Optimized polynomial expansions: Potentials and pp phase shift analyses

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We have examined the Cutkosky-Deo-Ciulli-Chao optimized polynomial expansion as applied to the proton-proton scattering amplitudes. We found that we could not reproduce Chao's positive numerical results for intermediate energy scattering data. Instead, we reached a negative conclusion as to the method's usefulness for that application. We also examined the method's ability to predict the higher angular momentum parts of the amplitudes for three different potentials having realistic parts. Again we found the method to have very meager successes.

[NUCLEAR REACTIONS Optimal polynomial expansions in nucleon-nucleon scattering; potential models, phase analyses; previous errors.]

I. INTRODUCTION

We have applied the optimized polynomial expansion (hereafter OPT) of the invariant scattering amplitudes proposed by Cutkosky and Deo,[1] Ciulli,[2] and Chao,[3] to the description of medium energy proton-proton scattering. We have attempted therein to gain some insight into the power and limitations of the method; specifically, into its claimed ability[3] to predict high frequency components (states of high angular momentum) from knowledge of the low frequency ones (states of low angular momentum). This program was carried out by examining OPT's predictive capabilities for potential models and for phase shift analyses of data. Our conclusion is that there is little predictive power for Chao's form of the method. This is a disappointing development for intermediate energy nucleon-nucleon physics. We had hoped to use the method to help meet the increase in non-1π-exchange phases as energy increases, phases which presently must be fixed by data.

In this paper we first review the formalism as set up by Chao.[3] We then report our investigation of OPT's predictive ability for: (i) the 1π-exchange potential; (ii) a superposition of Yukawa potentials with a smooth behavior at the 2π threshold and having central and tensor components; and (iii) a one boson exchange potential due to Bryan and Scott[4] (hereafter BS-III). Next we turn our attention to the possibility of using OPT to improve the nucleon-nucleon phase shift analyses of MacGregor *et al.* (MAW-X) (Ref. 5) and of Signell *et al.*[6] and we compare our results with those of Chao.[3] Finally, we summarize our main (negative) conclusions.

II. OPT FORMALISM IN pp SCATTERING

Throughout this paper we follow the phase conventions of Stapp, Ypsilantis, and Metropolis[7] (hereafter referred to as SYM). The partial waves with total spin S, orbital angular momentum L, and total angular momentum J are parametrized in terms of nuclear bar phase shifts.[7] The connection of the SYM α's and the parity conserving amplitudes f_0^J, f_1^J, f_{11}^J, f_{12}^J, and f_{22}^J of Goldberger, Grisaru, MacDowell, and Wong[8] (hereafter referred to as GGMW) is[9]

$$\alpha_J = if_0^J\,, \quad \alpha_{JJ} = if_1^J\,, \quad \begin{bmatrix} \alpha_{J-1,J} \\ \alpha^J \\ \alpha_{J+1,J} \end{bmatrix} = \frac{1}{2J+1}\begin{bmatrix} J & J+1 & 2\sqrt{J(J+1)} \\ \sqrt{J(J+1)} & -\sqrt{J(J+1)} & 1 \\ J+1 & J & -2\sqrt{J(J+1)} \end{bmatrix} \begin{bmatrix} if_{11}^J \\ if_{22}^J \\ if_{12}^J \end{bmatrix}\,. \tag{1}$$

Elastic pp amplitudes which have no kinematical singularities in $x=\cos\theta$, where θ denotes the center of mass scattering angle, are, according to Mandelstam's conjecture,[10] analytic in the whole cut x plane. Branch points are 2π exchange at $x=\pm x_c$, where $x_c=1+2m_\pi{}^2/p^2$, and 1π exchange at $x=\pm x_B$ where $x_B=1+m_\pi{}^2/2p^2$. Here p is the center-of-mass momentum of either particle.

Using the invariant amplitudes $(F_1,\ldots,F_5)$ of GGMW, Chao[3] introduced new amplitudes $(A,\ldots,E)$ which are free of kinematic singularities in $x=\cos\theta$. Mapping the cut x plane onto an ellipse in a complex z plane by means of the optimal conformal mapping of Cutkosky, Deo,[1] and Ciulli[2]

$$z(x)=\sin\left[\frac{\pi}{2}\frac{F(\sin^{-1}x,1/x_c)}{F\left[\frac{\pi}{2},1/x_c\right]}\right], \tag{2}$$

