Nonresonant approximations to the optical potential

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A new class of approximations to the optical potential, which includes those of the multiple-scattering variety, is investigated. These approximations are constructed so that the optical potential maintains the correct unitarity properties along with a proper treatment of nucleon identity. The special case of nucleon-nucleus scattering with complete inclusion of Pauli effects is studied in detail. The treatment is such that the optical potential receives contributions only from subsystems embedded in their own physically correct antisymmetrized subspaces. It is found that a systematic development of even the lowest-order approximations requires the use of the off-shell extension due to Alt, Grassberger, and Sandhas along with a consistent set of dynamical equations for the optical potential. In nucleon-nucleus scattering a lowest-order optical potential is obtained as part of a systematic, exact, inclusive connectivity expansion which is expected to be useful at moderately high energies. This lowest-order potential consists of an energy-shifted $(t\rho)$-like term with three-body kinematics plus a heavy-particle exchange or pickup term. The natural appearance of the exchange term additively in the optical potential clarifies the role of the elastic distortion in connection with the treatment of these processes. The relationship of the relevant aspects of the present analysis of the optical potential to conventional multiple scattering methods is discussed.

[NUCLEAR REACTIONS Approximate nuclear optical potential. Pauli principle in nuclear reaction theory. Multiple scattering approximations. Optical potential for nucleon-nucleus scattering.]

I. INTRODUCTION

Although the optical potential (OP) was introduced to describe elastic nucleon-nucleus scattering more than three decades ago, it has never been quite adequately described in nuclear reaction theory until very recently.[1,2] The main obstacle has always been the introduction of the Pauli principle in a satisfactory manner. For example, in the multiple scattering formalisms of Watson[3,4] and others,[5] the OP is defined *only after* sufficient approximations are introduced to reduce the problem to one which has the same form as the case without projectile-target antisymmetry so that it has been impossible to compare the exact and approximate optical potentials. Evidently, it is now possible to rectify this and our principal objective is to do this for a class of approximations including those of the multiple scattering type which we call *nonresonant.*

The Watson and Kerman-McManus-Thaler (KMT) formalisms have several intrinsic shortcomings when nucleon identity is taken into account (Appendix A). Not the least of these is the generation of optical potentials which include elastic flux. On the other hand, the theory of Refs. 1 and 2 possesses some serious apparent inadequacies of its own especially with regard to the convincing development of practical approximations. For example, it is not clear how one can recover $(t\rho)$-like approximations with "dressed" or energy-shifted two-nucleon transition operators. Also exchange terms appear whose physical significance is unclear.

In this article we develop, under what seem to be plausible assumptions, an approximation sequence for the antisymmetrized OP based on the theory of Refs. 1 and 2, which possesses all of the practical advantages of the Watson and KMT formalisms, but is free of their disadvantages. In particular, the absence of elastic flux in the OP is preserved *throughout* our approximation sequence. Also, we give a physically motivated statement of the impulse/closure approximation, which is used to obtain a definitive prescription for the energy-shifted two-nucleon transition operators with correct recoil-preserving kinematics and we demonstrate how the heavy-particle exchange potential appears in the low-order OP in a natural manner. We give similar prescriptions for the contributions of two- and higher-nucleon correlations. We are not aware of any previous treatment of the OP for nucleon-nucleus scattering with all of the preceding attributes.

The theory of the OP introduced in Refs. 1 and 2 is distinctive in three respects:

(1) A new definition of the antisymmetrized OP is proposed in Refs. 1 and 2, which is consistent with the two-body description of elastic two-fragment scattering. This definition explicitly incorporates the dependence of the OP upon the choice of off-shell extension for the multichannel transition operators. The recognition of this dependence is the key ingredient missing from previous work.

(2) A specific choice[6] of off-shell extension is shown to lead to an OP without elastic unitarity cuts. This guarantees, e.g., that the OP is real below the inelastic threshold although the unitarity property is, in general, important at all energies.

(3) A set of dynamical equations is obtained for determining the OP in a consistent manner as well as for constructing approximations which satisfy property 2.

All other methods for the antisymmetrized OP proposed before or since Refs. 1 and 2 fail to satisfy at least one of properties 1–3; the original multiple scattering theories,[3–5] e.g., fail on all three counts. It is emphasized in Ref. 1 that property 1 is formalism independent, while properties 2 and 3 represent constraints, which sharply delimit the ways one can go about formulating the problem. For instance, properties 1 and 2 can be realized using the methods of conventional nuclear reaction theory.[7] However, with such techniques it seems difficult to satisfy property 3 because the dynamical equations in this case are Lippmann-Schwinger-type equations, which possess unsatisfactory solution characteristics in contrast to the connected-kernel equations of Refs. 1 and 2. It is not easy to construct approximate OP operators which possess the correct unitarity properties using Lippmann-Schwinger dynamical equations.[7–9]

Several arguments are presented in Refs. 1, 2, and 7 in support of the choice of the Alt-Grassberger-Sandhas (AGS)[6] off-shell extension in connection with the OP for elastic two-fragment nuclear scattering. The more frequently used prior off-shell extension, e.g., is known to yield unwanted elastic singularities in the OP.[7] Subsequently, some distinctive features were claimed for the OP based on the prior off-shell extension and the definition of Refs. 1 and 2.[10] After a number of further analyses,[8,9,11,12] it appears that the prior off-shell extension possesses only one noteworthy feature, viz., it leads to an approximation sequence for the OP, wherein the lowest-order term is the same as in Ref. 5. However, this may not be a desirable quality (see Appendix A). There are now many indications (Refs. 1,2,7–9,11–13) that the AGS choice is the appropriate one for describing the nuclear OP, and, therefore, we adhere to this choice throughout this article in accord with Refs. 1 and 2.

II. ANTISYMMETRIZED OPTICAL POTENTIAL

Our treatment of the OP is based upon the antisymmetrization techniques of Bencze and Redish[14] and on sets of connected-kernel scattering integral equations which possess a multiple-scattering structure. The antisymmetrization formalism we employ is reviewed in Refs. 2, 7, 8, 15, and 16, while the relevant connected-kernel techniques are developed in Refs. 2 and 15–18. In this section we review the relevant aspects of this work.

The antisymmetrized transition operator for elastic scattering, $T(\hat{\beta})$, is related to the OP operator, $\mathscr{V}_{\rm opt}(\hat{\beta})$, by the two-body integral equation

$$T(\hat{\beta})=\mathscr{V}_{\rm opt}(\hat{\beta})+\mathscr{V}_{\rm opt}(\hat{\beta})G_{\bar{\beta}}(z)P_{\bar{\beta}}T(\hat{\beta}) \ . \quad (2.1)$$

Here $\hat{\beta}$ refers to the equivalence class of two-cluster partitions $\alpha,\beta,\lambda,\ldots,$ which are related to each other by permutations. The amplitude for the elastic scattering of the two composite fragments which correspond to any one, say $\bar{\beta}$, partition contained in $\hat{\beta}$ is

$$\langle \phi_{\bar{\beta}}(\vec{k}_f) \,|\, T(\hat{\beta}) \,|\, \phi_{\bar{\beta}}(\vec{k}_i) \rangle \ , \quad (2.2)$$

where $|\phi_{\bar{\beta}}(\vec{k})\rangle$ is the two-fragment ground-state wave function with relative momentum $\vec{k}$, which is an eigenstate of the channel Hamiltonian $H_{\bar{\beta}}$. The projector onto the space spanned by $\{|\phi_{\bar{\beta}}(\vec{k})\rangle\}$ is denoted as $P_{\bar{\beta}}$. Also $G_{\beta}(z)=(z-H_{\beta})^{-1}$ and we suppose that $z=E+i0$, as is appropriate for scattering. We remark that both $T(\hat{\beta})$ and $\mathscr{V}_{\rm opt}(\hat{\beta})$ depend on the choice of $\bar{\beta}$, but matrix elements of the form (2.2) do not.[17] Finally, we require that

$$R_{\bar{\beta}}\,|\,\phi_{\bar{\beta}}(\vec{k})\rangle=|\,\phi_{\bar{\beta}}(\vec{k})\rangle \ , \quad (2.3)$$

where $R_{\bar{\beta}}$ is the antisymmetrizer (internal to the fragments) with respect to all permutations which map $\bar{\beta}$ into itself.

In Refs. 1 and 2 it is shown that $\mathscr{V}_{\rm opt}(\hat{\beta})$ satisfies the dynamical integral equation

$$\mathscr{V}_{\rm opt}(\hat{\beta})=B(\hat{\beta})+C(\hat{\beta})\mathscr{V}_{\rm opt}(\hat{\beta}) \ . \tag{2.4}$$

The detailed structures of the operators $B(\hat{\beta})$ and $C(\hat{\beta})$ are presented in Ref. 2. In Appendix B we show that all of the subsystem scattering implicit in $B(\hat{\beta})$ and $C(\hat{\beta})$ takes place only in the properly antisymmetrized subspaces, and that $B(\hat{\beta})$ and $C(\hat{\beta})$ are explicitly free of $\hat{\beta}$-class elastic unitarity cuts. The remarkable aspect of these attributes is that they are *easily* maintained for a large class of approximations. Thus, we can construct approximate operators $\mathscr{V}_{\rm opt}(\hat{\beta})$, which possess no elastic unitarity cuts correponding to any of the Pauli equivalent channels $\beta\in\hat{\beta}$. The unitary content of $\mathscr{V}_{\rm opt}(\hat{\beta})$ then reflects only the flux passing into inelastic channels, which is generally agreed to be a characteristic of a properly constructed OP formalism. By way of contrast, we note that with conventional dynamical equations for $\mathscr{V}_{\rm opt}(\hat{\beta})$, even with the favorable unitary characteristics for the AGS off-shell extension, it is very difficult to maintain the correct unitarity structure of $\mathscr{V}_{\rm opt}(\hat{\beta})$ except for special approximations.[8,9,11,13]

In the application of (2.4) one is only interested in matrix elements of $\mathscr{V}_{\rm opt}(\hat{\beta})$ of the form (2.2) or, equivalently, the operator $\mathscr{V}\equiv P_{\bar{\beta}}\mathscr{V}_{\rm opt}(\hat{\beta})P_{\bar{\beta}}$. Since

$$C(\hat{\beta})P_{\bar{\beta}}=C(\hat{\beta}) \ , \tag{2.5}$$

(2.4) can be reduced to an integral equation for $\mathscr{V}$. Given the input $B(\hat{\beta})$ and $C(\hat{\beta})$ the solution of the resultant single-vector variable integral equation is relatively trivial.[2] It is easily shown that $P_{\bar{\beta}}C(\hat{\beta})$ is a connected operator, so $[C(\hat{\beta})]^2$ is connected and in all cases we can regard (2.4) as a connected-kernel equation.

