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Aspects of deuteron production in relativistic heavy ion collisions

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One criticism of the thermodynamic model for deuteron production has been that it may not be possible for deuterons to exist inside the fireball. We give arguments why deuterons can exist inside the fireball, although they cannot exist in cold nuclear matter at normal density. We redefine the Berkeley coalescence model and show that with this redefinition, the coalescence model is rather close to the thermodynamic prescription. We show that this redefined coalescence model is equivalent to the quantum mechanical sudden approximation model considered by Kapusta. We construct the grand canonical partition function for interacting fermions up to the second virial coefficient and show that the redefined coalescence model and the simple thermodynamic model are two limits of a more complete theory.

[NUCLEAR REACTIONS Deuteron production; relativistic heavy ions; thermodynamic model; coalescence model; sudden approximation model; second virial coefficient.]

I. INTRODUCTION

Thermodynamic models[1,2] have enjoyed some success[3,4] and popularity in the description of relativistic heavy ion collisions. In these models it is necessary to calculate the many body partition function or equivalently the available phase space. Since we are dealing with an interacting system this is a nontrivial problem. In most calculations it is assumed the interactions play two roles; first, they cause the system to thermalize, and second, they cause the appearance of bound states such as the deuteron and alpha as well as resonances such as the delta and excited states of the alpha. These bound states and resonances are then treated as separate species of particles in a noninteracting gas model. This procedure can be questioned on several grounds. First of all, does it make sense to consider a deuteron as existing inside a fireball since it is very weakly bound? For example, in zero temperature nuclear matter the deuteron becomes unbound at less than 10% of the saturation density. This problem is less severe for more strongly bound states such as the α particle. There is also an overcounting problem since, due to Levinson's theorem, we know that a potential cannot change the total number of states. This is an acute problem if we include all the scattering resonances in the various nucleon-nucleon and nucleon-nucleus channels as has been suggested.[2]

An alternate approach is possible. One can assume that inside the fireball no bound states exist and we have a purely free gas. The bound states then form at some breakup density. This leads to the coalescence model.[5,6]

In this paper we study the problem of how to calculate the partition function in the interacting system concentrating primarily on the role of the deuteron. In most of the discussion we ignore the higher-A bound states for simplicity. (The numbers of ^{3}He, ^{3}H, etc., are small.)

In the next section we briefly present the simple thermodynamic prescription of Ref. 4, where the deuteron is treated as a separate noninteracting particle. In Sec. III we argue that because of the low phase space density this is reasonable. In Sec. IV we present a redefined coalescence model which is shown in Sec. V to be equivalent to the quantum mechanical sudden approximation discussed by Kapusta.[7]

In Sec. VII we show that the partition function for the interacting system can be more properly obtained through the virial expansion. Our purpose here is not to do new calculations to compare with experiments but rather to discuss the validity of some widely used models. We do not know of any practical example where the second virial coefficient has been obtained for a relativistic quantum system, although the formalism for such calculations has been set up.[8] The nonrelativistic treatment done here, apart from clearly demonstrating the links between different prescriptions, will also show which simple prescription is more accurate. The prescription can then be made relativistic. We do not consider in this paper the question of whether the system can actually thermalize but consider only what happens if it thermalizes.

II. A SIMPLE THERMODYNAMIC PRESCRIPTION

Consider a fireball which contains nucleons. Because of interactions, a deuteron can form and we assume that the only role of the forces is to allow the deuterons to form and dissociate. Otherwise, the nucleons and deuterons are treated as free particles inside a box of volume V. The simple thermodynamic prescription is then

$$n_p=2\frac{V}{h^3}\left[\frac{2\pi m}{\beta}\right]^{3/2}e^{\lambda_p}\,, \qquad (1)$$

$$n_n=2\frac{V}{h^3}\left[\frac{2\pi m}{\beta}\right]^{3/2}e^{\lambda_n}\,, \qquad (2)$$

$$n_d=3\frac{V}{h^3}\left[\frac{4\pi m}{\beta}\right]^{3/2}e^{\lambda_p+\lambda_n}e^{\beta E_B}\,. \qquad (3)$$

