle amplitudes themselves have no such singularities, as already required. Time reversal invariance follows from two forms of the equations as before. Unitarity is immediate from an obvious generalization of the FLN proof. The reduction of the kernel to $N-2$ variables follows from standard applications of angular momentum techniques, which of course become increasingly tedious as the number of particles increases, but which have to be faced in any exact N-particle theory. We therefore claim to have proved that the N-particle zero range equations are always $N-2$ variable equations of the form

$$\begin{aligned}{}^{(N)}M^{C(N)C'(N)}_{C(N-1)\cdots} &= {}^{(N-1)}M^{C(N)}_{C(N-1)\cdots}\\ &\times\left[\delta_{C(N)C'(N)\cdots}-\sum_{C(N'')}\bar{\delta}_{CC''}\sum_{C''(N-1)\subset C''\cdots}{}^{(N)}\bar{R}^{CC''}_{C''(N-1)\cdots}\,{}^{(N)}M^{C''C'}_{C''(N-1)\cdots}\right]\end{aligned}\tag{6.9}$$

and in reverse order. Finally, the essential singularities can always be factored out by an obvious generalization of Eq. (13).

The physics lying behind the simple result we have obtained is that by sticking to two-particle on-shell scatterings of the pairs as the driving mechanism and making the angular momentum reduction, the only variable content left on which these amplitudes can depend, thanks to momentum conservation, is the appropriate analytic continuation to negative energies required by the uncertainty principle. The factorization is quite general for short range interactions as was proved long ago.[17] The simplification was conjectured a decade ago,[34] but could not be proved because of the reluctance of this author to abandon "left hand cuts" in the two particle input, which turns out to be the key to success. In the relativistic generalization of this approach, which we claim to be immediate and which has been shown to work in the three particle case,[3,4] this assumption turns out to be analogous to the "locality" assumption of quantum field theory. Our theory differs in that it can be kept consistently to sectors in which only a finite number of particles enter by using particle functions rather than field functions as the basis. The basic trick in the relativistic generalization[3,4] is simply to assume that "particle" and "quantum" bind to make a state which the same mass and quantum numbers as the particle. As in the nonrelativistic theory presented in this paper, unitarity and time reversal invariance are immediate. "Crossing" and relativistic spin are under investigation.

VII. CONCLUSIONS

We claim to have shown in this paper that by assuming that the two particle on-shell amplitude contains only the physical two-particle on-shell unitarity cut and bound state poles we can derive three and four particle equations which predict physical three and four particle on-shell amplitudes which are rigorously unitary, and are uniquely defined in terms of physical observables, subject to any parametrization that agrees with experiment over a finite energy range and is compatible with our basic restriction. We also show that these equations allow phenomenological extension capable of facing the problems of data analysis for systems with three particles in the final state and of three cluster nuclear reaction theory. The approach used here implies a relativistic generalization which already has produced a covariant model for the two particle, one quantum and particle, antiparticle, and quantum sectors of elementary particle theory describing both elastic scattering and single quantum production firmly grounded in the experimental results obtainable at low energy.[4]

ACKNOWLEDGMENTS

It is hard to make adequate acknowledgment here for help received in a piece of work that has extended over fifteen years. My most recent collaborators have been L. Dodd, whose critical acumen eradicated the last wisps of conjecture, and J. Lindesay who has already reduced the projected relativistic theory to practice.[3] Much is due to the hospitality of E. Schmid at Tübingen, and an award from the Humboldt Stiftung in 1979. The work of E. Zeiger and B. Karlsson was obviously critically important, as was that of T. Osborn at an earlier stage. E. Alt and W. Sandhas were most generous in terms of sympathetic criticism, and I have profited much from discussions with C. Alabiso, L. Biedenharn, H. Haberzettl, L. Kok, K. L. Kowalski, and V. Vanzani. L. Faddeev was helpful at a stage when I was floundering in mathematical subtleties. I could go on to name most of the active researchers in few particle physics who I have met, but this partial list must suffice. This work was also supported by the Department of Energy under Contract DE-AC03-75SF00515.

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