The nonorthogonality term[19]

$$\mathscr{N}\equiv\sum_{\beta\in\hat{\beta}}\mathscr{R}_\beta\bar{\delta}_{\beta,\bar{\beta}}P_{\bar{\beta}} \tag{2.6}$$

is ubiquitous in the AGS theory of the antisymmetrized OP (Refs. 1,2,7–9,11–13,15). We remark[9] that the projected operator,

$$\begin{aligned}\bar{\mathscr{N}}&\equiv P_{\bar{\beta}}\mathscr{N}\\ &=\sum_{\beta\in\hat{\beta}}\mathscr{R}_\beta\bar{\delta}_{\beta\bar{\beta}}P_{\beta}P_{\bar{\beta}}\end{aligned} \tag{2.7}$$

is a sum of bounded, connected operators, $P_\beta P_{\bar{\beta}}$, $\beta\neq\bar{\beta}$.

The operators

$$B\equiv P_{\bar{\beta}}B(\hat{\beta})P_{\bar{\beta}} \ , \tag{2.8}$$

$$C\equiv P_{\bar{\beta}}C(\hat{\beta})P_{\bar{\beta}} \ , \tag{2.9}$$

can be written in the respective forms (Ref. 2, Appendix B)

$$B=\bar{\mathscr{N}}G_{\bar{\beta}}{}^{-1}+B_0 \ , \tag{2.10}$$

$$C=-\bar{\mathscr{N}}+C_0 \ . \tag{2.11}$$

The preceding segregation of the linear $\bar{\mathscr{N}}$ dependence is nontrivial in that B_0 and C_0 are dynamical while $\bar{\mathscr{N}}$ is not. That is, in the limit of no interfragment interactions B_0 and C_0 vanish but $\bar{\mathscr{N}}$ does not. Evidently

$$\mathscr{V}=B+C\mathscr{V} \ . \tag{2.12}$$

In the next two sections we propose some specific methods for generating approximate solutions of (2.12). The structures of B_0 and C_0 are such that it is not obvious how multiple-scattering approximations, e.g., can be generated. Even more mysterious is the role of the nonorthogonality term $\bar{\mathscr{N}}$, particularly when one wants to use $\mathscr{V}$ to obtain the scattering amplitude from

$$T=\mathscr{V}+\mathscr{V}G_{\bar{\beta}}T \ , \tag{2.13}$$

where

$$T\equiv P_{\bar{\beta}}T(\hat{\beta})P_{\bar{\beta}} \ . \tag{2.14}$$

III. NONRESONANT APPROXIMATIONS

Let us first clarify the role of $\bar{\mathscr{N}}$ in the $\mathscr{V}$ integral equation (2.12) by taking the nondynamical limit $B_0=C_0=0$. In this case, $\mathscr{V}$ is simply $(1+\bar{\mathscr{N}})^{-1}\bar{\mathscr{N}}$, which when inserted into (2.13) yields the correct nondynamical limit $\bar{\mathscr{N}}G_{\bar{\beta}}{}^{-1}$ for T.[11,20] This indicates the interrelationship between $\bar{\mathscr{N}}$ terms in B and C and shows that whatever approximations we make should reside in B_0 or C_0.

Next let us, for example, suppose that we can neglect C_0. (This is, in essence, what we will eventually take to define a class of so-called nonresonant approximations.) Then

$$\mathscr{V}\simeq(1+\bar{\mathscr{N}})^{-1}B \ . \tag{3.1}$$

If (3.1) is employed in (2.13) we obtain the approximate integral equation

$$T=\bar{\mathscr{N}}G_{\bar{\beta}}{}^{-1}+B_0+B_0G_{\bar{\beta}}T \ . \tag{3.2}$$

In practice one solves (3.2) for the half-on-shell matrix elements $\langle \phi_{\bar\beta}(\vec{k}') | T | \phi_{\bar\beta}(\vec{k}) \rangle$. Because $\overline{\mathscr{N}}$ is a sum of bounded, connected operators, we have half-on-shell[21]

$$\overline{\mathscr{N}} G_{\bar\beta}^{-1} | \phi_{\bar\beta}(\vec{k}) \rangle = 0 \tag{3.3}$$

and

$$(1 - B_0 G_{\bar\beta})^{-1} \overline{\mathscr{N}} G_{\bar\beta}^{-1} | \phi_{\bar\beta}(\vec{k}) \rangle = 0 , \tag{3.4}$$

unless, of course, $(1 - B_0 G_{\bar\beta})^{-1}$ does not exist, which we suppose is not the case. Equivalently, the OP model wave function $P_{\bar\beta} | \psi_{\bar\beta}^{(+)}(\vec{k}) \rangle$, which is related to T, in general, by the half-on-shell relation

$$T | \phi_{\bar\beta}(\vec{k}) \rangle = \mathscr{V} P_{\bar\beta} | \psi_{\bar\beta}^{(+)}(\vec{k}) \rangle , \tag{3.5}$$

will, in the approximation (3.1), be generated by the effective OP, B_0. From (2.13), (3.5), and (3.1) it follows that

$$\begin{aligned}(G_{\bar\beta}^{-1} - B_0) P_{\bar\beta} | \psi_{\bar\beta}^{(+)}(\vec{k}) \rangle \\ = (1 + \overline{\mathscr{N}}) G_{\bar\beta}^{-1} | \phi_{\bar\beta} \rangle = 0 ,\end{aligned} \tag{3.6}$$

which supports our assertion. What this means is that we can regard (3.1) as equivalent to working with the effective equation

$$T_0 = B_0 + B_0 G_{\bar\beta} T_0 . \tag{3.7}$$

We keep in mind that B_0 is explicitly free of all $\hat\beta$-class elastic unitarity cuts (Appendix B).

Next we generalize the preceding argument. If $(1 - C)^{-1}$ exists, then $\mathscr{V}$ is given by

$$\mathscr{V} = (1 - C)^{-1} B . \tag{3.8}$$

Manipulations similar to those leading to (3.2) result in the (exact) integral equation for T:

$$T = \overline{\mathscr{N}} G_{\bar\beta}^{-1} + B_0 + [B_0 G_{\bar\beta} + C_0] T . \tag{3.9}$$

In the process of passing to (3.9) we seem to have lost the explicit two-body structure for the T equation which motivated the introduction of the OP in the first place. Let us examine how we can recover it. Evidently, the solution of (3.9) is

$$T = (1 - K)^{-1} (B_0 + \overline{\mathscr{N}} G_{\bar\beta}^{-1}) , \tag{3.10}$$

where

$$K = C_0 + B_0 G_{\bar\beta} \tag{3.11}$$

is the (connected) kernel of (3.9). Thus

$$T = T_0 + (1 - K)^{-1} \overline{\mathscr{N}} G_{\bar\beta}^{-1} , \tag{3.12}$$

where

$$T_0 = (1 - K)^{-1} B_0 , \tag{3.13}$$

so that

$$T_0 = \mathscr{V}_0 + \mathscr{V}_0 G_{\bar\beta} T_0 , \tag{3.14}$$

with

$$\mathscr{V}_0 = (1 - C_0)^{-1} B_0 . \tag{3.15}$$

If $(1 - K)^{-1}$ exists, then, again since $\overline{\mathscr{N}}$ is a sum of bounded, connected operators, we have half on shell[21]

$$T | \phi_{\bar\beta}(\vec{k}) \rangle = T_0 | \phi_{\bar\beta}(\vec{k}) \rangle , \tag{3.16}$$

and so for all practical purposes we may regard $\mathscr{V}_0$ as the effective OP. Again we remark that B_0 and C_0 are explicitly free of all $\hat\beta$-class elastic unitarity cuts and so is $\mathscr{V}_0$ except for possible spurious singularities associated with the inversion in (3.15).

The preceding formalizes what is immediately transparent if one were to use (3.9) in an explicit calculation of the elastic scattering amplitudes. Namely, one would write (3.9) as a half-on-shell equation and drop the $\overline{\mathscr{N}} G_{\bar\beta}^{-1}$ term. The resultant equation is then easily placed in the form of a two-body Lippmann-Schwinger equation provided $\mathscr{V}_0$ is identified as the effective potential.

In the present article we wish to consider a simple class of approximations, where we can apply the preceding considerations. Our principal approximation is that

$$C_0 \simeq 0 . \tag{3.17}$$

Then B_0 is the effective OP and the scattering amplitude is determined from (3.7).

Now C_0 is derived from the operator $[C(\hat\beta) + \mathscr{N}]$, which is very highly clustered.[22,23] Such highly correlated dynamics among all of the nucleons comprising the projectile and target is expected to be important at relatively low energies and may be associated with compound-nucleus-like resonance behavior.

We can elaborate upon this association of full clustering, or the completely connected part of $\mathscr{V}_{\rm opt}(\hat\beta)$, and compound resonances in the following manner. It is easy to show using the results of Ref. 7 that $\mathscr{V}_{\rm opt}(\hat\beta)$ satisfies the quasi-Lippmann-Schwinger equation

$$\mathscr{V}_{\rm opt}(\hat\beta) = V_e^{\bar\beta} + D_{\bar\beta} + V_e^{\bar\beta} G_{\bar\beta} Q_{\bar\beta} \mathscr{V}_{\rm opt}(\hat\beta) , \tag{3.18}$$

where

$$D_{\bar{\beta}}=\left[\sum_{\beta\in\hat{\beta}}\mathscr{R}_{\beta}\bar{\delta}_{\beta,\bar{\beta}}\right]Q_{\bar{\beta}}G_{\bar{\beta}}^{-1}\,, \tag{3.19}$$

$$Q_{\bar{\beta}}=I-P_{\bar{\beta}}\,, \tag{3.20}$$

and[8,9]

$$V_e^{\bar{\beta}}=(1+\mathscr{N})^{-1}(V^{\bar{\beta}}+\mathscr{N}G_{\bar{\beta}}^{-1})R_{\bar{\beta}}\,. \tag{3.21}$$

With standard manipulations one obtains from (3.18) the quasi-Feshbach form

$$\mathscr{V}_{\text{opt}}(\hat{\beta})=V_e^{\bar{\beta}}+D_{\bar{\beta}}+V_e^{\bar{\beta}}Q_{\bar{\beta}}(G_{\bar{\beta}}^{-1}-Q_{\bar{\beta}}V_e^{\bar{\beta}}Q_{\bar{\beta}})^{-1}\times Q_{\bar{\beta}}(V_e^{\bar{\beta}}+D_{\bar{\beta}})\,. \tag{3.22}$$

Evidently the $D_{\bar{\beta}}$ terms in (3.22) are of no physical consequence.[12] It has been shown[8,9,13] that the discrete (fully clustered) eigenstates of $(G_{\bar{\beta}}^{-1}-Q_{\bar{\beta}}V_e^{\bar{\beta}}Q_{\bar{\beta}})^{-1}$ correspond to the Feshbach resonances[24] for the elastic scattering of two nuclear fragments. Thus, these real-axis pole singularities reside entirely within the fully connected part of $\mathscr{V}_{\text{opt}}(\hat{\beta})$. It is consistent, then, to ignore the fully connected parts of $\mathscr{V}_{\text{opt}}(\hat{\beta})$ if we are in physical circumstances, where we expect resonance behavior to be unimportant. We are aware of the fact that there may be fully-clustered parts of B_0 and C_0 which have nothing to do with resonances, and which may be important at medium energies. At present there is no compelling way of identifying physically important processes with such structure.