Here, n_p, n_n, and n_d are the numbers of protons, neutrons, and deuterons, respectively; β is the inverse of the temperature τ; and E_B is the binding energy of the deuteron. The quantity e^{λ} is called the fugacity[9]; $\lambda=\beta\mu$, where μ is the chemical potential. The three constants λ_p, λ_n, and β can be eliminated using the following three equations:

$$n_p+n_d=\text{number of primordial protons}\,, \qquad (4)$$

$$n_n+n_d=\text{number of primordial neutrons}\,, \qquad (5)$$

$$\frac{3}{2\beta}(n_p+n_n+n_d)\quad=\text{energy determined from kinematics}\,. \qquad (6)$$

One can also derive the following equation:

$$\frac{d^3n_d}{d^3p_d}(\vec{p}_d)=\frac{3}{4}\frac{h^3}{V}\frac{d^3n_p}{d^3p}(\vec{p})\frac{d^3n_n}{d^3p}(\vec{p})\exp(\beta E_B)\,, \qquad (7)$$

where $\vec{p}_d=2\vec{p}$; we note that $(d^3n/d^3p)(\vec{p})$ here refers to the measured protons, not to the primordial protons.

Other composites can be included if needed. Resonances have sometimes been included in the calculation[3] and sometimes left out[4]. For example, in some calculations the $T=1$, $s=0$, $l=0$ unbound state of the deuteron (this state is, of course, also present in the n-n and p-p channels) is included when calculating the temperature. Later, the state decays into a neutron and a proton. We will show that it is better not to include the $T=1$, $s=0$ state in the simple thermodynamic prescription.

III. CAN DEUTERONS EXIST IN THE FIREBALL?

One often quoted criticism of the thermodynamic model for deuteron production is that it may not be possible for deuterons to exist inside the fireball. We will now argue that because of high temperature in the fireball, it makes sense to talk of a deuteron in the fireball. Consider a deuteron and another nucleon. If the third nucleon is outside the "volume" of the deuteron then clearly it is reasonable to talk of a deuteron and a nucleon.

Now switch from configuration space to momentum space. If in momentum space the third nucleon is outside the volume of the deuteron, then it is just as good to talk of a deuteron. At zero temperature and normal nuclear density, the probability of finding a third nucleon inside the momentum space volume occupied by a deuteron is large; but at high temperature and somewhat reduced density the occupation is uncrowded and the probability of finding a third nucleon in the momentum space volume is small. In such a case we still have a deuteron and third nucleon.

One might argue that in configuration space the nucleon-nucleon force is short range, whereas in momentum space it is not. But that only means that because of the third particle the deuteron can break up. But if it breaks up, it can recombine as well. According to two previous calculations[2,7] there is enough time for equilibration.

Although the above arguments should suffice, we will set up a small model calculation reinforcing what we have just said. The example is well known in the cluster model of nuclei.[10]

Consider a cluster enclosed in a box in which there are also free nucleons. We impose periodic boundary conditions. At zero temperature, the free nucleons will fill up to the Fermi level. For simplicity consider a zero net momentum cluster wave function in the box represented by $\chi(|x_1-x_2|)$. Since the cluster is in a box which also has other nucleons, we need to antisymmetrize between the cluster particles and the particles inside the Fermi sea. The process of antisymmetrization will drastically alter the nature of $\chi(|x_1-x_2|)$. The Fourier components that are already contained in the wave functions of the free uncorrelated nucleons will be removed from $\chi(|x_1-x_2|)$. The new normalized $\chi(|x_1-x_2|)$ will no longer look like the free $\chi(|x_1-x_2|)$.

To be concrete, let the cluster size be l and the cluster wave function be

$$\chi(|x_1-x_2|)=\frac{1}{\sqrt{2l}}[e^{iq(x_1-x_2)}+e^{-iq(x_1-x_2)}] .$$

The free nucleons inside the box are represented by $(l/\sqrt{L})e^{ikx}$. The square of the overlap of the wave function of particle 1 in the cluster with a wave function in the box is

$$|a_k|^2=\frac{l}{2L}\left\{\frac{\sin[(k-q)l/2]}{[k-q]l/2}+\frac{\sin[(k+q)l/2]}{[k+q]l/2}\right\}^2 .$$

If the occupation in the box is such that all orbitals for which $|a_k|^2$ is significant are occupied, then the intrinsic nature of the cluster is lost.

The situation is altered in the high temperature case. The occupation probability in any given momentum state is small; thus, the presence of other nucleons does not significantly alter the state of the cluster. We will now establish what is precisely the parameter that has to be small for the deuteron to exist in the fireball.

