Elastic isovector vibrations and the boundary conditions

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The elastic model, which successfully explains the isoscalar giant resonances, is now applied to study the isovector giant resonances. In solving the Lamé equation, we employ the boundary condition that the displacement shall vanish at the surface. The results are compared with those obtained under the contrasting boundary condition that the stresses shall be absent at the surface. It was found that the results are dependent on the type of boundary conditions. Future experimental data will allow the assessment of the correct choice of boundary conditions.

[NUCLEAR STRUCTURE Nuclear giant resonances, isovector vibrational nodes. Electric and magnetic multipoles.]

I. INTRODUCTION

In the description of the isovector giant dipole resonances, the hydrodynamical model of Steinwedel and Jensen has been fairly successful in explaining the experimental data.[1,2] Recent refinements in admixing the Steinwedel-Jensen states with the Goldhaber-Teller states improve the theoretical results,[3] but the basic hydrodynamical description of the fluid is unaltered. The hydrodynamical model is, however, not adequate in describing quantitatively other nuclear phenomena such as rotations and vibrations involving isoscalar giant resonances. The homogeneous elastic solid was introduced phenomenologically for rotational nuclear states.[4–6] Bertsch[7] treated the giant resonances as elastic vibrations for the first time. Subsequently, the Lamé equation was derived by using the concept of the quantum stress tensor.[8–10] Still later, it was derived by taking the moments of the Vlasov equation.[11,12] By solving the Lamé equation exactly, it was found that the observed isoscalar giant $E0$, $E2$, and $E3$ resonances can indeed be well explained in a unified manner as an elastic vibration of a nucleus.[10]

Based on the earlier success of the elastic model of the isoscalar giant resonances, Bertsch and Stricker[13] examined the giant electric dipole oscillations as an elastic vibration of the proton medium against the neutron medium. An irrotational flow was assumed in their study, and a trial form of displacement vector was used in a variational calculation.

Another approach to study isovector giant resonances is to solve the Lamé equation appropriate for the elastic sphere and to impose boundary conditions to obtain the eigenenergies and proper displacement vectors. In such an approach, the boundary conditions are very important. An examination of the importance of the boundary condition may proceed through two parallel paths: One can vary the boundary conditions and study how well the results agree or disagree with experiment, relying on present or future experimental results to favor one boundary condition over the other. This has been the traditional approach.[3] One can also take a theoretical approach to see which of the boundary conditions can be justified from a more fundamental point of view. We shall take the first path in this paper and leave the examination of the second for future investigations.

In the isovector giant resonances, there are two different boundary conditions which one can plausibly impose at the surface. There is the boundary condition that the total stress tensor at the nuclear surface must vanish so that there is no net force across the boundary. This means that the proton and neutron fluids are free to move away from each other at the surfaces of the nucleus. Results have been obtained previously for isovector giant resonances using such a boundary condition.[14] Besides the 1^- state at 80.9 MeV/$A^{1/3}$, there are now 2^+ states at 55.7 and 119.5 MeV/$A^{1/3}$ and an isovector monopole state at 144.8 MeV/$A^{1/3}$. Another plausible boundary condition is that of Steinwedel and Jensen which requires that the neutrons oscillate against protons with the surface fixed. The argument[2] is that an excessive energy would be involved in displacing protons beyond the volume occupied by the neutrons. Such a boundary condition is equivalent to a vanishing relative displacement vector for neutrons against protons at the nuclear surface. We shall examine in this paper the isovector giant resonances in the elastic vibration model using such a boundary condition. The true physical situation may be expected to be in between the two approaches as has been observed earlier by Myers *et al.*,[3] in the case of the hydrodynamical approach.

II. ELASTIC ISOVECTOR VIBRATION

Starting with the density and velocity fields of each spin and isospin component, we can show that the equation of motion for an isovector displacement is given by the Lamé equation[8,14]

$$m^* n_0 \frac{\partial^2 \vec{\mathrm{D}}}{\partial t^2} = (\lambda+\mu)\nabla(\nabla\cdot\vec{\mathrm{D}}) + \mu\nabla^2\vec{\mathrm{D}} \ . \tag{2.1}$$

The isovector displacement field $D(\vec{r},t)$ is a linear combination of the displacement fields $\vec{d}$ of neutrons and protons

$$\vec{\mathrm{D}}(\vec{r},t) = \vec{d}_{n\uparrow}(\vec{r},t) + \vec{d}_{n\downarrow}(\vec{r},t) - \vec{d}_{p\uparrow}(\vec{r},t) - \vec{d}_{p\downarrow}(\vec{r},t) \ , \tag{2.2}$$

where the first subscript of $\vec{d}$ denotes the isospin component (neutron or proton) and the second subscript the spin component. The Lamé coefficients, as derived from the quantum stress tensor, are given by

$$\mu = \frac{\hbar^2}{5m} k_F n_0 \ , \tag{2.3}$$

$$\lambda = \frac{n_0 K_3}{9} - \tfrac{2}{3}\mu \ , \tag{2.4}$$

where k_F is the Fermi momentum, n_0 is the equilibrium total density, and K_3 is the isospin "incompressibility" related to the isospin symmetry energy s_3, by

$$K_3 = 18 s_3 \ . \tag{2.5}$$

In Eq. (2.1) the effective mass m^* is introduced in order to take into account the nonlocality of the exchange interaction, the additional velocity dependence of the interaction, and the coupling of the phonon to the nucleons.