The possible close connection of $C(\hat{\beta})$ to resonance behavior is also indicated by (3.8). Since C is compact we know that the singularities of $\mathscr{V}$ associated with the discrete spectrum of C will possess factorizable residues. We have not investigated the relationship of these singularities to the Feshbach resonances present in $\mathscr{V}$.[8,9,13]

We remark that the resolvents of the kernels of the integral equations satisfied by $\Lambda_{\lambda}^{\hat{\alpha}}(\hat{\beta})$ (see Appendix B) may possess pole singularities. These poles would then appear in both B_0 and C_0, and it is possible that their effects in each would cancel out [cf. (3.8)]. In any case such singularities would also be associated with the fully-clustered parts of B_0 and C_0 and we are specifically interested in those situations where such effects are not important. The class of approximations which is defined by neglecting C_0 (3.17) *as well as* the fully-clustered parts of B_0 we refer to as *nonresonant.* We investigate some special cases of this type of approximation to the OP in the next section for nucleon-nucleus scattering.

IV. LOW-ORDER MULTIPLE SCATTERING APPROXIMATIONS

Our development up to this point holds for two arbitrary nuclear fragments. However, the case of nucleon-nucleus scattering has been most extensively investigated under those circumstances for which we expect the nonresonant assumption to be valid. We refer here to moderately high incident nucleon energies so the elastic nucleon-nucleus scattering is largely diffractive and the dominant elementary reaction mechanism is an impulsive direct interaction. Somewhat more specifically, we confine ourselves to incident nucleon energies from one to two orders of magnitude greater than the single-nucleon binding energies. We refer to this as the "medium energy" range.

If one is primarily interested in the description of diffractive effects the dominant effects of the Pauli principle are relatively simple at medium energies. The physical reasons for this are clear.[3] One has several (and for heavy nuclei very many) elastic scattering channels which are related by permutation symmetry. These are the Pauli equivalent channels. The salient fact is that these channels communicate via nucleon exchanges. As a consequence for low-momentum transfers the effects of the indistinguishability of these channels are likely to be greatly suppressed when compared to the diffractive scattering within one (any one) of these channels. Thus the major effects of the Pauli principle involve the fewest nucleon exchanges so that the dominant effect of identity is simply the exchange symmetry associated with each two-nucleon scattering. However, higher-order Pauli effects are expected to be important in the backward scattering hemisphere, where they become competitive with or dominate the "direct" scattering processes.

The preceding physical picture underlies the widely used prescription of Takeda and Watson.[3] However, the analytical development of this prescription is not without ambiguity, particularly in regard to the identification of the so-called "target-exchange" terms which are presumed to be negligible.[25–27]

It is reasonable to conjecture that the segregation of the major Pauli effects from the minor ones will be difficult if one adopts a standard multiple-scattering point of view. The reason for this is that from such a standpoint one is attempting to describe an *intrinsically multichannel* problem using a formalism and physical picture which is unambiguous only for single-channel situations. The channels referred to here are the various Pauli-equivalent elastic channels. However, an approach such as in Refs. 1 and 2 in which Pauli-equivalent channels are treated symmetrically throughout would appear to offer considerable advantages for the development of physically motivated approximations. A signature of the symmetrical treatment of these channels is an approximate OP which possesses no $\hat{\beta}$-elastic unitarity cuts.

The results of Refs. 2 and 17 offer an alternative to the standard multiple scattering picture, which may be more appropriate in dealing with the antisymmetrized OP. We refer here to the clustering ideas discussed in Ref. 17 in connection with the new dynamical equations proposed there. We next develop a specific example of these ideas as applied to the antisymmetrized version of these equations found in Refs. 1 and 2.

The nonresonant approximation of Sec. III is essentially equivalent to the assumptions (see Appendix B)

$$\Lambda_{\hat{\gamma}}^{\hat{\beta}}(\hat{\beta})\simeq\delta_{\hat{\beta},\hat{\gamma}} , \tag{4.1}$$

$$C(\hat{\beta})\simeq-\mathscr{N} . \tag{4.2}$$

If (4.1) and (4.2) are valid, then we obtain using (B19) and (B23)

$$\mathscr{V}_{\rm opt}(\hat{\beta})\simeq\widetilde{\mathscr{W}}_{\rm MS}^{\hat{\beta},\hat{\beta}}+\widetilde{\mathscr{W}}^{\hat{\beta},\hat{\beta}}+\widetilde{\mathscr{N}}(\hat{\beta})G_{\overline{\beta}}^{-1}+V_{\overline{\beta}}^{\hat{\beta}}[1-P_{\overline{\beta}}\widetilde{\mathscr{N}}(\hat{\beta})]-\mathscr{N}\mathscr{V}_{\rm opt}(\hat{\beta}) . \tag{4.3}$$

The term $\widetilde{\mathscr{N}}(\hat{\beta})G_{\overline{\beta}}^{-1}$ in (4.3) is converted into $\overline{\mathscr{N}}G_{\overline{\beta}}^{-1}$ in the OP formalism. The significance of this last quantity is discussed in great detail in the preceding section, where it is shown that it is effectively "recycled" out of consideration. Similar remarks apply to the kernel term $(-\mathscr{N})\mathscr{V}_{\rm opt}(\hat{\beta})$. The fully-connected term $V_{\overline{\beta}}^{\hat{\beta}}P_{\overline{\beta}}\widetilde{\mathscr{N}}(\hat{\beta})$ will be dropped as part of our nonresonant assumption although it does not present serious calculational difficulties. Thus our *effective* nonresonant OP is simply

$$\mathscr{V}_{\rm opt}(\hat{\beta})\simeq\widetilde{\mathscr{W}}_{\rm MS}^{\hat{\beta},\hat{\beta}}+\widetilde{\mathscr{W}}^{\hat{\beta},\hat{\beta}}+V_{\overline{\beta}}^{\hat{\beta}} . \tag{4.4}$$

Our objective in this section is to consider some low-order approximations to (4.4) in the special case of nucleon-nucleus scattering, where we take the canonical partition to be $\overline{\beta}=(1)\ (2\cdots N)$. Because of its parametric energy independence and highly unclustered structure the interaction $V_{\overline{\beta}}^{\hat{\beta}}$ is a necessary component of any low-order approximation to the nonresonant OP. Now

$$V_{\overline{\beta}}^{\beta}=\sum_{i'}\overline{\Delta}_{\beta,i'}V_{i'}\Delta_{\overline{\beta},i'} , \tag{4.5}$$

where $V_{i'}$ is the two-nucleon potential acting between the two clustered nucleons in the $(N-1)$-cluster partition i'. For example, i' may refer to the partition (12) (3) (4) $\cdots$ (N) so $V_{i'}=V_{(12)}$ in this case. In (4.3), $\Delta_{\overline{\beta},i'}=1$, if i' is contained in $\overline{\beta}$ and is zero otherwise. Also, $\overline{\Delta}_{\beta,i'}=1-\Delta_{\beta,i'}$. If we combine (4.5) and the expression (B9) for $V_{\overline{\beta}}^{\hat{\beta}}$ we find using the antisymmetry of the target wave functions that

$$\langle\phi_{\overline{\beta}}(\vec{k}')\,|\,V_{\overline{\beta}}^{\hat{\beta}}\,|\,\phi_{\overline{\beta}}(\vec{k})\rangle=-\overline{\delta}_{1,j}\overline{\delta}_{l,j}\overline{\delta}_{1,l}\times(N-1)(N-2)\langle\phi_{\overline{\beta}}(\vec{k}')\,|\,\mathscr{E}_{1,j}V_{(l,j)}\,|\,\phi_{\overline{\beta}}(\vec{k})\rangle , \tag{4.6}$$

where l and j refer to some arbitrary, but definite, target ($\overline{\beta}$) nucleons and $\mathscr{E}_{1,j}$ is the $1,j$ exchange operator.

We see then that $V_{\overline{\beta}}^{\hat{\beta}}$ is the pickup or heavy-particle exchange term. We learn that such terms appear linearly in the nucleon-nucleus OP. The consequence of this is that some of the distortions which are associated with the ultimate contribution of the pickup to the scattering amplitude are taken care of automatically by the OP formalism. The appearance and the significance of the pickup term has often been a point of some ambiguity in investigations of nucleon-nucleus scattering.

Since $\overline{N}_{\hat{\beta},\hat{\beta}}=1$, we have from (B36)

$$\widetilde{\mathscr{W}}_{\rm MS}^{\hat{\beta},\hat{\beta}}=\sum_{\beta\in\hat{\beta}}\mathscr{R}_{\beta}\sum_{a\notin\hat{\beta}}{}'\,W^{\beta,\overline{\beta}}(a)R_{\overline{\beta}} . \tag{4.7}$$

The properties of the a-connected cluster terms $W^{\beta,\overline{\beta}}(a)$ are reviewed in Appendix B. Physically, $W^{\beta,\overline{\beta}}(a)$ consists of all those scatterings which begin and end with a two-nucleon interaction external to $\overline{\beta}$ and β, respectively, which can be grouped into the cluster a. It represents a selective summation of the elementary multiple scatterings, each of which is represented by the two-particle transition operator

$$t_{i'}=V_{i'}+V_{i'}G_0t_{i'} , \tag{4.8}$$

where $G_0=(z-H_0)^{-1}$ is the *free-particle* propagator, and H_0 is the N-particle kinetic energy.[17] The lowest-order approximation to (4.7),

$$(\widetilde{\mathscr{W}}^{\hat{\beta},\hat{\beta}}_{\rm MS})_{\rm LO}=\sum_{\beta\in\hat{\beta}}\mathscr{R}_{\beta}\sum_{i'}\overline{\Delta}_{\beta,i'}t_{i'}\overline{\Delta}_{\overline{\beta},i'}\ , \tag{4.9}$$

is appropriate only in a low-density situation.
For moderate energies and any but the lightest nuclei it is unlikely that (4.9) will be a good approximation because the interactive effects of the residual nucleus are not negligible.[5,17,18] It is easy to show that

$$\langle\phi_{\overline{\beta}}(\vec{k}')\,|\,(\widetilde{\mathscr{W}}^{\hat{\beta},\hat{\beta}}_{\rm MS})_{\rm LO}\,|\,\phi_{\overline{\beta}}(\vec{k})\rangle =(N-1)\langle\phi_{\overline{\beta}}(\vec{k}')\,|\,\widetilde{t}\,|\,\phi_{\overline{\beta}}(\vec{k})\rangle\ , \tag{4.10}$$

where

$$\widetilde{t}=(1-\mathscr{E}_{1,j})\overline{\delta}_{1,j}t_{(1,j)}\ , \tag{4.11}$$

is the antisymmetrized form of (4.8). Results similar to (4.10) realize also for the dressed and energy-shifted two-nucleon transition operators encountered later.