Let V be the volume of the fireball, τ the temperature, and n the number of nucleons. For simplicity, $n_p=n_n=n/2$. Consider a deuteron in the $s=1$, $M=1$ state. Let Δ^3p be the characteristic volume of a deuteron in momentum space; $\tilde{v}$ is the characteristic volume in the configuration space. Then the probability that a given proton (or a neutron) with spin up appears in a certain configuration space $\tilde{v}$ and within a momentum volume Δ^3p around p is

$$P(\vec{p})=\frac{\tilde{v}}{V}\frac{l}{(n/2)}\frac{1}{2}\Delta^3p\frac{d^3n_p}{d^3p}(\vec{p}) . \tag{8}$$

The factor $n/2$ is the number of protons; $\frac{1}{2}$ is inserted because of spin up. Since the phase space for one quantum state is h^3 we put $\Delta^3p\tilde{v}=h^3$. Now,

$$\frac{d^3n_p}{d^3p}(\vec{p})=2\frac{V}{h^3}e^{\lambda}e^{-p^2/(2m\tau)} , \tag{9}$$

$$\frac{n}{2}=\int\frac{d^3n_p}{d^3p}d^3p=2\frac{V}{h^3}e^{\lambda}(2\pi m\tau)^{3/2} . \tag{10}$$

Thus, Eq. (8) reduces to

$$P(\vec{p})=\frac{h^3}{2V}\frac{1}{(2\pi m\tau)^{3/2}}e^{-p^2/(2m\tau)} . \tag{11}$$

At the most probable velocity,

$$P=\frac{1}{e}\frac{h^3}{2V}\frac{1}{(2\pi m\tau)^{3/2}} . \tag{12}$$

The probability that a neutron and proton appear in this phase space and no other nucleons appear is

$$\frac{n}{2}P(1-P)^{n/2-1}\frac{n}{2}P(1-P)^{n/2-1}=\frac{n^2}{4}P^2(1-P)^{n-2} . \tag{13}$$

For the deuteron to exist we need $(1-P)^{n-2}$ to be close to unity. This is guaranteed if $(n-2)P\approx np<<1$. From previous calculations,[4] this number is about 0.1 in fireballs. Let us now consider the case for zero temperature. Once again [Eq. (8)],

$$P=\frac{h^3}{nV}\frac{d^3n_p}{d^3p} .$$

In the present case of zero temperature,

$$\frac{d^3n_p}{d^3p}=2\frac{V}{h^3} , \quad \text{for } p<p_F$$

$$P=\frac{2}{n} .$$

Thus, $nP=2$ and the cluster picture is invalid.

Returning to the case of the fireball we see that the worst case of interference from antisymmetry will occur when the deuteron is emitted with zero velocity in the fireball. The quantity $P(\vec{p})$ of Eq. (11) is now $P=(h^3/2V)\,[1/(2\pi m\tau)^{3/2}]$; the quantity $(n/2)P$ is simply e^{λ} [see Eq. (10)], the fugacity of the protons. The smallness of this quantity determines how well the Fermi-Dirac or Bose-Einstein distribution degenerates into the

Maxwell-Boltzmann distribution. Indeed, it was pointed out[11] that unless the Maxwell-Boltzmann limit is achieved, the idea of composite formation in the fireball does not have much meaning. In a typical fireball the fugacity is about 0.1. In a classical gas, the fugacity at room temperature and atmospheric pressure is much closer to zero. Thus, of course, we do not expect a simple thermodynamic prescription to be as valid for the fireball as for the classical gas. Nonetheless, an expansion in powers of fugacity appears meaningful. Experimentally, one sees more protons (goes similar to e^{λ}) than deuterons (goes similar to $e^{2\lambda}$) and more deuterons than tritons (goes similar to $e^{3\lambda}$). A cluster expansion seems useful and we will do such an expansion later.