where $F(v,k)$ is the incomplete elliptic integral of the first kind,[11] Chao expanded his amplitudes in polynomials of $z(x)$ convergent in the ellipse. These amplitudes and their expansions, called optimized polynomial expansions (OPT), can be written

$$\begin{aligned}
\widetilde{A}&=(2+\beta)\widetilde{F}_1-2x\beta\widetilde{F}_2-(6+4\beta)\widetilde{F}_3+\beta\widetilde{F}_5\ ,\\
&=\sum_{\text{even } n}a_nz^n(x)\ ,\\
\widetilde{B}&=2(\widetilde{F}_1+\widetilde{F}_3)=\sum_{\text{even } n}b_nz^n(x)\ ,\\
\widetilde{C}&=\widetilde{F}_1-2\widetilde{F}_3-\widetilde{F}_5=\sum_{\text{even } n}c_nz^n(x)\ ,\\
\widetilde{D}&=2(\widetilde{F}_2+\widetilde{F}_4)=\sum_{\text{odd } n}d_nz^n(x)\ ,\\
\widetilde{E}&=2\widetilde{F}_2=\sum_{\text{odd } n}e_nz^n(x)\ ,
\end{aligned} \tag{3}$$

where $\beta=(p/M)^2$, $M=$ proton mass, and $\widetilde{F}_i\equiv F_i-F_i(1\pi)$ with $F_i(1\pi)$ being the contribution of the one pion pole. The relation between the helicity amplitudes $(\phi_1,\ldots,\phi_5)$ of GGMW and the Chao amplitudes (A, B, C, D, E) is[3]

$$\begin{bmatrix}\frac{1}{2}(\phi_1-\phi_2)\\ \frac{1}{2}(\phi_1+\phi_2)\\ \dfrac{\phi_3}{1+x}\\ \dfrac{\phi_4}{1-x}\\ \dfrac{M}{E}\dfrac{\phi_5}{\sqrt{1-x^2}}\end{bmatrix}=\frac{M^2}{8\pi E}\begin{bmatrix}1&0&0&0&0\\0&-\beta&+\beta&x&x\beta\\0&0&-\beta&1+\beta&-\beta\\0&0&\beta&1+\beta&-\beta\\0&0&0&-1&0\end{bmatrix}\begin{bmatrix}A\\B\\C\\D\\E\end{bmatrix}, \tag{4}$$

where $E=(p^2+M^2)^{1/2}$. The spin singlet is contained in $(\phi_1-\phi_2)$ and the triplets are contained in the other helicity amplitudes. Because $z(-x)=-z(x)$, the restrictions to even or odd terms in the summations of Eq. (3) follow from (4). The partial wave expansions of the helicity amplitudes on the left hand side of (4) are given in terms of d functions by GGMW.[8] Introducing Chao's coefficients[3]

$$\begin{aligned}
W_{Jn}&=\int_{-1}^{+1}d_{00}^J(x)z^n(x)\,dx\ ,\\
X_{Jn}&=\int_{-1}^{+1}x d_{00}^J(x)z^n(x)\,dx\ ,\\
Y_{Jn}&=\int_{-1}^{+1}(1-x)d_{10}^J(x)z^n(x)\,dx\ ,\\
Z_{Jn}&=\int_{-1}^{+1}(1+x)d_{11}^J(x)z^n(x)\,dx\ ,
\end{aligned}$$

one can project the partial waves on both sides of (4) via the GGMW partial wave expansions. One finds[3]

$$\begin{aligned}
i\widetilde{f}_0^J&=\sum_{\text{even } n}W_{Jn}a_n\ ,\\
i\widetilde{f}_1^J&=-\beta\sum_{\text{even } n}Z_{Jn}c_n+(1+\beta)\sum_{\text{odd } n}Z_{Jn}d_n-\beta\sum_{\text{odd } n}Z_{Jn}e_n\ ,\\
i\widetilde{f}_{11}^J&=-\beta\sum_{\text{even } n}W_{Jn}b_n+\beta\sum_{\text{even } n}W_{Jn}c_n+\sum_{\text{odd } n}X_{Jn}d_n+\beta\sum_{\text{odd } n}X_{Jn}e_n\ ,\\
i\widetilde{f}_{22}^J&=-\beta\sum_{\text{even } n}Z_{Jn}c_n+(1+\beta)\sum_{\text{odd } n}Z_{Jn}d_n-\beta\sum_{\text{odd } n}Z_{Jn}e_n\ ,\\
i\widetilde{f}_{12}^J&=-\frac{E}{M}\sum_{\text{odd } n}Y_{Jn}d_n\ .
\end{aligned} \tag{5}$$

Again, in (5), we have subtracted the 1π-exchange pole as indicated by the notation $\tilde{f}$. Subtracting this pole from the SYM amplitudes α as well, one can compute $\tilde{f}_0^J$, etc., by inverting (1).