We now develop a more sophisticated approximation sequence to $\widetilde{\mathscr{W}}^{\hat{\beta},\hat{\beta}}_{\rm MS}$ of the general type suggested in Ref. 17. In an impulsive situation for elastic scattering the parameter of smallness is the average single-particle binding energy compared to the energy of the incident nucleon. A simple hierarchy of impulsive approximations is then suggested via the successive "ionization" of clusters of the target particles, where only these ionized nucleons interact with the projectile. The term ionization means that the nucleon is dynamically free of the residual target nucleons, but it does imply that the latter are bound together in all intermediate states. The projectile interaction with the ionized nucleons is made unambiguous if we require that it be described by a fully connected subamplitude. For one ionized target nucleon the requirement is trivial. If we are considering two ionized target nucleons the interactions with the projectile are represented by a fully connected three-to-three amplitude. The dynamics of un-ionized nucleons are entirely arbitrary. As a consequence we term this decomposition of $\widetilde{\mathscr{W}}^{\hat{\beta},\hat{\beta}}_{\rm MS}$, the *inclusive connectivity expansion.*

First, let us recall that a partition b is said to be *contained* in another, a, if b can be obtained from a by subdividing one or more of its clusters. We denote this situation as $b\subseteq a$, where we include the possibility of equality. If $b\not\subseteq a$ then b *is not contained in* a. The relevant component of $\widetilde{\mathscr{W}}^{\hat{\beta},\hat{\beta}}_{\rm MS}$ is the connectivity expansion in terms of the operators $W^{\beta,\overline{\beta}}(a)$ [cf. (B14)], where $a\not\subseteq\beta,\overline{\beta}$. Choose a set of two-cluster partitions α such that $\alpha\notin\hat{\beta}$. Then we can write

$$\sum_{a\notin\hat{\beta}}{}'\,W^{\beta,\overline{\beta}}(a)=\sum_{\alpha}\sum_{a}\Delta_{\alpha,a}W^{\beta,\overline{\beta}}(a) +\sum_{\alpha}\sum_{a\notin\hat{\beta}}\overline{\Delta}_{\alpha,a}W^{\beta,\overline{\beta}}(a)\ , \tag{4.12}$$

or

$$\sum_{a\notin\hat{\beta}}{}'\,W^{\beta,\overline{\beta}}(a)=\sum_{\alpha}t^{\beta,\overline{\beta}}_{\alpha} +\sum_{\alpha}\sum_{a\notin\hat{\beta}}{}'\,\overline{\Delta}_{\alpha,a}W^{\beta,\overline{\beta}}(a)\ , \tag{4.13}$$

where

$$t^{\beta,\overline{\beta}}_{\alpha}=V^{\beta,\overline{\beta}}_{\alpha}+V^{\beta}_{\alpha}G_{\alpha}V^{\overline{\beta}}_{\alpha}\ . \tag{4.14}$$

We exploit (4.13) shortly.

If we "ionize" particle j (relative to $\overline{\beta}$) then the corresponding "residual" partition is

$$r(1,j)\equiv(1,j)(2,\ldots,j-1,j+1,\cdots N)\ , \tag{4.15}$$

which has two clusters. One cluster, $(1,j)$, refers to the (connected) interaction with the projectile (1), while the other serves as a reference for all of the dynamical states (bound and continuum) of the residual nucleus. As the first step in the inclusive connectivity expansion we identify α as the set of all $r(i,j)$. Obviously,

$$t^{\beta,\overline{\beta}}_{r(1,j)}=\overline{\Delta}_{\beta,(1j)}V_{(1j)}\{1+[E-K_1-K_j-V_{(1j)}-H_{r(1,j)} +i0]^{-1}V_{(1j)}\}\overline{\Delta}_{\overline{\beta},(1j)}\ . \tag{4.16}$$

Here $H_{r(1,j)}$ is the Hamiltonian of the residual nucleus and

$$H_{r(1,j)}=K_{r(1,j)}+H^{I}_{r(1,j)}\ , \tag{4.17}$$

where $K_{r(1,j)}$ is the kinetic energy operator corresponding to the center-of-mass (c.m.) motion of the residual nucleus, and $H^{I}_{r(1,j)}$ is the internal Hamiltonian.

We observe that [cf. (4.7)] $t^{\beta,\overline{\beta}}_{r(1,j)}$ is going to operate to the right on $R_{\overline{\beta}}$. However, we see from (B4) that

$$R_{\overline{\beta}}=R_r[1,j]R_{\overline{\beta}}\ , \tag{4.18}$$

where $R_r[1,j]$ is the antisymmetrizer on the subspace of the states of the residual nucleus (but not 1 and j). Since $R_r[1,j]$ commutes with both $H_{r(1,j)}$ and $V_{(1j)}$, we can absorb $R_r[1,j]$ into the Green's function in (4.16), which, therefore, involves *only physical residual nucleus states.*

Suppose we insert into the Green's function in

(4.15) a complete set of states which are direct products of the eigenstates of $K_1+K_j+V_{(1j)}$, $K_{r(1,j)}$, and $H^I_{r(1,j)}$. The subsystem dynamics, which are represented by the sum over the physical eigenstates of $H^I_{r(1,j)}$, present the real problem associated with the dressed two-nucleon operator (4.16). Except for few-nucleon targets or some especially simple model for $H^I_{r(1,j)}$, approximations are required to handle (4.16) in practical calculations. One such approximation, which is consistent with the impulsive nature of $t^{\beta,\bar{\beta}}_{r(1,j)}$, is to suppose that closure can be imposed with respect to the eigenstates of the residual nucleus. This mathematical prescription amounts to the physical assumption that we can set

$$\epsilon_n \simeq \epsilon_0 \tag{4.19}$$

for all n, where n labels the energy eigenvalues, ϵ_n, of $H^I_{r(1,j)}$, and $n=0$ denotes the ground state. Approximation (4.19) corresponds to the situation, where in comparison to the large initial projectile energy there are only relatively small differences among those (internal) energies of those states of the residual nucleus which play a significant role in the impulsive two-nucleon collision.

Assuming (4.19) we obtain from (4.16)

$$t^{\beta,\bar{\beta}}_{r(1,j)} \simeq \bar{\Delta}_{\beta,(1j)} t_D(1,j) \bar{\Delta}_{\bar{\beta},(1j)} , \tag{4.20}$$

where

$$t_D(1,j) = V_{(1j)}\{1+[E-\epsilon_0-K_1-K_j-V_{(1j)} -K_{r(1,j)}+i0]^{-1} V_{(1j)}\} . \tag{4.21}$$

Evidently, $t_D(1,j)$ is a two-nucleon transition operator with three-body kinematics and with a parametric energy shifted by the ground-state energy of the *residual* nucleus. It is important to recognize that we have executed a closure approximation in such a way that the recoil motion of both the struck nucleon *and* the residual nucleus are correctly represented. Now

$$K_{r(1,j)} = \left[\frac{\vec{P}^2}{2M} \right]_{r(1,j)} , \tag{4.22}$$

where $\vec{P}$ refers to the total momentum of the recoil nucleus and M is its mass. In the limit of very large M the Green's function in (4.21) reduces to the commonly used form

$$[E-\epsilon_0-K_1-K_j-V_{(1j)}+i0]^{-1} . \tag{4.23}$$

In general, however, the use of (4.23) represents an improper treatment of recoil.

We obtain an apparently more realistic low-order approximation to $\widetilde{\mathscr{W}}^{\beta,\bar{\beta}}_{\rm MS}$ by using $t_D(1,j)$ instead of $t_{i'}$ in (4.9). We find using the same techniques that are used to obtain (4.10)

$$\widetilde{\mathscr{W}}^{\beta,\bar{\beta}}_{\rm MS} \simeq (N-1)\langle \phi_{\bar{\beta}}(\vec{k}') \,|\, \tilde{t}_D \,|\, \phi_{\bar{\beta}}(\vec{k}) \rangle , \tag{4.24}$$

where

$$\tilde{t}_D = (1-\mathscr{E}_{1,j})\delta_{1,j} t_D(1,j) \tag{4.25}$$

is the antisymmetrized two-nucleon transition operator with *three-body* kinematics .

We can extend the argumentation which led to (4.13) to obtain an *exact* inclusive connectivity expansion for $\widetilde{\mathscr{W}}^{\beta,\bar{\beta}}_{\rm MS}$. We make two preliminary observations. If $a \not\subset \bar{\beta}$ or $a \not\subset \beta$, then $t^{\beta,\bar{\beta}}_a = 0$. Also, if $\bar{\beta} = (1)\,(2, \ldots, N-1)$, and $a \not\subset \bar{\beta}$, then

$$a = (1i\,\cdots)\cdots(\cdots) , \tag{4.26}$$

that is, the projectile 1 is *always* included in a cluster containing at least one other particle.