IV. A REDEFINED COALESCENCE MODEL

The assumption of the coalescence model is that a neutron and a proton leaving the fireball with about the same momentum will coalesce into a deuteron. Following the arguments of Kapusta we formalize this statement and show that with a slight change in the formulation the coalescence model becomes much closer to the thermodynamic model. Consider again $n_p = n_n = n/2$ and the formation of a deuteron in the $s=1$, $M=1$ state. Consider a volume element $\Delta^3 p$ centered at $\vec{p}$. The probability of finding a given proton with spin up in this sphere is

$$\tilde{P}(\vec{p}) = \frac{1}{(n/2)}\frac{1}{2}\Delta^3 p \frac{d^3 n_0}{d^3 p}(\vec{p}) \ . \tag{14}$$

However, in order to form a deuteron the nucleons have to be correlated not only in momentum space but in configuration space as well. Thus, we consider the probability that the proton is not only in the given momentum space volume but also in a given configuration space volume $\tilde{v}$ in the fireball. This volume $\tilde{v}$ is the characteristic volume of the deuteron. We are thus led to consider

$$P(\vec{p}) = \frac{\tilde{v}}{V}\frac{1}{(n/2)}\frac{1}{2}\Delta^3 p \frac{d^3 n_p}{d^3 p}(\vec{p}) \ . \tag{15}$$

We have assumed that the characteristic volume of the deuteron is smaller than the volume of the fireball. Unless this is true there is no hope of finding a correspondence with statistical mechanics which is based on the premise that the container is larger than the objects contained. The measured root mean square radius of the deuteron is 2.1 fm; typical radii of fireballs are $3-5$ fm. The probability that both a neutron and a proton with spin up appear in the same part of the phase space with no other spin up nucleons in the same phase space is once again

$$\frac{n}{2}P(1-P)^{n/2-1}\frac{n}{2}P(1-P)^{n/2-1} \approx \frac{n^2}{4}P^2 \ . \tag{16}$$

We are, however, not interested in knowing in which of the volumes $\tilde{v}$ in the fireball the two nucleons appeared. The number of such elements is $(V/\tilde{v})$. We thus obtain

$$\frac{V}{\tilde{v}}P^2 n^2/4 = \frac{\tilde{v}}{V}\frac{1}{4}\Delta^3 p_1 \Delta^3 p_2 \left[\frac{d^3 n_p}{d^3 p}(\vec{p})\right]^2 \ .$$

To obtain $d^3 n_d / d^3 p_d$ note that $d^3 p_1\, d^3 p_2 = d^3 p_d\, d^3 p_r$, where $\vec{p}_d$ and $\vec{p}_r$ are center of mass and relative momenta. We now equate $d^3 p_r \tilde{v} = h^3$, the phase space volume of one quantum state. We only considered the $s=1$, $M=1$ state. Including other M states, we obtain

$$\frac{d^3 n_d}{d^3 p_d}(\vec{p}_d) = \frac{3}{4}\frac{h^3}{V}\left[\frac{d^3 n_p}{d^3 p}(\vec{p})\right]^2 \ . \tag{17}$$

The above equation is now quite close to Eq. (7); the difference is the factor $\exp(\beta E_B)$ and also that $d^3 n_p / d^3 p$ in Eq. (17) refers to the primordial protons, not to the measured protons. Note that in contrast to the usual coalescence model we have a factor $1/V$.

V. THE SUDDEN APPROXIMATION MODEL

We now show that the redefined coalescence model is equivalent to the quantum mechanical sudden approximation model considered by Kapusta. In this model, deuterons arise because proton and neutron wave functions in the fireball have an overlap with a deuteron wave function; the absolute square of this overlap gives the number of deuterons. Let us denote the nucleon wave functions in the fireball by $(1/L^{3/2})\, e^{(i/\hbar)\vec{p}\cdot\vec{r}}$; here, L is a physical parameter and L^3 is the volume of the fireball. We ignore the spin factors for the moment and put them in later. The number of protons in a momentum interval is $(d^3 n_p / d^3 p)\,(\vec{p}_1)\, d^3 p_1$ and the number of neutrons is $(d^3 n_n / d^3 p)\,(\vec{p}_2) d^3 p_2$. Let the deuteron internal wave function be $\chi(r)$. The overlap of a two particle wave function with the deuteron wave function can be calculated from

$$\int \chi^*(r)\frac{1}{L^3}e^{(i/\hbar)\vec{p}_1\cdot\vec{r}_1}e^{(i/\hbar)\vec{p}_2\cdot\vec{r}_2}d^3r=\frac{1}{L^3}\int \chi^*(r)e^{(i/\hbar)(\vec{p}_1+\vec{p}_2)\cdot\vec{R}}e^{(i/\hbar)[(\vec{p}_1-\vec{p}_2)\cdot\vec{r}]/2}d^3r$$
$$=\frac{1}{L^3}\int e^{(i/\hbar)\vec{p}_d\cdot\vec{R}}\chi^*(r)e^{(i/\hbar)\vec{p}\cdot\vec{r}}d^3r$$
$$=\left[\frac{2\pi\hbar}{L}\right]^{3/2}\frac{1}{L^{3/2}}e^{(i/\hbar)\vec{p}_d\cdot\vec{R}}\psi^*(p)\ . \tag{18}$$