The point of these maneuvers was the hope that one could terminate the expansions in (3) after a few terms. Then the nonvanishing coefficients could be found from a few "unknown" low partial wave α's. These would be "input" to the calculation of the $\tilde{f}^J$'s, and hence to the coefficients $(a_n, b_n, c_n, d_n, e_n)$ through (5). The δ's for the higher partial waves could then be predicted by utilizing some unitarization scheme. In applying this procedure to pp scattering, we used the unitarization scheme and the ordering of amplitudes and coefficients proposed by Chao[3]

singlets: $(\alpha_0, \alpha_2, \alpha_4, \ldots)$, singlet coefficients: $(a_0, a_2, a_4, \ldots)$,

triplets: $(\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha^2, \alpha_{32}, \alpha_{33}, \alpha_{34}, \alpha^4, \alpha_{54}, \ldots)$, triplet coefficients: $(b_0, c_0, d_1, e_1, b_2, c_2, d_3, e_3, \ldots)$.

The leftmost members were input; those to the right, "output."

III. POTENTIAL MODELS

In this section we report our tests of the power of OPT to predict the high L phases of several potential models with realistic features: (i) the 1π-exchange potential; (ii) a superposition of Yukawa potentials; and (iii) a Bryan-Scott one boson exchange potential.[4] Table I shows the exact low angular momentum phase values $\tilde{\delta} \equiv \delta - \delta(1\pi)$ used to calculate the OPT coefficients for each of the potentials.

A. The 1π-exchange potential

This potential's form is

$$V_{1\pi}(r) = \frac{g^2 m_\pi^2}{4\pi 4M^2}\left(\tfrac{1}{3}(\sigma_1\cdot\sigma_2)\phi(r) + S_{12}X(r)\right), \quad (6)$$

where

$$\phi(r) = \frac{e^{-m_\pi r}}{r};$$

$$X(r) = \frac{1}{3}\left[1 + \frac{3}{m_\pi r} + \frac{3}{m_\pi^2 r^2}\right]\phi(r);$$

and

$$S_{12} = 3\sigma_1\cdot\hat{r}\sigma_2\cdot\hat{r} - \sigma_1\cdot\sigma_2 .$$

In Table II, we show the phases $\tilde{\delta} \equiv \delta - \delta(1\pi)$ with $m_\pi = 135$ MeV, $M = 938.256$ MeV, $g^2/4\pi = 14.4$, and $T_{\rm lab} = 210$ MeV. The first column shows the potential model values and the next columns various OPT predictions. In the first OPT column the exact values of all phases with $J \le 3$, and the ^3F $\tilde{\delta}$'s, were used as input to calculate the OPT coefficients. These in turn, were used to predict the higher L $\tilde{\delta}$'s shown in the same column. Seeing

TABLE I. Exact non-1π-pole nuclear bar values $\tilde{\delta}$, in degrees, at 210 MeV used to compute OPT coefficients which appear in Tables II, III, and IV. The potentials used for columns (i)–(iii) are described in the text. The last column shows the MAW "experimental" values (Ref. 5) of $\tilde{\delta}$ to which Chao compared his predictions.

State	(i) 1π potential	(ii) ΣY potential	(iii) BS-III potential	MAW-X Phase shift analysis
$^3P_0$	−37.30	−24.52	−47.61	−56.97±0.56
$^3P_1$	+ 9.92	+ 0.03	+ 5.01	+ 7.97±0.32
$^3P_2$	−4.35	+ 33.18	+ 10.87	+ 11.34±0.23
ϵ_2	+ 0.85	+ 5.22	+ 1.54	+ 2.57±0.16
$^3F_2$	−0.51	−0.74	−0.55	−0.97±0.33
$^3F_3$	−0.04	+ 2.29	+ 0.43	+ 1.04±0.20
$^3F_4$	+ 0.023	+ 0.295	+ 0.582	+ 1.38±0.19
ϵ_4	−0.057	+ 0.386	−0.066	+ 0.25±0.09
$^3H_4$	−0.002	−0.015	+ 0.031	−0.12±0.21
$^3H_5$	−0.037	+ 0.150	−0.008	−0.16±0.18

TABLE II. Nuclear bar OPT values of $\tilde{\delta}$ at 210 MeV for the 1π-exchange potential (6). Values enclosed in brackets are calculated exactly via the Schrödinger equation.

