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Critical properties of a generalized pseudospin Hamiltonian

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A two-parameter pseudospin Hamiltonian is introduced which contains both the Meshkov-Glick-Lipkin and the Abecasis-Fässler-Plastino Hamiltonians as special cases. The ground state critical properties of the two-parameter Hamiltonian are determined: It exhibits both first and second order phase transitions. The ground state critical properties of any one-parameter pseudospin Hamiltonian contained in the two-parameter model can immediately be determined by inspecting the interaction parameter space of the two-parameter model. The critical set in this space is identified with that of the cusp catastrophe.

[NUCLEAR STRUCTURE Pseudospin Hamiltonian, critical properties, first and second order phase transitions.]

I. INTRODUCTION

The Meshkov-Glick-Lipkin (MGL) pseudospin Hamiltonian[1] (1a) and the Abecasis-Fässler-Plastino[2] (AFP) pseudospin Hamiltonian (1b)

$$\mathscr{H}_{\mathrm{MGL}}=\epsilon J_Z+\tfrac{1}{2}V(J_+{}^2+J_-{}^2)\ , \tag{1a}$$

$$\mathscr{H}_{\mathrm{AFP}}=\epsilon J_z+\tfrac{1}{2}V'(J^2-J_x{}^2+J_x)\ , \tag{1b}$$

have been studied for their ground state critical properties. Studies in the MGL model have been carried out both numerically[1,3] and analytically.[3] The model exhibits a second order ground state energy phase transition as the parameter V describing the strength of monopole coupling increases through the critical value $|V_c|=\epsilon/N$. At this value of the interaction parameter, the ground state energy $E_g(V)$ exhibits a discontinuity in its second derivative in the thermodynamic limit $N\rightarrow\infty$. Studies on the AFP model were carried out numerically because of the difficulty of analytic treatment.[4] A rapid variation in $\partial^2E_g(V')/\partial V'^2$ was observed for $NV'\sim\epsilon$, and it was inferred that the AFP model exhibits a ground state energy phase transition similar to that of the MGL model.[4]

Both the MGL and the AFP pseudospin models can be regarded as special cases of a slightly more general two-parameter pseudospin model

$$\mathscr{H}_{\mathrm{cusp}}/N=\epsilon(J_z/N)+a\,(J_x/N)^2+b\,(J_x/N)\ . \tag{2}$$

The parameter ϵ may be set equal to 1 by appropriately scaling the energies. Pseudospin operators J always appear in the combination (J/N) for well defined thermodynamic reasons.[5,6] Neither of the original models[1,2] satisfied the thermodynamic criteria, although the Hamiltonian used for studies of the ground state critical properties did.[3] The MGL Hamiltonian (1a) can be seen to be a special case of (2) by setting $b=0$ and rotating the coordinate system of (1a) through $\pi/4$ radians about the z axis. The AFP Hamiltonian (1b) can be seen to be a special case of (2) by dropping the term $\frac{1}{2}V'J^2$, which simply adds a constant energy to all states in the ground state manifold. It is this term which accounts for the increase[4] in the ground state's energy $E_g(V')$ as V' increases, rather than the decrease that is observed in the MGL model and that is expected in realistic models of physical systems.

The pseudospin model (2) is simple to treat analytically. As the subscript suggests, it is closely related to the cusp catastrophe.[7,8] As a result, the critical properties of (2) are simply those of the cusp catastrophe, which are well documented.

II. GROUND STATE CRITICAL PROPERTIES

To determine the ground state critical properties of (2), we proceed according to a rigorous algorithm which is derived from coherent states and a classical limit[8,9]

(i) Replace $J_\pm/N$ by $\frac{1}{2}\sin\theta e^{\pm i\phi}$ and replace J_z/N by $\frac{1}{2}\cos\theta$.

(ii) Minimize the resulting function for a fixed value of the parameters (a,b). The minimum value is the ground state energy per particle, E_g/N.

(iii) Search for discontinuities in the ground state energy, or its various derivatives, as the interaction parameters (a,b) are varied.