Now the partitions (4.15) codify all the terms of various connectivitites $W^{\beta,\bar{\beta}}(a)$, which enter into the two-particle-connected portion of the inclusive connectivity expansion. Thus, it is clear that the three-particle-connected part corresponds to the partition $(i<j)$,

$$r(1ij) \equiv (1,i,j)(2,..,i-1,i+1,\ldots,j-1,j+1,\ldots N) , \tag{4.27}$$

which also has two clusters. However, here $(1,i,j)$ refers to a connected three-particle interaction, while the other cluster refers to the states of the (new) $(N-3)$-particle residual nucleus in the same undifferentiated manner as with (4.15). Then [cf. (4.13)]

$$\sum_{a \notin \bar{\beta}}{}' W^{\beta,\bar{\beta}}(a) = \sum_i t^{\beta,\bar{\beta}}_{r(1,i)} + \sum_{i\neq j} \sum_{a \notin \bar{\beta}}{}' [\Delta_{r(1ij),a}]_{\rm sym} \bar{\Delta}_{r(1i),a} W^{\beta,\bar{\beta}}(a) + \sum_{i\neq j} \sum_{a \notin \bar{\beta}}{}' [\bar{\Delta}_{r(1ij),a}]_{\rm sym} \bar{\Delta}_{r(1i),a} W^{\beta,\bar{\beta}}(a) , \tag{4.28}$$

where

$$[\Delta_{r(1ij),a}]_{\text{sym}}=\Delta_{r(1ij),a},\ i<j\ ,$$
$$=\Delta_{r(1ji),a},\ j<i\ . \quad (4.29)$$

We note that

$$\bar{\Delta}_{r(1i),a}=1-\Delta_{r(1i),a}\ , \quad (4.30)$$

and, e.g., with $i<j$,

$$\Delta_{r(1ij),a}\Delta_{r(1i),a}=\Delta_{r(1i,j),a}\ , \quad (4.31)$$

where $r(1i,j)$ is the *three-cluster* partition

$$r(1i,j)\equiv(1i)(j)(\cdots)\ . \quad (4.32)$$

With the aid of (4.29)–(4.32) the second term on the right side of (4.28) reduces to

$$\sum_{i\neq j}\sum_{a\notin\hat{\beta}}{}'[\Delta_{r(1ij),a}]_{\text{sym}}\bar{\Delta}_{r(1i),a}W^{\beta,\bar{\beta}}(a)=\sum_{i<j}t^{\beta,\bar{\beta}}_{r(1ij)}-\sum_{i\neq j}t^{\beta,\bar{\beta}}_{r(1i,j)}\ . \quad (4.33)$$

The right side of (4.33) represents a collection of amplitudes whose $(1ij)$ parts are fully connected. The terms $t^{\beta,\bar{\beta}}_{r(1i,j)}$ subtract off the $(1i)$ and $(1j)$ disconnected parts of $t^{\beta,\bar{\beta}}_{r(1ij)}$. Note that while the last operator does not contain a (ij) disconnected piece, it does contain intermediate (ij) interactions. Thus (4.33) represents the fully connected interaction of the projectile with a fully correlated but ionized pair of target nucleons. In this way of developing approximations to the OP, (4.33) corresponds to the two-nucleon correlation contribution. One can again introduce a closure approximation on the residual nucleus which then yields a correction to (4.24) consisting of an antisymmetrized connected three-nucleon transition operator with four-body kinematics, and with a parametric energy shifted by the binding energy of the $(N-2)$-particle residual nucleus.

One can continue on from (4.28) in a similar fashion, although the higher-order terms are probably of marginal practical significance. An apparently more elegant way of characterizing the entire procedure is to begin from the identity (B29):

$$W^{\beta,\bar{\beta}}_{\text{MS}}=\sum_{f}{}'C_f t_f^{\beta,\bar{\beta}}\ . \quad (4.34)$$

We note that *all* two-cluster partitions $f_2\not\subset\bar{\beta}$, have the form

$$f_2=(1i\cdots)(\cdots)\ . \quad (4.35)$$

Also note that $C_\alpha=1$ for any two-cluster partition α. Thus it may seem that we have merely written out an f_2-biased version of (4.34). However, (4.34) contains contributions from $W^{\beta,\bar{\beta}}(a)$ with $a\in\hat{\beta}$ and in our formulation such terms are treated separately. An inclusive connectivity type of expansion is obtained for $W^{\beta,\bar{\beta}}_{\text{MS}}$ if the expansion on the right of (4.34) is grouped according to each f_2 such that there are no $(1i\cdots)$ disconnected parts. The counting coefficients C_f supply the correct number of terms so that this can be done. One can use these combinatorial groupings in $W^{\beta,\bar{\beta}}_{\text{MS}}$ to identify their counterparts in the expansion (4.12) and thus obtain an exact statement of the inclusive connectivity expansion for $\tilde{\mathscr{W}}^{\hat{\beta},\bar{\beta}}_{\text{MS}}$. For all practical purposes our previous arguments suffice.

The $W^{\beta,\bar{\beta}}(a)$, $a\in\hat{\beta}$, terms, of course, contribute to $\tilde{\mathscr{W}}^{\hat{\beta},\bar{\beta}}$, although without their elastic discontinuities [cf. (B34)–(B36)]. The contributions to the OP of the excited states of the target nucleus are contained in $\tilde{\mathscr{W}}^{\hat{\beta},\bar{\beta}}$. These excited states appear only in the various channels $\beta\in\hat{\beta}$, $\beta\neq\bar{\beta}$, which are Pauli equivalent to $\bar{\beta}$. The contributions of a few of these (discrete) states may not be particularly difficult to calculate. The reason for this is that if $\lambda\in\hat{\beta}$, then [cf. (B12)]

$$W^{\beta,\bar{\beta}}(\lambda)=(V^{\beta}_{\lambda}G_\lambda V^{\bar{\beta}}_{\lambda})_\lambda\ . \quad (4.36)$$

The eigenstates, $|\lambda'\rangle$, of H_λ, which correspond to bound states of the target nucleus give rise to the λ-connected contributions:

$$\left[V^{\beta}_{\lambda}G_\lambda|\lambda'\rangle\langle\lambda'|V^{\bar{\beta}}_{\lambda}\right]_\lambda=V^{\beta}_{\lambda}G_\lambda|\lambda'\rangle\langle\lambda'|V^{\bar{\beta}}_{\lambda}\ . \quad (4.37)$$

Effects of the target excited states appear in $\Lambda^{\hat{\alpha}}_{\lambda}(\beta)$ for all $\beta\in\hat{\beta}$. Again the evaluation of the contributions from the excited bound states of the target is relatively easy. For example, in the approximation

$$W^{\alpha,0}(\gamma)G_0\simeq\left[\sum_{\gamma'}V^{\alpha}_{\gamma}|\gamma'\rangle\langle\gamma'|G_\gamma\right]_\gamma=\sum_{\gamma'}V^{\alpha}_{\gamma}|\gamma'\rangle\langle\gamma'|G_\gamma\ , \quad (4.38)$$

the kernels of the integral equations (B6) take on a very simple separable structure.

The preceding analysis indicates an interesting aspect of the present dynamical equations approach to the OP. Namely, if one were, for example, to approximate all the two-cluster terms $W^{\alpha,0}(\gamma)G_0$ by finite sums of separable terms as in (4.38) the inversion of (B6) yields $\Lambda^{\hat{\alpha}}_{\lambda}(\hat{\beta})$ with analytical structure reminiscent of, but by no means identical to, the so-called Feshbach[28] form of the OP. A difference of pivotal importance between the Feshbach-form generated approximations and those that we have just discussed, is that we avoid expressing the problem in terms of eigenstates of unphysical projected Hamiltonians such as $Q_{\bar{\beta}}HQ_{\bar{\beta}}$.

We do not pursue the question of the effects of the excited target states. If these contributions are a comparatively unimportant part of (4.3) then we can neglect the term $\mathscr{W}^{\bar{\beta},\bar{\beta}}$, which is also consistent with our neglect of other highly clustered pieces of $\mathscr{V}_{\rm opt}(\hat{\beta})$. It is useful to summarize our results by concluding this section with the final two-body integral equations for the elastic scattering amplitude $T(\vec{k}'\,|\,\vec{k})$, which are to be solved in our (conjectured) lowest-order nonresonant (NR) approximation to $\mathscr{V}_{\rm opt}(\hat{\beta})$ for nucleon-nucleus scattering:

$$T(\vec{k}'\,|\,\vec{k})=\mathscr{V}^{(0)}_{\rm NR}(\vec{k}'\,|\,\vec{k}) + \int (d\vec{k}'')\frac{\mathscr{V}^{(0)}_{\rm NR}(\vec{k}'\,|\,\vec{k}'')T(\vec{k}''\,|\,\vec{k})}{E-E''+i0}\,, \tag{4.39}$$

where $(A=N-1)$,

$$\mathscr{V}^{(0)}_{\rm NR}(\vec{k}'\,|\,\vec{k})=A\langle\phi_{\bar{\beta}}(\vec{k}')\,|\,\tilde{t}_D\,|\,\phi_{\bar{\beta}}(\vec{k})\rangle - A(A-1)\langle\phi_{\bar{\beta}}(\vec{k}')\,|\,V_{\rm EX}\,|\,\phi_{\bar{\beta}}(\vec{k})\rangle\,, \tag{4.40}$$

and $V_{\rm EX}$ is the two-nucleon exchange potential

$$V_{\rm EX}\equiv\mathscr{E}_{1,j}V_{(lj)}\,, \tag{4.41}$$

where 1 denotes the incident nucleon and j and l refer to two distinct target nucleons. Equation (4.39) is written in the total c.m. system so that

$$E''=\frac{\hbar^2(\vec{k}'')^2}{2\mu}+\epsilon(\hat{\beta})\,, \tag{4.42}$$

where μ is the reduced mass and $\epsilon(\hat{\beta})$ is the target binding energy. The parametric energy, E, has a similar decomposition so that the energy denominator in (4.39) is simply

$$(2\mu/\hbar^2)(\vec{k}^2-\vec{k}''^2+i0)^{-1}\,, \tag{4.43}$$

if $\vec{k}$ is the incident relative wave vector. We also emphasize that $\tilde{t}_D$ refers to an antisymmetrized two-nucleon transition operator with three-body kinematics.

This section has been confined entirely to nucleon-nucleus elastic scattering. The approximations proposed here as well as all of the work of the preceding sections also holds for the case of *particle*-nucleus scattering, where the projectile is *not* a nucleon or a nuclear fragment. In such a case one has merely a single partiton $\bar{\beta}$ comprising the entire $\hat{\beta}$ class; that is, $\hat{\beta}=\bar{\beta}$ in all of our sums over the channels Pauli equivalent to the incident channel $\bar{\beta}$. In this instance our approach also possesses some distinct advantages, namely all of the dynamics involving subsystems of the target nucleons are carried out on the physical, antisymmetrized subspaces. Also the progressive clustering sequence which is represented by the (exact) inclusive connectivity expansion yields a systematic, unitarity-preserving series of approximations to the optical potential, which take into account the antisymmetry of the target nucleus in a consistent manner. We remark that even in this well-worked example, the manner of our deduction of the lowest-order approximation to the OP as the energy-shifted two-particle transition operator with three-body kinematics (the pickup term vanishes) is essentially new.

One of the important innovations by KMT[5] is the exploitation of the simplifications which result from the explicit treatment of target antisymmetry. An associated difficulty, however, is that quasi-two-particle operators appear, which are defined in terms of the symmetrized Green's functions $R_{\bar{\beta}}G_{\bar{\beta}}$. (See Ref. 5 and Appendix A.) This complicates the justification of the replacement of such operators by true two-particle amplitudes.[5] We emphasize that this last remark holds whether one chooses to formulate the impulse approximation via the Watson or KMT prescriptions (Appendix A). When applied to the case of projectile-nucleus scattering our approach seems to handle these problems in a comparatively straightforward manner, while still preserving the simplicity of the final result.