We seek solutions for the displacement vector in the form

$$\vec{\mathrm{D}}(r,t) = \vec{\mathscr{D}}(\vec{r})e^{i\omega t} \ . \tag{2.6}$$

Then Eq. (2.1) becomes

$$(\nabla^2 + k^2)\vec{\mathscr{D}} = (1 - k^2/h^2)\nabla(\Delta) \ , \tag{2.7}$$

where $\Delta = \nabla\cdot\mathscr{D}$ is the dilatation, and k and h are related to the frequency ω by

$$h^2 = m^* n_0 \omega^2/(\lambda+2\mu) \tag{2.8}$$

and

$$k^2 = m^* n_0 \omega^2/\mu \ . \tag{2.9}$$

After taking the divergence of Eq. (2.7) we see that the dilatation Δ satisfies the Helmholtz equation

$$(\nabla^2 + h^2)\nabla = 0 \ . \tag{2.10}$$

Thus, the dilatation Δ will be a linear combination of products of regular Bessel functions and spherical harmonics. A particular solution of (2.7) is given by

$$\vec{\mathscr{D}}_1(\vec{r}) = -\frac{1}{h^2}\nabla(\Delta) \tag{2.11}$$

or

$$\vec{\mathscr{D}}_1(r) = -\frac{1}{h^{2+l}}\nabla\sum_{lm} j_l(hr)Y_{lm}(\theta,\phi)(-1)^l \ . \tag{2.12}$$

The homogeneous equation

$$(\nabla^2 + k^2)\vec{\mathscr{D}} = 0 \tag{2.13}$$

has two independent solutions $\vec{\mathscr{D}}_2$ and $\vec{\mathscr{D}}_3$ such that their corresponding dilatation Δ is zero, that is, they are isovolumetric solutions. They may be written as

$$\vec{\mathscr{D}}_2(\vec{r}) = \psi_l(kr)(\vec{r}\times\nabla)\Omega_{lm}(r,\theta,\phi) \tag{2.14}$$

and

$$\vec{\mathscr{D}}_3(\vec{r})=\psi_l(kr)\nabla\times[(\vec{r}\times\nabla)\Omega_{lm}(r,\theta,\phi)]\,, \tag{2.15}$$

where ψ_l and Ω_{lm} are related to the Bessel function and the spherical harmonics by

$$\psi_l(kr)=(-1)^l j_l(kr)/(kr)^l \tag{2.16}$$

and

$$\Omega_{lm}(r,\theta,\phi)=r^l Y_l(\theta,\phi)\,. \tag{2.17}$$

While $\vec{\mathscr{D}}_1$ and $\vec{\mathscr{D}}_3$ give rise to electric multipole states with natural parity, $\vec{\mathscr{D}}_2$ generates magnetic multipole states of unnatural parity. It is easy to see that the compressional perturbation will travel with speed

$$C_1=\left[\frac{\lambda+2\mu}{m^*n_0}\right]^{1/2}, \tag{2.18}$$

and the pure shear perturbation will travel with speed

$$C_2=\left[\frac{\mu}{m^*n_0}\right]^{1/2}. \tag{2.19}$$

Owing to the fact that we use the Thomas-Fermi approximation to derive the Lamé constants,[10] the speed of shear elastic waves C_2 depends only on the Fermi momentum and the effective mass.

The general solution of (2.1) is then given by

$$\vec{\mathscr{D}}(\vec{r})=A_1\vec{\mathscr{D}}_1(\vec{r})+A_2\vec{\mathscr{D}}_2(\vec{r})+A_3\vec{\mathscr{D}}_3(\vec{r})\,, \tag{2.20}$$

where $\nabla\cdot\vec{\mathscr{D}}_1\neq 0$, $\nabla\cdot\vec{\mathscr{D}}_2=0$, and $\nabla\cdot\vec{\mathscr{D}}_3=0$.

In order to make some comparison with the available experimental data, we shall concern ourselves with the electric multipole only, that is, the case when A_2 is assumed zero. The boundary condition that the displacement is zero at $r=R$ yields the eigenvalue equation

$$j_l'(kR)[j_l(kR)/kR+j_l'(kR)]-\frac{l(l+1)}{hkR^2}j_l(hR)j_l(kR)=0\,, \tag{2.21}$$

where

$$j'(x)=\frac{d}{dx}j_l(x)\,. \tag{2.22}$$

TABLE I. The isovector giant multipole eigenenergies are shown. The predictions given in the two first columns were obtained using the boundary condition of zero displacement at the nuclear surface while those of the third column correspond to the boundary condition of no stress at the surface.