This algorithm determines the ground state critical properties in the thermodynamic limit $N\to\infty$. The ground state critical properties for finite N are identical, but phase transitions (if any) occur at renormalized values of the interaction parameters, i.e.,

$$a_C(N)/a_c(N\to\infty)=1+\mathscr{O}(N^{-1}),$$
$$b_c(N)/b_c(\infty)=1+\mathscr{O}(N^{-1}) .$$

Applying the first step of this algorithm to the Hamiltonian (2) leads to the following function, defined over the sphere surface ($0\le\theta\le\pi$, $0\le\phi\le 2\pi$):

$$\langle \mathscr{H}/N\rangle=\tfrac{1}{2}\cos\theta+a(\tfrac{1}{2}\sin\theta\cos\theta)^2+b(\tfrac{1}{2}\sin\theta\cos\phi) . \tag{3}$$

The minima of (3) are determined by the usual methods:

$$\frac{\partial}{\partial\phi}\langle \mathscr{H}/N\rangle=\tfrac{1}{2}(-\sin\phi)(a\sin^2\theta\cos\phi+b\sin\theta)=0 . \tag{4}$$

This has solutions

$$\cos\phi=\pm 1 \tag{5a}$$

and

$$\cos\phi=\frac{-b}{a\sin\theta} \text{ if } \left|\frac{b}{a\sin\theta}\right|\le 1 . \tag{5b}$$

Differentiating with respect to θ leads to

$$\frac{\partial}{\partial\theta}\langle \mathscr{H}/N\rangle=-\tfrac{1}{2}\sin\theta+\frac{a}{2}\sin\theta\cos\theta\cos^2\phi+\frac{b}{2}\cos\theta\cos\phi=0 . \tag{6}$$

If ϕ determined by (5b), then (6) reduces to

$$\frac{\partial}{\partial\theta}\langle \mathscr{H}/N\rangle=-\tfrac{1}{2}\sin\theta=0 . \tag{7}$$

This violates the condition stated in (5b), so that $\cos\phi=\pm 1$. As a result of seeking a minimum, $b\cos\phi=-|b|$. Thus, the expectation value (3) reduces to

$$\langle \mathscr{H}/N\rangle=\tfrac{1}{2}\cos\theta+\tfrac{1}{4}a\sin^2\theta-\tfrac{1}{2}|b|\sin\theta . \tag{8}$$

In the half planes $b\ge 0$ and $b\le 0$, the function (8) is locally diffemorphic with the cusp catastrophe. This is most easily seen by expanding (8) up to fourth order in $(\theta-\pi)$. As a result, the ground state energy depends smoothly on the interaction parameters (a,b) except possibly for $b=0$.

To determine the continuity properties of $E_g(a,b)$, we set $b=0$ in (8) (MGL model). It is easily verified that (8) is minimized by

$$\begin{aligned}&\theta=\pi, \quad a\ge -1\\ &\cos\theta=+1/a, \quad a\le -1\end{aligned} \tag{9}$$

and that a second order ground state energy phase transition occurs at $(a,b)=(-1,0)^3$.

To compute the critical properties elsewhere on the symmetry axis $b=0$, we compute $N^{-1}\partial E_g(a,b)/\partial b$ in the right half plane $b\ge 0$, and evaluate the result at $b=0$:

$$\frac{1}{N}\frac{\partial}{\partial b}E_g(a,b)=\left[\tfrac{1}{2}\sin\theta+\tfrac{1}{2}a\sin\theta\cos\theta-\frac{b}{2}\cos\theta\right]\frac{\partial\theta}{\partial b}-\tfrac{1}{2}\sin\theta . \tag{10}$$

The expression within parentheses vanishes at the minimum by (6). The remaining term, $-\frac{1}{2}\sin\theta$, vanishes at $b=0$ for $a\ge -1$, and is $-\frac{1}{2}(1-a^{-2})^{1/2}$ for $a\le -1$ by (9). As a result, there is a discontinuity in $\partial E_g(a,b)/\partial b$ on crossing the half line $a<-1$, $b=0$:

$$\Delta\left[\frac{\partial E_g}{\partial b}\right]=\frac{\partial E_g(a,b=0^+)}{\partial b}-\frac{\partial E_g(a,b=0^-)}{\partial b}=-(1-a^{-2})^{1/2} . \tag{11}$$

The critical properties of the pseudospin model Hamiltonian are summarized in Fig. 1.