State	(1π potential)	$L \le 3$ input	OPT predictions $J \le 4$ input	$J \le 5$ input
ϵ_4	[−0.057]	+0.006	[−0.057]	[−0.057]
$^3H_4$	[−0.002]	−0.027	[−0.002]	[−0.002]
$^3H_5$	[−0.037]	+0.004	+0.010	[−0.037]
$^3H_6$	[+0.014]	+0.007	+0.008	+0.007
ϵ_6	[−0.020]	−0.0005	−0.007	−0.003
$^3K_6$	[+0.003]	−0.002	+0.002	+0.001
$^3K_7$	[−0.013]	+0.0007	+0.002	−0.005
$^1G_4$	[+0.064]	+0.144	[+0.064]	
$^1I_6$	[+0.014]	+0.0135	+0.002	

that the OPT predictions did not seem to approach the exact values as L increased, we tried using more and more terms in the expansion with results shown in the last two columns. When we reached the $J \le 5$ expansion, we finally found correct signs for the higher J predictions but the magnitudes were disappointingly poor. The singlet results, not shown, were also poor.

B. Superposition of Yukawas

Our second potential, a superposition of Yukawa potentials (hereafter ΣY), had central and tensor components and amplitudes with a cut at $x = x_c$

$$V_{\Sigma Y}(r) \equiv g^2[1+S_{12}]v(r) , \qquad (7)$$

with

$$v(r) \equiv \int_{2m_\pi}^{\infty} \rho(\mu) \frac{e^{-\mu r}}{r} d\mu ,$$

and

$$\rho(\mu) = \frac{m_\pi}{\pi} \frac{\gamma(\mu^2 - 4m_\pi^2)^{1/2}}{(\mu^2-\mu_a^2)^2 + \gamma^2(\mu^2 - 4m_\pi^2)} .$$

Here

$$\gamma = \mu_a \Gamma_a / (\mu_a^2 - 4m_\pi^2)^{1/2}$$

and we have chosen $\mu_a = 3.5 m_\pi$ and $\Gamma_a = m_\pi$. The results are shown in Table III for $g=4$ and $T_{\text{lab}} = 210$ MeV. Notice that there is no 1π pole, so $\tilde{\delta} = \delta$.

As Tables I and III show, the superposition of Yukawa potentials produces phase shifts having a definite trend from low L to high L for each fixed $L-J$ relationship. Nevertheless, we need up to and including the $^3H_4$ $\tilde{\delta}$ as input in order to obtain reasonable predictions for higher L's. For this potential, ΣY, the singlet results are good.

TABLE III. Nuclear bar values of $\tilde{\delta}$ at 210 MeV for the potentials $V_{\Sigma Y}$, Eq. (7). Values enclosed in brackets were calculated exactly via the Schrödinger equation.

State	Exact (ΣY)	OPT predictions $L \le 3$ input	$J \le 4$ input
ϵ_4	[+0.386]	−0.008	[+0.386]
$^3H_4$	[−0.015]	−0.048	[−0.015]
$^3H_5$	[+0.150]	+0.136	+0.104
$^3H_6$	[+0.012]	+0.001	+0.005
ϵ_6	[+0.033]	−0.007	+0.040
$^3K_6$	[+0.012]	−0.005	−0.000
$^1G_4$	[+0.183]	+0.229	[+0.183]
$^1I_6$	[−0.014]	+0.022	+0.015

C. One boson exchange

As a last model we examined a Bryan-Scott one boson exchange potential.[4] In Table IV we show the OPT predictions for this potential's $\tilde{\delta}$'s, and again for $T_{\text{lab}} = 210$ MeV. The Coulomb interaction is included. Tables I and IV reveal that this potential is very irregular, in the sense that there is no trend visible when going from low L to high L for a fixed $L-J$ relationship. Nevertheless, the predictions of the triplet coupled phases are similar to those of the other two potentials. That the $\tilde{\delta}$ ($^3H_5$) comes out with the wrong (positive) sign is

TABLE IV. Values of $\tilde{\delta}$ at 210 MeV for the BS-III potential. Values enclosed in brackets were calculated exactly via the Schrödinger equation.