V. SUMMARY AND DISCUSSION

Our principal results can be summarized as follows:

(1) We have proposed a class of nonresonant approximations to the completely antisymmetrized optical potential for elastic two-fragment nuclear scattering based upon a consistent set of dynamical equations. This sequence of approximations preserves the elastic unitarity properties which customarily provide the motivation for the introduction of the optical potential. Namely, the optical potential is presumed to account for the flux only into the inelastic channels.

(2) We have shown in the special case of medium

energy nucleon-nucleus scattering how our nonresonant approximation contains multiple-scattering-like structure. Specifically, we show in detail on the basis of some reasonable physical assumptions, that the lowest-order optical potential is the sum of an antisymmetrized, energy-shifted two-nucleon transition operator with three-body kinematics, plus a heavy-particle exchange (or pickup) amplitude. The appearance of this exchange term in the optical potential clarifies the role of distortion in connection with this part of the elastic scattering amplitude. [See Eqs. (4.38)–(4.42) for our final results for low-order nucleon-nucleus scattering.]

(3) We have shown how the machinery developed in Refs. 1 and 2 is to be used in connection with realistic problems. A major aspect of this is the interpretation of several mysterious terms which appear in the dynamical equations of Refs. 1 and 2.

(4) Besides the favorable unitarity characteristics, our general development as well as the various approximation sequences we have proposed are formulated so that all intermediate-state subsystem propagation takes place only in the appropriate antisymmetrized subspaces.

(5) It is pointed out that the special case of particle-nucleus scattering is contained in our results. In this instance, where there is no projectile-nucleus antisymmetrization, our results are also distinctive in their preservation of the correct reality properties of the optical potential, as well as ensuring that all of the dynamics involving subsystems of the target nucleons is carried out on the physical, antisymmetrized subspaces.

Some of the preceding results appear to be complex renditions of conclusions which have already been drawn using seemingly simpler conventional methods. Our impulse approximation $\tilde{t}_D$ seems a case in point. There are, however, some significant distinctions. Most importantly, we begin from the optical potential appropriate to the physical situation and approximate it. The conventional formal approaches[3–5] first approximate the physical problem to a point to where a definition of an approximate optical potential is obvious. As a consequence, the statement, development, and possible improvement of our approximations are quite different than previous treatments. Also, the fact that we work with sets of scattering integral equations with well-defined inversions plays an important part in our method of approximation as do the detailed structural characteristics of the constituent operators.

We discuss (Appendix A) several aspects of the conventional multiple scattering treatments of the nucleon-nucleus OP which we believe are pertinent to assessing our work. One of these questions is the choice between the versions of the impulse approximation proposed by Watson *et al.*[3,4,29] and by KMT.[5,30–38] *To the extent that a comparison is possible* our results appear to be more in accord with the Watson approach rather than that of KMT. We compare some recent versions[10,11,26] of the KMT and Watson formalisms, which include the full effects of the Pauli principle with the present approach. The principal differences, at least for the low-order approximation, lie in the satisfaction of the unitarity constraints on the optical potential. The theory of Refs. 1 and 2 as applied in this article is the only one in which these constraints are satisfied at all stages.

Finally, the results of this article lend further support to the proposal of Refs. 1 and 2 that the AGS off-shell extension be considered the appropriate one for dealing with nuclear scattering problems of any type, but especially in connection with considerations of the optical potential.

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APPENDIX A: PAULI PRINCIPLE IN MULTIPLE SCATTERING THEORIES

If we ignore the projectile-target antisymmetrization, the the lowest-order KMT expression for the OP is, for scattering in the single elastic channel $\beta(=\bar{\beta})$ with $R_\beta P_\beta = P_\beta$,[5]

$$U_1^{\mathrm{KMT}} = A T_{0j}(1+P_\beta G_\beta T_{0j})^{-1} , \tag{A1}$$

where A is the number of target nucleons,

$$T_{0j} = v_{0j} + v_{0j} R_\beta G_\beta T_{0j} , \tag{A2}$$

and v_{0j} is the two-particle interaction between the projectile, 0, and a typical target nucleon j. Equivalently, we have

$$U_1^{\mathrm{KMT}}=A\tau_{0j} , \tag{A3}$$

where

$$\tau_{0j}=v_{0j}+v_{0j}R_\beta G_\beta Q_\beta \tau_{0j} . \tag{A4}$$

Since τ_{0j} possesses no P_β space discontinuities, U_1^{KMT} has no elastic unitarity cuts. This demonstrates the well-known equivalence of the first-order Watson and KMT version of the OP *before* the approximation of τ_{0j} or T_{0j}, respectively, by two-particle transition matrices.[33,35] When this last step is taken, however, the Watson OP retains its elastic unitary character, while the KMT OP picks up elastic cuts.[35,39–44]

Things change when incident nucleon antisymmetrization is even only minimally taken into account. We quote the low-order result of Ref. 10 corresponding to (A1):

$$\hat{U}_1^{\mathrm{KMT}}=A\hat{T}_{0j}(1+P_\beta G_\beta \hat{T}_{0j})^{-1} , \tag{A5}$$

where

$$\hat{T}_{0j}=v_{0j}(1-\mathscr{E}_{0j})(1+R_\beta G_\beta T_{0j}) , \tag{A6}$$

and $\mathscr{E}_{0j}$ is the projectile-target nucleon exchange operator. Unlike U_1^{KMT}, the operator $\hat{U}_1^{\mathrm{KMT}}$ *does possess* elastic unitarity cuts. This situation persists when $\hat{T}_{0j}$ is approximated by antisymmetrized two-nucleon transition operators. Although Eqs. (A5) and (A6) are obtained in Ref. 10 using (as in Ref. 3) the prior off-shell extension, a possible equivalence to the Watson case such as (A1)–(A4) is no longer obvious. It appears then that for nucleon-nucleus scattering the justification for the Takeda-Watson[3] prescription[27] is in general different for the Watson and the KMT formalisms. This observation is pertinent to the comparative calculations for nucleon-nucleus scattering, which are carried out in Refs. 33 and 37. The presence in $\hat{U}_1^{\mathrm{KMT}}$ of elastic unitarity cuts (in contrast to U_1^{KMT}) is a strong signal that Pauli effects are not being handled correctly in (A5), and represents the major deviation of a KMT-type approximation sequence from one of the type developed in Sec. IV.[42]

Since there are intrinsic differences in the implementation of antisymmetry between the KMT and Watson approaches, a comparison of their distinctive approximation characteristics is most easily done when the projectile's identity with the target nucleons is ignored. [Equations (A1)–(A4).] If we let $t_{0j}^{(2)}$ denote some particular *two*-particle transition operator, then the KMT (U_{KMT}^a) and Watson (U_W^a) approximations to U_1^{KMT} are distinguished by the replacements $T_{0j}\to t_{0j}^{(2)}$ and $\tau_{0j}\to t_{0j}^{(2)}$, respectively. It should now be clear in what sense our results (4.39)–(4.43), resemble the Watson formulation.

Although the differences between U_W^a and U_{KMT}^a are of the $(1/A)$ type, they can be competitive with Pauli effects for large momentum transfers, where diffractive mechanisms are no longer dominant. Also U_{KMT}^a possesses a discontinuity across the elastic unitarity cut for *all* energies, while $U_W^{(a)}$ does not. We discuss next two not entirely dynamical arguments which have been used to motivate the use of U_{KMT}^a rather then U_W^a.

It has been asserted[37] that the KMT formulation is designed to avoid the double-counting error (dce) implicit in the replacement of τ_{0j} by T_{0j}, where it is assumed that only the latter operator can be sensibly approximated by $t_{0j}^{(2)}$.[45] One of the reasons for supposing that there is such a dce follows from the approximation to $Q_\beta G_\beta$ in (A4), which is implied by the replacement $\tau_{0j}\to t_{0j}^{(2)}$. For example, suppose the operator $Q_\beta G_\beta$ is approximated by its fully disconnected part G_0. Then, since $P_\beta G_0\neq 0$ and $G_0 P_\beta\neq 0$, one has seemingly reintroduced some of the P_β space back into the intermediate states by truncating the connectivity expansion for $Q_\beta G_\beta$. However, there is nothing manifestly inconsistent about making such a "Q_β-space violating" approximation to $Q_\beta G_\beta$, as long as the result is still continuous across the elastic unitarity cut. The only significant aspect about $P_\beta G_\beta$ in the theory of the OP is its generation of the elastic unitarity cut, but the OP formalism is designed to take care of this cut properly despite approximations to the OP. Thus, the judgment of the effectiveness of a particular truncation of $Q_\beta G_\beta$ as an approximation depends entirely upon how well one has approximated the OP and not on naive P_β- and Q_β-space assessments for a *single* channel β.

In view of these remarks it is interesting to note that if $G_0^{(2)}$ is the Green's function appearing in $t_{0j}^{(2)}$, then

$$U_{\mathrm{KMT}}^a=A\tau^a , \tag{A7}$$

where

$$\tau^a=v_{0j}+v_{0j}(G_0^{(2)}-P_\beta G_\beta)\tau^a . \tag{A8}$$

For any of the usual two-particle approximations $P_\beta(G_0^{(2)}-G_\beta)\neq 0$, e.g., and thus we have an introduction of P_β-space components in much the same way as in the Watson case. Moreover, we see from (A8) that part of the elastic-unitarity-generating part of G_β has been reintroduced back into the OP. Thus, in any reasonable approximation the KMT prescription *always* leads to an OP which possesses an elastic unitarity cut along with an apparent dce peculiar to itself.

Another, basically nondynamical, argument is often used to introduce the KMT approximation scenario. This argument begins from the observation that on the R_β-projected space the exact elastic transition operator, $T_{\rm EL}$, satisfies[5]

$$T_{\rm EL}=AT_{0j}+(A-1)T_{0j}G_\beta R_\beta T_{\rm EL}\ , \qquad \text{(A9)}$$

where we continue to neglect the identity of the incident nucleon. The $(A-1)$ factor has its origin in the fact that the Foldy-Watson multiple-scattering series is arranged so that successive scatterings from the same target nucleon are forbidden.[36] The approximation (A7) yields an operator $T_{\rm EL}^{(a)}$ which satisfies

$$T_{\rm EL}^{(a)}=At_{0j}^{(2)}+(A-1)t_{0j}^{(2)}P_\beta G_\beta T_{\rm EL}^{(a)}\ . \qquad \text{(A10)}$$

The multiple scattering series generated by (A10) appears to be consistent with the omission of successive scatterings from the same target nucleon in virtue of the $(A-1)$ factor. A loophole in this argument is that the definition of what one refers to as a "scattering" has changed in the passage from (A9) to (A10). Namely, the propagation between successive scatterings is governed by G_β in (A9), but by $P_\beta G_\beta$ in (4.25). However, while T_{0j} involves G_β, the two-particle operator $t_{0j}^{(2)}$ is defined in terms of $G_0^{(2)}$ and *not* $P_\beta G_\beta$. This mismatch produces quite legitimate self-scattering terms. Evidently, consistency in the representation of the elementary projectile-target-nucleon scatterings is necessary for a meaningful counting argument.