III. DISCUSSION

Model (2) is structurally stable[7,8] in the sense that any perturbation of it obtained by the inclusion of additional pseudospin operators (e.g., $\{J_z,J_x\}$) will not qualitatively change its critical properties. Model (2) is canonical in the sense that its classical limit is diffeomorphic with the canonical mathematical function called the cusp catastrophe.[8]

In the construction of pseudospin models exhibiting ground state energy phase transitions, it is sufficient to study two-parameter models (the cusp catastrophe is a canonical two-parameter function). One parameter must be coupled to an operator exhibiting at least a twofold symmetry: In the present case $J_x{}^2\to(-J_x)^2$. The other parameter must be coupled to a symmetry-breaking term: In the present case $J_x\to -J_x$. Thus, on one hand model (2) is a symmetry-breaking extension of the MGL model, and on the other hand it is a direct pseudospin operator analog of the canonical cusp catastrophe[8]:

$$J_z+aJ_x{}^2+bJ_x\leftrightarrow x^4+ax^2+bx . \tag{12}$$

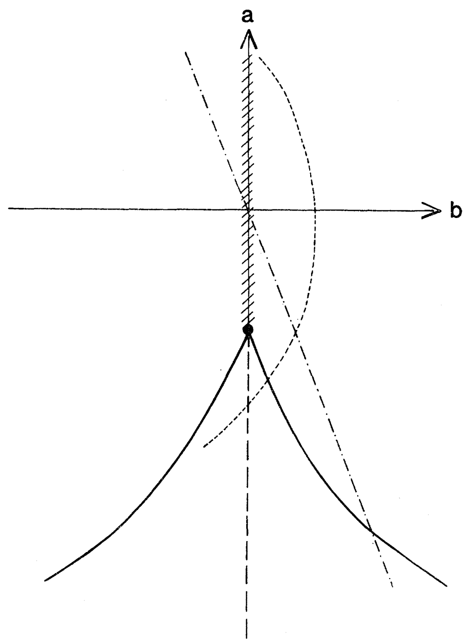

FIG. 1. The interaction parameter space (a,b) describes the critical properties of the canonical pseudospin model (3) and all of its one-parameter submodels. The cusp point (●) describes a second order phase transition. The long dashed half-line ($a<-1$, $b=0$) describes the locus of first order phase transitions. The solid fold lines $(a+\frac{1}{3})^3=(b/2)^2$ describe the spinodal lines [limits of metastability (Ref. 10)] of the model. One parameter submodels of $\mathscr{H}_{\text{cusp}}(a,b)$ are described by curves $[a(s),b(s)]$ (— — —) in the interaction parameter space. If the curve intersects the critical set (— — or ●) the model exhibits a (first or second order) phase transition. Generically, one parameter trajectories will not pass through the cusp point $(-1,0)$ unless the model itself is symmetry restricted ($b=0$). This is just the MGL case. The parameter trajectory for the AFP model is the —·—·—· line $(a,b)=(NV'/2,V'/2)$, which minimizes the critical set. This model does not exhibit phase transitions, although it will have excited metastable states for sufficiently large values of the interaction parameter V'. The parameter trajectory for the MGL model is along the line, ////, the cusp point ●, and the long dashed line. This corresponds to a trajectory along the $b=0$ line.

The catastrophe function x^4+ax^2+bx [right-hand side of (12)] exhibits both first and second order phase transitions, as does the pseudospin Hamiltonian $\mathscr{H}_{\text{cusp}}$ [left-hand side of (12)].[8] It is remarkable that the ground state critical properties of the two-parameter pseudospin Hamiltonian (2) are more easily accessible than the critical properties of a one-parameter special case (1b) of this model. In fact, catastrophe theory was developed in part to effect such simplifications.[7,8]

The particular cases (1a) and (1b) of (2) are indicated in Fig. 1. Parameter variations in the symmetry-restricted MGL model may drive the model through the cusp point $(a,b)=(-1,0)$. In this case, a second order ground state energy phase transition occurs. Parameter variations in the AFP model drive this model along a straight line through the origin. This line does not cross the half line $a\le -1$, $b=0$, so there are no discontinuities in the ground state energy or any of its derivatives as a function of the interaction parameter V'.[8] The rapid variation in $\partial^2 E_g(V')/\partial V'^2$ observed previously is due entirely to the nonthermodynamic scaling[5,6] used in the original AFP model (1b).

A large number of one-parameter special cases of (2) can be studied simply by allowing the parameters a,b to be functions of a single parameter s. The corresponding model is

$$\mathscr{H}(s)/N=J_z/N+a(s)(J_x/N)^2+b(s)(J_x/N)\ . \tag{13}$$

The ground state critical properties of (13) are completely determined simply by following the path $[a(s),b(s)]$ in the interaction-parameter plane. This path will generically miss the cusp point $(-1,0)$ unless the model is symmetry restricted ($b=0$), in which case the MGL model is recovered. Model (13) will then have a first order phase transition if the path $[a(s),b(s)]$ crosses the half line $a<-1$, $b=0$; otherwise, it will not exhibit any phase transitions.

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