In the last step we have assumed that the fireball is bigger than the characteristic volume of the deuteron, since we have equated

$$\int_{\text{fireball}}\chi^*(r)e^{(i/\hbar)\vec{p}\cdot\vec{r}}d^3r=(2\pi\hbar)^{3/2}\psi^*(p)\ ,$$

where $\psi(p)$ is the Fourier transform of the deuteron internal wave function. The number of deuterons is

$$d^3n_d=\int\frac{d^3n_p}{d^3p}(\vec{p}_1)\frac{d^3n_n}{d^3p}(\vec{p}_2)d^3p_1d^3p_2\frac{h^3}{L^3}\psi^*(p)\psi(p)$$
$$=\frac{h^3}{L^3}\int\frac{d^3n_p}{d^3p_1}(\vec{p}_d/2+\vec{p})\frac{d^3n_n}{d^3p_2}(\vec{p}_d/2-\vec{p})d^3p_d d^3p\,\psi^*(p)\psi(p)\ .$$

Assuming that the neutron or ptoton distributions do not change significantly within the range of p for which $\psi(p)$ is significant and putting in the spin factor, we have

$$\frac{d^3n_d}{d^3p_d}(\vec{p}_d)=\frac{3}{4}\frac{h^3}{V}\frac{d^3n_p}{d^3p}(\vec{p}_d/2)\frac{d^3n_n}{d^3p}(\vec{p}_d/2)\ . \tag{19}$$

This is identical to Eq. (17).

VI. THE UPPER BOUND FOR FUGACITY IN THE COALESCENCE MODEL

The approximation of Eq. (16) has to be reasonably good for either the thermodynamic prescription or the coalescence model. In the thermodynamic model, it is needed so that the effects of antisymmetry be small. In the coalescence model it is needed both for antisymmetry and for obtaining a simple final answer. The thermodynamic prescription does not demand that d^3n_d is proportional to $(n^2/4)P^2$; the coalescence model does. It is not difficult to see that there will be a severe overcounting problem in the coalescence model unless nP is quite small. The limit can be easily obtained.

We consider again $n_p=n_n=n/2$. We remind the reader that in Eq. (17), d^3n_p/d^3p refers to primordial protons. Now

$$\frac{d^3n_p}{d^3p}=2\frac{V}{h^3}e^{\lambda}e^{-p^2/(2m\tau)}\ ,$$

$$n_p=2\frac{V}{h^3}e^{\lambda}(2\pi m\tau)^{3/2}\ .$$

Equation (17) now gives

$$\frac{d^3n_d}{d^3p_d}=3\frac{V}{h^3}e^{2\lambda}e^{-p_d^2/(4m\tau)}\ ,$$

so that

$$n_d=3\frac{V}{h^3}e^{2\lambda}(4\pi m\tau)^{3/2}\ .$$

Since the number of deuterons cannot be greater than the number of primordial protons we obtain an upper limit for fugacity by setting $n_d=n_p$. This gives the limit at which the approximation of neglecting higher-A clusters completely break down:

$$e^{\lambda}=\frac{2}{3}\frac{1}{\sqrt{8}}=0.236\ . \tag{20}$$

VII. STATISTICAL MECHANICS OF INTERACTING PARTICLES AT FINITE TEMPERATURE

We will now show that the simple thermodynamic prescription of Sec. II. and the coalescence model of Sec. IV are two limits of a more complete theory. The quantity of central importance in equilibrium statistical mechanics is the grand partition function. An expansion of the quantum mechanical grand partition function up to the second virial coefficient is well known.[12] We consider a box of nucleons interacting by a two body force; for simplicity, the force acts only in the $l=0$ state. The grand partition function can be written as