The preceding problems of conventional multiple-scattering theories persist or are enhanced when full antisymmetry is imposed. We have already commented on a few of these difficulties. Picklesimer[11] has found a KMT-type approximation sequence in the AGS case which is identical to that of Ref. 10 through terms of order $(1+\mathscr{N})^{-1}\mathscr{N}$. The arguments of Ref. 11 do not seem to provide any criteria for preferring the prior over the AGS off-shell extensions in this regard. However, the formalisms of Ref. 10 and Ref. 11 share a common feature: they yield order by order antisymmetrized OP's which possess elastic unitarity cuts.[46] In the AGS case, this violates the original purpose for introducing that off-shell extension as well as the constraints the OP must satisfy in this case.[1,2]

In summary, we see that with or without Pauli effects a KMT approximation sequence is really an algorithm for constructing an *effective interaction* for elastic nucleon-nucleus scattering rather than an *optical potential,* where the latter is distinguished from the former by the absence of elastic flux. For this reason, the theory of Refs. 1 and 2 does not lend itself to KMT-type approximations. On the whole, the arguments which presently exist in the literature for preferring a KMT to a Watson approximation sequence do not appear to be very compelling.

APPENDIX B: ANTISYMMETRIZED FORMS

We follow the notation of Ref. 2 and use the results of Refs. 2 and 15–18. However, the contents of this Appendix represent new results which are required in support of the arguments in Secs. II–IV.

If $a,b,c\ldots$ refer to partitions, of the N particles the operators $M^{a,b}$, $K^{a,c}$, and $T^{c,b}$ are label transforming,[14] and if

$$M^{a,b}=\sum_c K^{a,c}T^{c,b}\ , \qquad \text{(B1)}$$

then we have the product rule[2,14]

$$\widetilde{M}^{\hat a,\hat b}=\sum_{\hat c}\mathscr{K}^{\hat a,\hat c}\widetilde{T}^{\hat c,\hat b}\ . \qquad \text{(B2)}$$

The quantities appearing in (B2) are the antisymmetrized sums

$$\widetilde{M}^{\hat a,\hat b}=\bar N_{\hat a,\hat b}\sum_{a\in\hat a}\mathscr{R}_a M^{a,\bar b} \qquad \text{(B3)}$$

with similar definitions for $\mathscr{K}^{\hat a,\hat c}$ and $\widetilde{T}^{\hat c,\hat b}$. Here, $\bar a,\bar b,\bar c$ refer to canonical partitions and the normalization constants, $\bar N_{\hat a,\hat b}$, are defined in Ref. 2. We note that $\mathscr{R}_c=R_{\bar c}\mathscr{R}_c$.

Let S denote the permutation group on N objects. The subgroup $S_f\subset S$ consists of all permutations which leave f unchanged. It is easy to show that if $f\subset a$, then $S_f\subset S_a$; evidently S_a can be decomposed into cosets of S_f yielding results analogous to those of Ref. 14. It then follows that

$$R_fR_a=R_aR_f=R_a\ , \qquad \text{(B4)}$$

where, e.g., R_f is the antisymmetrizer with respect to S_f.

The quantities $\Lambda_{\lambda}^{\alpha}(\hat{\beta})$, which appear in $B(\hat{\beta})$ and $C(\hat{\beta})$ satisfy the connected-kernel equations[1,2]

$$\Lambda_{\lambda}^{\alpha}(\hat{\beta})=\delta_{\alpha,\lambda}+\sum_{\gamma}[W^{\alpha,0}(\gamma)G_0-V_{\gamma}^{\alpha}G_{\gamma}P_{\gamma}\delta(\gamma\in\hat{\beta})]\Lambda_{\lambda}^{\gamma}(\hat{\beta}) . \tag{B5}$$

In the preceding equations V_{β}^{α} refers to those interactions external to partition α but internal to partition β and $\delta(\gamma\in\hat{\beta})=1$ if $\gamma\in\hat{\beta}$ and is zero otherwise. The kernels in (B5) are free of $\hat{\beta}$-class elastic unitarity cuts[1,2] if P_{γ} includes the ground states of H_{γ}. We can transform (B5) into an explicitly antisymmetric form with the aid of the product rule (B2):

$$\Lambda_{\hat{\lambda}}^{\hat{\alpha}}(\hat{\beta})=\tilde{\delta}_{\hat{\alpha},\hat{\lambda}}+\sum_{\hat{\gamma}}[\tilde{W}^{\hat{\alpha},0}(\hat{\gamma})G_0-V_{\hat{\gamma}}^{\hat{\alpha}}G_{\bar{\gamma}}P_{\bar{\gamma}}\delta_{\hat{\gamma},\hat{\beta}}]R_{\bar{\gamma}}\Lambda_{\hat{\lambda}}^{\hat{\gamma}}(\hat{\beta}) . \tag{B6}$$

Here

$$\tilde{\delta}_{\hat{\alpha},\hat{\lambda}}\equiv\bar{N}_{\hat{\alpha},\hat{\lambda}}R_{\bar{\lambda}} , \tag{B7}$$

$$\tilde{W}^{\hat{\alpha},0}(\hat{\gamma})\equiv\bar{N}_{\hat{\alpha},\hat{\gamma}}\sum_{\alpha\in\hat{\alpha}}\mathscr{R}_{\alpha}W^{\alpha,0}(\bar{\gamma})R_{\bar{\gamma}} , \tag{B8}$$

$$V_{\hat{\gamma}}^{\hat{\alpha}}\equiv\bar{N}_{\hat{\lambda},\hat{\gamma}}\sum_{\lambda\in\hat{\lambda}}\mathscr{R}_{\lambda}V_{\bar{\gamma}}^{\lambda}R_{\bar{\gamma}} . \tag{B9}$$

Now

$$W^{\alpha,0}(\bar{\gamma})G_0=(V_{\bar{\gamma}}^{\alpha}G_{\bar{\gamma}})_{\bar{\gamma}} , \tag{B10}$$

where $[\mathscr{O}]_a$ denotes the a-connected part of the operator $\mathscr{O}$. We see then that only the physical elastic singularities of $W^{\alpha,0}(\bar{\gamma})G_0R_{\bar{\gamma}}$, namely those defined on the completely antisymmetrized subsystem space projected out by $R_{\bar{\gamma}}$, enter into the kernel of (B6). Also, it is precisely these $\hat{\beta}$-class singularities which are subtracted off by $V_{\hat{\gamma}}^{\hat{\alpha}}G_{\bar{\gamma}}P_{\bar{\gamma}}R_{\bar{\gamma}}\delta_{\hat{\gamma},\hat{\beta}}$. This is an important point because an R_{γ} symmetrized projector P_{γ} in (B5) does not subtract off the wrong-symmetry $\hat{\beta}$-class elastic singularities of $W^{\alpha,0}(\gamma)G_0$. Once the explicitly antisymmetrized form (B6) is reached, however, we see that only the physically relevant states and their corresponding singularities are involved. Now we see that (B6) is a connected-kernel integral equation (see Theorem 3 of Appendix A, Ref. 2) whose kernel is *manifestly* free of all $\hat{\beta}$-class singularities.

Now

$$W_{\rm MS}^{\gamma,\beta}=\sum_{a}{}' W^{\gamma,\beta}(a) , \tag{B11}$$

where the prime denotes the omission of the single-cluster partition,

$$W^{\gamma,\beta}(a)=(V^{\gamma,\beta}+V^{\gamma}GV^{\beta})_a$$
$$=(V_a^{\gamma,\beta}+V_a^{\gamma}G_aV_a^{\beta})_a , \tag{B12}$$

and G is the full Green's function. Let us make the following label-transforming-invariant decomposition

$$W_{\rm MS}^{\gamma,\beta}=\mathscr{W}_{\rm MS}^{\gamma,\beta}+\sum_{\lambda\in\beta}W^{\gamma,\beta}(\lambda) , \tag{B13}$$

where for example, it is easily verified that

$$\mathscr{W}_{\rm MS}^{\gamma,\beta}\equiv\sum_{a\notin\beta}{}' W^{\gamma,\beta}(a) \tag{B14}$$

is label transforming.

The combination of operators

$$\mathscr{W}_{\rm MS}^{\gamma,\beta}+\sum_{\lambda\in\beta}[W^{\gamma,\beta}(\lambda)-V_{\lambda}^{\gamma}G_{\lambda}P_{\lambda}V_{\lambda}^{\beta}] \tag{B15}$$

appears in $B(\hat{\beta})$ [cf. Eqs. (5.11a) and (5.16) of Ref. 2]. This expression contains no $\hat{\beta}$-class elastic singularities provided P_{λ} includes the ground states of H_{λ}. We then have a situation similar to that encountered with (B6). That is, it is not manifest that we are dealing only with *physical* singularities corresponding to antisymmetrized subsystem states and indeed we are not until we antisymmetrize.

We can split $W^{\gamma,\beta}(\lambda)$ into two pieces,

$$W^{\gamma,\beta}(\lambda)=\mathscr{W}^{\gamma,\beta}(\lambda)+V_{\lambda}^{\gamma}G_{\lambda}P_{\lambda}^{\rm US}V_{\lambda}^{\beta} , \tag{B16}$$

one of which, $\mathscr{W}^{\gamma,\beta}(\lambda)$, contains no λ-channel elastic unitarity cuts.[2] $P_{\lambda}^{\rm US}$ denotes the unsymmetrized (US) projector onto the ground states of H_{λ} including those of improper symmetry. A short calculation using the product rule (B2) yields

$$\bar{N}_{\hat{\gamma},\hat{\beta}}\sum_{\gamma\in\hat{\gamma}}\mathscr{R}_{\gamma}\sum_{\lambda\in\hat{\beta}}V_{\lambda}^{\gamma}G_{\lambda}P_{\lambda}^{\rm US}V_{\lambda}^{\bar{\beta}}R_{\bar{\beta}}$$
$$=V_{\hat{\beta}}^{\hat{\gamma}}G_{\bar{\beta}}(P_{\bar{\beta}}^{\rm US}R_{\bar{\beta}})(V')_{\hat{\beta}}^{\hat{\beta}} , \tag{B17}$$

where

$$(V')_{\hat{\gamma}}^{\hat{\lambda}}\equiv\bar{N}_{\hat{\lambda},\hat{\gamma}}\sum_{\lambda\in\hat{\lambda}}\mathscr{R}_{\lambda}V_{\lambda}^{\bar{\gamma}}R_{\bar{\gamma}} . \tag{B18}$$

Since $P_{\bar{\beta}}=P_{\bar{\beta}}^{\rm US}R_{\bar{\beta}}$ we see that only properly antisymmetrized eigenstates enter into the pole term in (B17).