	Eigenenergies (MeV)$A^{1/3}$		
$l\pi$	Boundary condition $\vec{\mathscr{D}}=0$ $(m^*/m)_\infty=0.64$ $k_F=1.00\ \mathrm{fm}^{-1}$	Boundary condition $\vec{\mathscr{D}}=0$ $(m^*/m)_\infty=0.87$ $k_F=1.10\ \mathrm{fm}^{-1}$	Boundary condition $\delta\vec{P}_{ij}=0$ $(m^*/m)_\infty=0.64$ $k_F=1.15\ \mathrm{fm}^{-1}$
0^+	123.6	120.2	144.8
1^-	80.5	79.2	80.9
	112.3	107.5	174.2
	167.7	155.7	223.4
2^+	113.0	108.0	55.7
	153.6	145.0	119.5
	197.7	182.5	209.5
3^-	138.8	130.6	88.2
	192.0	179.0	160.8
	230.6	212.8	244.4
4^-	162.8	151.6	116.9
	224.1	206.6	201.6

When the eigenvalue is obtained, the coefficient A_3 is given by

$$\frac{A_3}{A_1}=-\left[\frac{k}{h}\right]^{l+1}\frac{j_l'(hR)kR}{l(l+1)j_l(kR)} . \tag{2.23}$$

III. EIGENENERGIES OF ISOVECTOR GIANT RESONANCES

In order to solve for the eigenenergies of isovector giant resonances, we need to specify the Fermi momentum k_F, the isospin symmetry energy s_3, and the effective mass m^*. Myers and Swiatecki[16] give $s_3=27.612$ MeV which leads to an isovector incompressibility of $K_3=497.02$ MeV. For the effective mass, we chose the form prescribed by Brown and Speth.[17]

$$\frac{m^*}{m}=\left[\frac{m^*}{m}\right]_\infty+\frac{1-(m^*/m)_\infty}{\left[1+\frac{E_{\text{vib}}(\text{MeV})A^{1/3}}{82}\right]^2} .$$

It has been observed by Mahaux *et al.*[18] that the prescription for the effective mass given by Brown and Speth may not be correct and consequently, our results will vary accordingly. We examine two sets of parameters. When the value $(m^*/m)_\infty$ is set to be 0.64 as given by Brown and Speth, k_F must be small to give an electric dipole close to the experimental value of 80 $A^{-1/3}$ MeV. The eigenenergies for electric multipole states obtained with $(m^*/m)_\infty=0.64$ and $k_F=1.00$ fm^{-1}, and also with $(m^*/m)_\infty=0.87$ and $k_F=1.10$ fm^{-1}, are given in Table I. For comparison we also provide the results of previous solutions obtained with the boundary condition of vanishing stress at the surface, calculated with $(m^*/m)_\infty=0.64$ and $k_F=1.15$ fm^{-1}.

IV. RESULTS AND DISCUSSIONS

The results in Table I indicate that while the two solutions for the two different sets of parameters with the boundary condition $\vec{\mathscr{D}}=0$ are very similar, the solutions with different boundary conditions are significantly different. Looking at the results in detail, we see that the two different boundary conditions give distinctly different predictions on the isovector 2^+ and 3^- states.

With the boundary condition $\delta\vec{P}_{ij}=0$ there are 2^+ and 3^- states at 55.7 and 88.2 (MeV)$A^{-1/3}$. These states are absent in the case of the other boundary condition of $\vec{\mathscr{D}}=0$. There are the tentative results of Pitthan *et al.*[19] which suggest the possible existence of an isovector $E2$ resonance at 53 (MeV)$A^{-1/3}$. Recently, Drake *et al.*[20] found an E_2 isovector giant resonance at about 23 MeV for ^{208}Pb which corresponds to $\hbar\omega_2\sim 136\ A^{-1/3}$(MeV). So far, the experimental data are not sufficient to draw conclusive statements in favor of one boundary condition over another. There is uncertainty experimentally in the existence of the lower E_2 isovector states and theoretically in the effective mass as a function of the multiple excitation energy. Thus, the location of the isovector resonances of various multipolarities with different experimental probes will be of great help in testing the different boundary conditions. Theoretically, it is not enough to calculate only the locations of the collective states; it is necessary to obtain the transition matrix elements for collective excitation to the different states and to compare with the observed transition strengths.

As expected, the boundary conditions play an important role in the isovector giant resonances. The actual physical situation may require an admixture of the boundary conditions, analogous to the situation in hydrodynamics as was observed previously.[3] It is desirable to investigate how a more fundamental theory allows a better determination of the boundary condition. Looking at the second order differential equation (2.7), one can impose the more natural boundary condition that $\vec{\mathscr{D}}(r)\to 0$ at $r\to\infty$. Such a boundary condition is not unlike the one we encounter in the search for eigenstates in a potential well. Whether such a boundary condition leads to a better description and less arbitrariness remains to be seen.

There is an additional ambiguity with regard to the velocity dependence of the interactions which may alter the conclusions we draw here. Investigation into these problems will enhance our understanding of the giant resonances.

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