State	Exact (BS-III)	OPT predictions $L\leq 3$ input	$J\leq 4$ input	$J\leq 5$ input
ϵ_4	[−0.066]	+0.011	[−0.066]	[−0.066]
$^3H_4$	[+0.031]	−0.020	[+0.031]	[+0.031]
$^3H_5$	[−0.008]	+0.043	+0.051	[−0.008]
$^3H_6$	[+0.046]	+0.052	+0.053	+0.052
ϵ_6	[−0.020]	−0.009	−0.007	−0.003
$^3K_6$	[+0.004]	−0.001	+0.007	+0.006
$^1G_4$	[+0.216]	+0.215	[+0.216]	
$^1I_6$	[+0.021]	+0.020	+0.020	

not surprising in view of the fact that both $\tilde{\delta}$ ($^3P_1$) and $\tilde{\delta}$ ($^3F_3$) are positive.

One sees from Tables I–IV that the predictive power of the OPT method is rather weak for the potentials examined. At 210 MeV one needs to know nearly all the partial waves which are treated as unknown in the usual phase shift analysis in order to make reasonable predictions for higher partial waves. This is not what had been expected for OPT.

IV. APPLICATION TO *pp* PHASE SHIFT ANALYSES

In this section we turn our attention to phase shift analyses and in particular to those of MAW-X (Ref. 5) in order to compare our results with Chao's.[3] Our own results (hereafter referred to as MSU) and those of Chao are displayed in Tables V and VI. Here the input was this set of nine phases: $^1S_0$, $^1D_2$, $^3P_0$, $^3P_1$, $^3P_2$, ϵ_2, $^3F_2$, $^3F_3$, and $^3F_4$. In obtaining the results shown in the tables, we have used Chao's form factor[3] and unitarization[3] for the 1π-exchange contribution. In direct contrast to Chao's report, we found little effect from these latter two refinements, especially for the triplet waves.

It is clear from the tables that our results for the triplet phases are in disagreement with Chao's. The disparity in $\tilde{\delta}(\epsilon_4)$ is particularly obvious. Just as for the potentials examined earlier, we were not able to generate satisfactory higher L $\tilde{\delta}$'s from lower L ones.

It is possible that much of the difference between our results and Chao's may originate in a subtle phase convention difference between SYM and GGMW. When we tried (improperly) ignoring that difference, as perhaps Chao might have done, our $\tilde{\delta}$'s moved substantially toward his. This was especially true for $\tilde{\delta}(\epsilon_4)$. Unfortunately, Chao's working computer program containing the phase definition did not seem to be available to us. Each of the

TABLE V. Comparison of the values of δ predicted by Chao and MSU with the results of the energy dependent (ED) phase shift analyses of MAW-VII (Ref. 5). $g^2/4\pi=15$.

State	MAW-VII	Chao	MSU	1π
		140 MeV		
$^1G_4$	0.71±0.02	0.72	0.72	0.57
ϵ_4	−0.77±0.03	−0.76	−0.88	−0.88
$^3H_4$	0.20±0.02	0.21	0.18	0.21
$^3H_5$	−0.57±0.04	−0.62	−0.54	−0.57
$^3H_6$	0.11±0.01	0.12	0.11	0.08
		220 MeV		
$^1G_4$	1.15±0.05	1.18	1.18	0.80
ϵ_4	−1.06±0.06	−1.05	−1.38	−1.34
$^3H_4$	0.38±0.07	0.36	0.32	0.40
$^3H_5$	−0.98±0.11	−0.99	−0.88	−1.00
$^3H_6$	0.27±0.03	0.27	0.25	0.16
		320 MeV		
$^1G_4$	−1.65±0.10	1.70	1.70	0.96
ϵ_4	−1.24±0.11	−1.22	−1.80	−1.77
$^3H_4$	0.58±0.16	0.51	0.45	0.62
$^3H_5$	−1.43±0.25	−1.33	−1.19	−1.46
$^3H_6$	0.51±0.07	0.49	0.47	0.27
		400 MeV		
$^1G_4$	2.01±0.14	2.09	2.09	1.03
ϵ_4	−1.30±0.16	−1.27	−2.24	−2.04
$^3H_4$	0.72±0.16	0.61	0.54	0.79
$^3H_5$	−1.73±0.38	−1.52	−1.32	−1.78
$^3H_6$	0.72±0.11	0.69	0.66	0.36

TABLE VI. Comparison of the values of δ predicted by Chao and MSU with the results of the single energy (SE) phase shift analyses of MAW-X (Ref. 5). $g^2/4\pi=14.43$.