$$\ln Z_{\rm gr}=A+B\ , \tag{21}$$

$$A=\frac{V}{h^3}[e^{\lambda_p}2(2\pi m\tau)^{3/2}+e^{\lambda_n}2(2\pi m\tau)^{3/2}]\ , \tag{22}$$

$$B=\frac{V}{h^3}3e^{\lambda_p+\lambda_n}(4\pi m\tau)^{3/2}\left[e^{\beta E_B}-1+\frac{\beta}{\pi}\int\delta_t(t)e^{-\beta t}dt\right] + \frac{V}{h^3}(e^{2\lambda_p}+e^{2\lambda_n}+e^{\lambda_p+\lambda_n})(4\pi m\tau)^{3/2}\frac{\beta}{\pi}\int\delta_s(t)e^{-\beta t}dt\ . \tag{23}$$

For noninteracting nucleons, B is zero. The virial expansion stops at B by assuming that only two-body clusters are important. Higher order clusters contribute to higher order virial coefficients. For the second order virial coefficient one just has to solve the two body bound and scattering problem; in Eq. (23), δ_t and δ_s refer to $l=0$ triplet and singlet phase shifts. At zero temperature all scattering states are occupied and because of Pauli exclusion principle, two body scattering at zero temperature is more complicated.[13] At high temperature (because of low occupation probability in any given single particle state), the Pauli exclusion problem disappears and one just has to solve the two body scattering problem. Note that up to the order written down in Eq. (21) deuterons are treated as free particles; an interaction of a deuteron with another nucleon would be a three body cluster. One now proceeds in the following way: Given the number of primordial protons, neutrons, and the nonrelativistic energy, one sets

$$\frac{\partial \ln Z_{\rm gr}}{\partial\lambda_p}=\text{number of primordial protons}\ ,$$

$$\frac{\partial \ln Z_{\rm gr}}{\partial\lambda_n}=\text{number of primordial neutrons},$$

$$\frac{-\partial \ln Z_{\rm gr}}{\partial\beta}=\text{energy}\ .$$

These equations eliminate λ_p, λ_n, and β. The number of deuterons is now obtained from

$$\frac{3V}{h^3}e^{\lambda_p+\lambda_n}e^{\beta E_B}(4\pi m\tau)^{3/2}=\text{number of deuterons}\ .$$

The difference between the number of primordial protons and deuterons is the number of the measured protons.

The simpler thermodynamic prescription of Sec. II is obtained by setting

$$B=3\frac{V}{h^3}e^{\lambda_p+\lambda_n}(4\pi m\tau)^{3/2}e^{\beta E_B}\ . \tag{24}$$

In treatments where the singlet state is also included as a particle, at a positive energy E' one has set

$$B=3\frac{V}{h^3}e^{\lambda_p+\lambda_n}(4\pi m\tau)^{3/2}e^{\beta E_B} + \frac{V}{h^3}(4\pi m\tau)^{3/2}e^{-\beta E'}(e^{2\lambda_p}+e^{2\lambda_n}+e^{\lambda_p+\lambda_n})\ . \tag{25}$$

Since, for high temperature, $e^{\beta E_B}\approx 1$, the coalescence model can be recovered by setting

$$B=3\frac{V}{h^3}e^{\lambda_p+\lambda_n}(4\pi m\tau)^{3/2}(e^{\beta E_B}-1)\ . \tag{26}$$

The various widely used models are thus just simple approximations to the full second virial coefficient and we will now show how accurate the various approximations are. It should be kept in mind that there is also the question of convergence of the virial expansion. One can mock up higher order terms in the virial expansion by modifying the second virial coefficient.

In order to estimate the relative importance of the phase shift integrals we take the nucleon nucleon potential to be hard core followed by an attractive square well[14]; the potentials have the correct scattering lengths and effective ranges as well as having the zeros of the phase shifts at the experimental energies for both singlet and triplet scattering. To make the calculation more realistic we include the $\Delta(3,3)$ as a particle. We use the two fireball model,[15] consider 800 MeV/nucleon laboratory energy and equal ion collisions. Compared to the full calculation [Eqs. (21)–(23)] the simple thermodynamic prescription [Eq. (24)] overestimates the temperature by 4% and underestimates the deuteron cross section by about 35%. The reason the agreement is so good is because the minus one in Eq. (23) is cancelled by phase shifts in both the single and triplet channels. Including also the singlet state as a particle [Eq. (25)] the temperature goes up by another 4% and the number of deuterons further drops. Thus, in the simple thermodynamic prescription it is better to leave out the interactions in the singlet state. One expects that including all the resonances in the three- and four-body channels (excited states of ^{3}H, ^{3}He, and ^{4}He) will also lead to over counting.