We next use the form (B16), the result (B17), and the product rule (B2) to obtain from Eqs. (5.11a) and (5.16) of Ref. 2 the manifestly antisymmetrized form:

$$B(\hat{\beta})R_{\bar{\beta}}=\sum_{\hat{\gamma}}\Lambda_{\hat{\gamma}}^{\hat{\beta}}(\hat{\beta})[\tilde{\mathscr{W}}_{\mathrm{MS}}^{\hat{\gamma},\hat{\beta}}+\tilde{\mathscr{W}}^{\hat{\gamma},\hat{\beta}}]+\tilde{\mathscr{N}}(\hat{\beta})G_{\bar{\beta}}^{-1}+\left[V_{\bar{\beta}}^{\hat{\beta}}-\sum_{\hat{\gamma}}\Lambda_{\hat{\gamma}}^{\hat{\beta}}V_{\hat{\beta}}^{\hat{\gamma}}P_{\bar{\beta}}\tilde{\mathscr{N}}(\hat{\beta})\right]. \tag{B19}$$

Here

$$\tilde{\mathscr{V}}(\hat{\beta})=\sum_{\beta\in\hat{\beta}}\mathscr{R}_{\beta}\bar{\delta}_{\beta,\bar{\beta}}R_{\bar{\beta}}\ , \tag{B20}$$

$$\tilde{\mathscr{W}}_{\mathrm{MS}}^{\hat{\gamma},\hat{\beta}}=\bar{N}_{\hat{\gamma},\hat{\beta}}\sum_{\gamma\in\hat{\gamma}}\mathscr{R}_{\gamma}\mathscr{W}_{\mathrm{MS}}^{\gamma,\bar{\beta}}R_{\bar{\beta}}\ , \tag{B21}$$

$$\tilde{\mathscr{W}}^{\hat{\gamma},\hat{\beta}}=\bar{N}_{\hat{\gamma},\hat{\beta}}\sum_{\gamma\in\hat{\gamma}}\mathscr{R}_{\gamma}\mathscr{W}^{\gamma,\bar{\beta}}R_{\bar{\beta}}\ . \tag{B22}$$

The properties of $\tilde{\mathscr{W}}_{\mathrm{MS}}^{\hat{\gamma},\hat{\beta}}$ and $\tilde{\mathscr{W}}^{\hat{\gamma},\hat{\beta}}$, which are considered below, and in Sec. IV are such that the manifestly antisymmetrized form (B19) also represents an expression for $B(\hat{\beta})R_{\bar{\beta}}$, which is explicitly free of all $\hat{\beta}$-class elastic unitarity cuts.

Similar methods can be utilized to obtain a manifestly antisymmetrized form for $C(\hat{\beta})$. One finds from Eqs. (5.19) and (5.22) of Ref. 2 that

$$C(\hat{\beta})=-\{\mathscr{N}+[N_{\hat{\beta}}(\Lambda_{\hat{\beta}}^{\hat{\beta}}-1)+\sum_{\hat{\gamma},\hat{a}}\Lambda_{\hat{\gamma}}^{\hat{\beta}}(\hat{\beta})\tilde{W}^{\hat{\gamma},0}(\hat{a})G_0\bar{\delta}_{\hat{a},\hat{\beta}}\bar{N}_{\hat{a},\hat{\beta}}N_{\hat{a}}]RP_{\bar{\beta}}\}\ , \tag{B23}$$

where

$$\tilde{W}^{\hat{\gamma},0}(\hat{a})=\bar{N}_{\hat{\gamma},\hat{a}}\sum_{\gamma\in\hat{\gamma}}\mathscr{R}_{\gamma}W^{\gamma,0}(\bar{a})R_{\bar{a}}\ , \tag{B24}$$

and we note the appearance of the *full* antisymmetrizer R. Again we emphasize that (B23) represents a form of $C(\hat{\beta})$, which is manifestly free of all $\hat{\beta}$-class unitarity cuts.

We next investigate

$$W_{\mathrm{MS}}^{\hat{\gamma},\hat{\beta}}=\bar{N}_{\hat{\gamma},\hat{\beta}}\sum_{\gamma\in\hat{\gamma}}\mathscr{R}_{\gamma}W_{\mathrm{MS}}^{\gamma,\bar{\beta}}R_{\bar{\beta}}\ . \tag{B25}$$

First we note that

$$W^{\gamma,\beta}(a)=\sum_{f}{}'(\Delta^{-1})_{a,f}t_f^{\gamma,\beta}\ , \tag{B26}$$

where $t_f^{\gamma,\beta}$ is given by (4.14) and Δ^{-1} is the inverse of the matrix $\Delta_{a,b}$, which has elements $\Delta_{a,b}=1$ if $b\subseteq a$ and zero otherwise.[2] Now if f has more than one cluster[23]

$$\sum_{a}{}'(\Delta^{-1})_{a,f}=C_f\ , \tag{B27}$$

where

$$C_f=(-1)^{n_f}(n_f-1)!\ , \tag{B28}$$

and n_f is the number of clusters corresponding to the partition f. We then learn from (B26) that

$$W_{\mathrm{MS}}^{\gamma,\beta}=\sum_f{}'C_ft_f^{\lambda,\beta}=\sum_f{}'C_fV_f^{\lambda,\beta}+\sum_f{}'\mathscr{M}_{\gamma,f}V_f^{\beta}\ , \tag{B29}$$

where

$$\mathscr{M}_{\gamma,f}=V_f^{\lambda}C_fG_f\ . \tag{B30}$$

Thus

$$W_{\mathrm{MS}}^{\hat{\gamma},\hat{\beta}}=\bar{N}_{\hat{\gamma},\hat{\beta}}\left[\sum_{\gamma\in\hat{\beta}}\mathscr{R}_{\gamma}\sum_f{}'C_fV_f^{\gamma,\bar{\beta}}R_{\bar{\beta}}+\sum_{\hat{f}}\mathscr{M}_{\hat{\gamma},\hat{f}}(V^t)_{\hat{f}}^{\hat{\beta}}\right]. \tag{B31}$$

The aspect of (B31) we wish to point out concerns the structure of

$$\mathscr{M}_{\hat{\gamma},\hat{f}}=\sum_{\gamma\in\hat{\beta}}\mathscr{R}_{\gamma}V_{\bar{f}}^{\gamma}C_{\bar{f}}G_{\bar{f}}R_{\bar{f}}\ , \tag{B32}$$

which we see involves propagation only in the antisymmetrized subsystem spaces. This property is not explicit in some of the relations which follow.

It is shown in Ref. 2 that the part of $W^{\gamma,\beta}(a)$ designated as $\mathscr{D}[(W^{\gamma,\beta}(a)]_{\lambda}$, which possesses a discontinuity across the λ-elastic cut is

$$\mathscr{D}[W^{\gamma,\beta}(a)]_{\lambda}=V_{\lambda}^{\gamma}G_{\lambda}P_{\lambda}^{\mathrm{US}}V_{\lambda}^{\beta}\delta_{a,\lambda}\ . \tag{B33}$$

Therefore, the operator [cf. (B16)],

$$\mathscr{W}^{\gamma,\beta}(\lambda)\equiv W^{\gamma,\beta}(\lambda)-V_{\lambda}^{\gamma}G_{\lambda}P_{\lambda}^{\mathrm{US}}V_{\lambda}^{\beta}\ , \tag{B34}$$

contains no λ-channel elastic unitarity cuts. If we call

$$\mathscr{W}^{\gamma,\beta}\equiv\sum_{\lambda\in\hat{\beta}}\mathscr{W}^{\gamma,\beta}(\lambda)\ , \tag{B35}$$

then

$$\tilde{\mathscr{W}}^{\hat{\gamma},\hat{\beta}}=\bar{N}_{\hat{\gamma},\hat{\beta}}\sum_{\gamma\in\hat{\gamma}}\mathscr{R}_{\gamma}\sum_{\lambda\in\hat{\beta}}\mathscr{W}^{\gamma,\bar{\beta}}(\lambda)R_{\bar{\beta}}\ . \tag{B36}$$

In the form (B36) we do not have explicit antisymmetry on the cluster subspaces corresponding to $\mathscr{W}^{\gamma,\bar{\beta}}(\lambda)$, $\lambda \neq \bar{\beta}$. The same is true for

$$\widetilde{\mathscr{W}}^{\hat{\gamma},\hat{\beta}}_{\rm MS} = \bar{N}_{\hat{\gamma},\hat{\beta}} \sum_{\gamma \in \hat{\gamma}} \mathscr{R}_{\gamma} \sum_{a \notin \hat{\beta}}{}' \, W^{\gamma,\bar{\beta}}(a) R_{\bar{\beta}} \,. \tag{B37}$$

However, if in (B26) we use the expression (B12) for $W^{\gamma,\beta}(\lambda)$ it is clear that $\widetilde{\mathscr{W}}^{\hat{\gamma},\hat{\beta}}$ involves propagation and projectors only in the properly antisymmetrized subspaces. That is, only the $R_{\bar{\beta}}P_{\bar{\beta}}$ projectors, e.g., enter into (B36) rather than the unphysical, unsymmetrized projectors $P_{\lambda}^{\rm US}$, as seems to be indicated by (B34) [cf. (B17)].

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[19]Here $\bar{\delta}_{\beta,\bar{\beta}} = 1 - \delta_{\beta,\bar{\beta}}$ and $\mathscr{R}_{\beta}$ is the product of $R_{\bar{\beta}}$ and the adjoint of the parity-weighted unitary transformation which maps $\bar{\beta}$ into β. These definitions hold for an arbitrary partition, say c.

[20]The existence of $(1+\mathscr{N})^{-1}$ is proven in Ref. 9. Since $\mathscr{N}^2$ is connected, the existence of $(1+\mathscr{N})^{-1}$ is the simplest example of the use of connected-kernel equations in the AGS version of the OP.

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[39]This point is not in contradiction with the discussion Ref. 40 of the unitarity properties of the optical potential in conventional multiple-scattering approximations. The relevant G_β propagators are not approximated in Ref. 40 so that the question does not arise.

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