State	MAW-X	Chao	MSU	1π
		210 MeV		
$^1G_4$	1.00±0.10	1.11	1.11	0.75
ϵ_4	−0.99±0.09	−1.02	−1.25	−1.24
$^3H_4$	0.24±0.21	0.39	0.35	0.36
$^3H_5$	−1.07±0.18	−0.90	−0.79	−0.91
$^3H_6$	0.15±0.13	0.31	0.29	0.15
		330 MeV		
$^1G_4$	1.22±0.28	1.68	1.68	0.95
ϵ_4	−1.11±0.28	−1.30	−1.83	−1.76
$^3H_5$	−1.80±0.46	−1.34	−1.21	−1.45
$^3H_6$	0.77±0.15	0.54	0.52	0.27

TABLE VII. Values of χ^2 from conventional phase shift analyses compared to those from OPT modified analyses at 210 MeV. In the 8-parameter search $^1S_0$, $^3P_0$, $^3P_1$, $^3P_2$, ϵ_2, $^3F_2$, and $^3F_3$ were allowed to be free. Phases successively added as free parameters are listed in the first column.

Additional phase parameters ($\tilde{\delta}$) set free in the search	Number of searched parameters	χ^2	χ^2 (OPT)
(base: see caption)	8	96.9	82.0
$^3F_4$	9	76.1	62.1
$^3F_4$, $^1G_4$	10	65.7	61.6
$^3F_4$, $^1G_4$, ϵ_4, $^3H_4$	12	46.7	50.8
$^3F_4$, $^1G_4$, ϵ_4, $^3H_4$, $^3H_5$, $^3H_6$	14	45.0	44.5

present authors checked our calculation independently and also checked those parts of Chao's calculation which were given explicitly in his thesis.[3] We were unable to find any change in our calculation that would cause it to reproduce Chao's result exactly.

In Table VII we compared values of χ^2 resulting from a conventional phase shift analysis[6] at 210 MeV to those resulting from OPT-modified analyses. In the conventional analysis the high L phases are set equal to the 1π-exchange values, whereas in the OPT-modified analysis they are predicted from the lower L phases. An examination of Table VII shows that no reduction in the number of free parameters can be achieved through utilization of OPT. We infer from results reported by Chao that this conclusion would be unaltered if the contribution from Cutkosky's ϕ^2 function[3] was included.

V. SUMMARY

Whereas the predictions of the high frequency components from the lower ones using Chao's optimized polynomial expansions are satisfactory for the singlet states, they are not so for the triplet states, either with the potential models we tested or with Chao's data selection. This is in contrast to Chao's results, but there are indications that Chao's calculation may have included an inconsistent phase definition. We conclude that there is not sufficient physical information contained in the analytic properties of the amplitudes to significantly reduce the number of free parameters in pp phase shift analyses at moderate energies. Thus the only advantage of using the OPT expansion at moderate energies would seem to lie in its provision of a smooth transition between low L "searched" and high L "nonsearched" phases.

Note added. This manuscript was sent to Prof. R. E. Cutkosky for comment, who with Chao reconstructed Chao's calculation and concluded (private communication from R. E. Cutkosky): (i) Chao did err in the aforementioned sign convention[9] and also at another point in the calculation; and (ii) the numbers in the present paper are the correct ones.

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[1]R. E. Cutkosky and B. B. Deo, Phys. Rev. 174, 1859 (1968).

[2]S. Ciulli, Nuovo Cimento 61A, 787 (1969).

[3]Y. A. Chao, Phys. Rev. Lett. 25, 309 (1970); thesis, Carnegie-Mellon University, 1970 (unpublished).

[4]R. A. Bryan and B. Scott, Phys. Rev. 177, 1435 (1969).

[5]M. M. Macgregor, R. A. Arndt, and R. M. Wright, Phys. Rev. 182, 1714 (1969); 169, 1128 (1968).

[6]M. D. Miller, P. S. Signell, and N. R. Yoder, Phys. Rev. 176, 1724 (1968).

[7]H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. 105, 302 (1957).

[8]M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. 120, 2250 (1960).

[9]The SYM states with definite J, M, L, and S are related to those of GGMW by

$$i^L \phi_M^J(L,S)_{\rm SYM} = \phi_M^J(L,S)_{\rm GGMW} \ ,$$

which means that the two conventions differ by a minus sign in the off-diagonal elements α^J.

[10]S. Mandelstam, Phys. Rev. 112, 1344 (1958).

[11]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).