In the coalescence model [Eq. (26)], the temperature is underestimated by about 5% and the number of deuterons overestimated by 45%. The coalescence model will improve as the temperature increases.

Given the grand partition function [Eqs. (21)–(23)] all other thermodynamic quantities can be calculated. Correction to the entropy[15] due to the interactions can be obtained simply. It is harder to calculate d^3n/d^3p if all the terms in Eq. (23) are retained. The reason is not hard to find; for interacting fermions, the inclusive cross section is not simple. Nonetheless, it is possible to obtain an expression for d^3n/d^3p. Since the discussion of this requires some mathematical complexity, we consider this problem in the Appendix.

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APPENDIX

In this Appendix we develop the framework for calculating a deuteron and proton inclusive spectrum when we have an interacting gas. We shall consider only nonrelativistic kinematics with only a few comments on possible relativistic generalizations.

We start not from the grand canonical ensemble but from the microcanonical ensemble and proceed as in Ref. 16. The inclusive cross section for deuterons can be written as a ratio

$$\frac{d^3n_d}{d^3p}=\frac{N}{D}\,, \tag{A1}$$

where D is the total number of states available to the system subject only to the constraints on total energy baryon number, charge, and momentum. The denominator can be expressed in the form

$$D=\mathrm{Tr}\,\delta(E-\hat{H})\delta(\vec{p}-\hat{\vec{p}})\delta(Q-\hat{Q})\delta(B-\hat{B})\,, \tag{A2}$$

where the ^ denotes operators. Thus, $\hat{B}$ is the baryon number operator. The trace is unrestricted.

The numerator N is the number of states available to the system when there is a deuteron of momentum $\vec{p}_d$ present. It can be written as

$$N=\mathrm{Tr}\,\delta(E-\hat{H})\delta(\vec{p}-\hat{p})\delta(Q-\hat{Q})\delta(B-\hat{B})P_D\,, \tag{A3}$$

where P_D projects out all states that contain a deuteron of momentum $\vec{p}_d$. More will be said about this operator later.

We now proceed to evaluate D. To do this we Laplace transform on E, P, Q, and B to obtain

$$L(D)=Z_D(\alpha,\beta,\gamma,\vec{q})=\mathrm{Tr}\,e^{-\beta\hat{H}}e^{-\hat{p}\cdot\vec{q}}e^{-\alpha\hat{B}}e^{-\gamma\hat{Q}}\,. \tag{A4}$$

This differs from the canonical partition function only due to the factor $e^{-\vec{p}\cdot\vec{q}}$, which came from total momentum conservation. We can approximately recover D from Z_D by using the saddle point approximation to the Laplace inversion integral.[16] Then we have only to evaluate Z_D, which can be done using the virial expansion, and we have to second order

$$\ln Z_D(\alpha,\beta,\gamma,\vec{q})=\sum_i e^{-\alpha B_i}e^{-\gamma Q_i}\frac{V}{h^3}\int d^3p\,e^{-\beta\epsilon_i(p)}e^{-\vec{q}\cdot\vec{p}} + \frac{V}{h^3}\sum_{i,j}e^{-\alpha(B_i+B_j)}e^{-\gamma(Q_i+Q_j)}\int d^3p_{\mathrm{c.m.}}\,e^{-\beta\epsilon_{ij}(p_{\mathrm{c.m.}})}e^{-\vec{q}\cdot\vec{p}_{\mathrm{c.m.}}} \times\left[\sum_K e^{\beta|\epsilon_K|}+\sum_S\frac{2S+1}{\pi}\int e^{-\beta\epsilon}\frac{d\delta_{ij}}{d\epsilon}d\epsilon\right], \tag{A5}$$

where i,j label the different species. Note that, in principle, the Δ resonance can be included through the phase shifts in the πN channel. This would correctly include the effects of the width.

For the numerator we proceed as previously. The situation simplifies for deuterons since a deuteron is already a two body cluster, and hence, a nucleon deuteron interaction is a three body cluster and neglected. We have

$$\ln Z_N = -\alpha-\gamma-\beta\epsilon_d(p_d)-\vec{q}\cdot\vec{p}_d+\ln Z_D \ .$$

In the limit when the saddle point for numerator and denominator occur at the same value we recover the usual grand canonical ensemble result.

An additional complication arises in the case of protons and in this case Z_N is given by

$$\begin{aligned}\ln Z_N = {} & -\alpha-\gamma-\beta\epsilon_p(p)-\vec{q}\cdot\vec{p}+\ln Z_D \\ & +\sum_i e^{-\alpha B_i} e^{-\gamma Q_i} \sum \frac{2S+1}{\pi}\int dp_r \frac{d\delta_l}{dp_r} e^{-\beta\epsilon_r(p_r)} \\ & \times \frac{1}{4\pi}\int d\Omega_{p_r} \exp\left\{-\beta\left[\epsilon_{\text{c.m.}}\left[\vec{p}-\frac{m_i}{m_p+m_i}\vec{p}_r\right]-\epsilon_p\right]\right\} \\ & \times \exp\left[\vec{q}\cdot\vec{p}_r\frac{m_i}{m_r+m_i}\right] ,\end{aligned} \tag{A6}$$

where the subscript r refers to relative quantites (momentum, etc.). Specializing to the case of only interacting neutrons and protons we have for the inclusive spectrum for protons in the grand canonical ensemble limit

$$\begin{aligned}\frac{d^3n}{d^3p}(p) = {} & \frac{V}{h^3}e^{-\alpha}e^{-\gamma}e^{-\beta\epsilon_p}\left\{2+e^{-\alpha}\frac{3}{\pi}\int dp_r\frac{d\delta_s}{dp_r}e^{-\beta\epsilon(p_r)}\frac{1}{4\pi}\int d\Omega_{p_r}e^{-\beta[\epsilon_{\text{c.m.}}(2p-2p_n)-\epsilon(p)]}\right. \\ & \left. +e^{-\alpha}(1+e^{-\gamma})\frac{1}{\pi}\int dp_r\frac{d\delta_t}{dp_r}e^{-\beta\epsilon(p_r)}\frac{1}{4\pi}\int d\Omega_{p_r}e^{-\beta[\epsilon_{\text{c.m.}}(2p-2p_r)-\epsilon_p(p)]}\right\} .\end{aligned} \tag{A7}$$

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[1]A. Z. Mekjian, Phys. Rev. Lett. 38, 640 (1977).
[2]A. Z. Mekjian, Phys. Rev. C 17, 1051 (1978).
[3]J. Gosset, J. I. Kapusta, and G. D. Westfall, Phys. Rev. C 18, 844 (1978).
[4]S. Das Gupta and A. Z. Mekjian, Phys. Rep. 72, 131 (1981).
[5]J. Gosset, H. H. Gutbrod, W. G. Meyer, A. M. Poskanzer, A. Sandoval, R. Stock, and G. D. Westfall, Phys. Rev. C 16, 629 (1977).
[6]M. C. Lemaire, S. Nagamiya, S. Schnetzer, H. Steiner, and I. Tanihata, Phys. Lett. 85B, 38 (1979).
[7]J. I. Kapusta, Phys. Rev. C 21, 1301 (1980).
[8]R. Dashen, S. Ma, and H. J. Bernstein, Phys. Rev. 187, 345 (1969).
[9]K. Huang, *Statistical Mechanics* (Wiley, New York, 1965), p. 164.
[10]K. Wildermuth and W. McClure, *Cluster Representations of Nuclei* (Springer, New York, 1966), p. 57.
[11]S. Das Gupta, Phys. Rev. C 18, 2773 (1978).
[12]See, for example, L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley, Reading, Massachusetts, 1970), pp. 215–234.
[13]See, for example, M. A. Preston and R. K. Bhaduri, *Structure of the Nucleus* (Addison-Wesley, Reading, Massachusetts, 1975), Chap. 8.
[14]H. A. Enge, *Introduction to Nuclear Physics* (Addison-Wesley, Reading, Massachusetts, 1966), pp. 40–64.
[15]S. Das Gupta and C. S. Lam, Phys. Rev. C 20, 1192 (1979).
[16]B. K. Jennings, L. Satpathy, S. Das Gupta, Phys. Rev. C 24, 440 (1981).