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# Relationship between the Bohr-Mottelson model and the interacting boson model

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The interacting boson model was invented in two independent modes: The Schwinger mode using six bosons ($s$ and five $d$ bosons) and the Holstein-Primakoff mode using five quadrupole quasibosons. We show that the mathematical equivalence of the two modes can be used to define a number conserving quadrupole boson (the $b$ boson). Two equivalent bases, the usual $s$-$d$ basis and a new $s$-$b$ basis, are exhibited. By an exercise of (possibly objectionable) physical license, the result can be interpreted as a proof of equivalence of interacting boson model I with the Bohr-Mottelson model. In the $s$-$b$ basis, the Hamiltonian and other operators depend only on the $b$ boson. In this form, all the topics usually associated with the Bohr-Mottleson model can be discussed: potential energy surface, shape parameters, vibrations vs rotations, etc. The precise relationship of our method to that employed in previous work is exposed. The latter is shown to correspond to the use of the Dyson generators of SU(6).

[NUCLEAR STRUCTURE Interacting bosons, Bohr-Mottelson form of IBM, potential energy surface from IBM, generator coordinates and IBM.]

## I. INTRODUCTION

The success, widely (if not universally) acknowledged, of the interacting boson model[1-4] (IBM) in the interpretation of nuclear band structure raises a basic question concerning its relationship to the classic Bohr-Mottelson model[5-7] (BMM). This question has already been explored from several standpoints. Meyer-ter-Vehn,[8] for example, has argued for the equivalence of the O(5) limit of the IBM with the $\gamma$-unstable version[9] of the BMM. This kind of specialized study had its stimulus in the very origins of the IBM where the equivalence of the two models in the SU(5) limit was almost a truism.[10]

In this paper, we are more interested in efforts to understand the relationship between the two models on a more general basis. Thus several authors[11-17] have shown that the method of coherent states (intrinsic state or condensate state) yields the classical limit of the IBM in the form of a potential energy surface (given as a function of the intrinsic shape variables) whose properties can be compared directly with the corresponding surface well known for the BMM. At the same time, all the limiting symmetries of the IBM can be identified.

This work still leaves open the problem of comparing the models in the fullest possible sense. But we must try first to distinguish such a comparison from a comparison of the two independently postulated forms of the IBM.[1,2] The form proposed in Ref. 1 involving six bosons, one carrying angular momentum zero (the $s$ boson) and five carrying angular momentum two (the $d$ boson), is usually referred to as the IBM. The form proposed in Ref. 2 involving five bosons carrying angular momentum two is sometimes called the truncated quadrupole phonon model (TQM). Each of these SU(6) postulations (remarkably) puts restrictions on the form of the Hamiltonian which renders the two models exactly equivalent as phenomenological models, as has been remarked by several authors.[18,19] However, if one credits the "derivation" of Ref. 2, the bosons introduced there map quasiparticle pairs and therefore are *necessarily* quasibosons with no sharp selection rule *vis a vis* the particle number operators. Even if one argues that the TQM is a subclass of

the BMM, does it follow that the quadrupole phonons of the latter are necessarily quasibosons? This is one of the questions we shall address in our work (and answer in the negative). Thus although our mathematical tools may, in part, appear indistinguishable from those of Refs. 18 and 19, our physics is different.

Other aspects of the possible relationship between the IBM and the BMM have been explored without having provided a definitive solution to the problem. Thus in Ref. 10, it is shown that the $s$ boson may be eliminated, using the properties of the special (seniority) basis used in this work. It is then argued that in the limit that the number of bosons ($N$) increases without limit, we regain the forms of the BMM. Another consequence of the present work is that this argument is incomplete[20]; in particular, the way the limit of large $N$ is to be taken depends on whether one is in the vibrational or rotational regime.

A work close in spirit to the present work, but differing greatly in technique, was carried out by Moshinsky[21] who showed how to express the standard IBM Hamiltonian in terms of the BMM variables in the *intrinsic* coordinate system. Here there is no discussion of different physical regimes, nor is there a precise statement of operator equivalence. Both of these subjects are highlights of our discussion.

Finally we must consider the work of Ginocchio and Kirson (GK).[11,16] There, the same coherent state used to calculate the classical limit is utilized once more as an intrinsic state in an application of the method of generator coordinates. This establishes a connection of the type sought in that the generator coordinates, five in number, are identified with the coordinates of the BMM in the intrinsic coordinate system, and the Hamiltonian appears, at least superficially, of the BMM type. We shall demonstrate, however, that they have obtained the Hamiltonian in a basis which is not orthonormal.

We turn then to the contents of the present work.[22] First of all, we have shown how to establish a unitary correspondence between the original IBM representation of SU(6) in terms of $s$ and $d$ bosons (the Schwinger representation[23]), and a representation in terms of a five-dimensional $b$ boson which *conserves* the number of nucleons and is associated with a Holstein-Primakoff (HP) representation.[24] We have given a *new* basis in terms of $s$ and $b$ bosons associated with this representation and shown its unitary equivalence to the original basis. The new Hamiltonian and other operators depend only on the $b$ boson. One interpretation of this set of results is that it represents a Bohr-Mottelson-like form of the IBM (i.e., a precise phenomenological rendering of the TQM). We have, however, chosen another presentation which is somewhat controversial. (Any reader who prefers the more conservative interpretation is strongly encouraged to make that choice.) We have chosen to extend the conventional definition of the IBM and restrict the conventional definition of the BMM, thereby doing no real violence to physics, so that the mathematical equivalence between the two representations becomes a theorem of physical equivalence. This presentation is developed in Secs. II–IV.

Whichever one of the points of view offered above is the more acceptable, the Hamiltonian in the BMM form which emerges from the transformation is generally not one of the well-studied forms. In Secs. V and VI, we explain how the standard forms utilized in the spherical and deformed regimes may be regained. This material is included for the sake of continuity and completeness, despite the fact that it has been discussed previously.[22] We emphasize that this account provides the only known method of computing the potential energy surface distinct from that given in Refs. 11–17 and also completes the work of those references in providing a suitable kinetic energy function.

We turn finally to a study of the connection of our work with that of Refs. 11–17. In Sec. VII we describe briefly the essential result by which the utilization of a coherent state yields the potential energy surface. In Sec. VIII, we study the application of the generator coordinate method to this problem and reach the conclusion that this method is an expression of a mapping of the Schwinger SU(6) generators onto a form of the generators originally proposed by Dyson.[25] This version, in contrast to the HP version, lacks formal Hermiticity and is shown (and is also known) to correspond to an orthogonal but unnormalized basis is the real five-dimensional space of the BMM model. In a very recent communication, Castanos *et al.*[26] have pointed out that the self-adjointness can be restored by use of Bargman complex Hilbert space.[27] Similar ideas have been developed in the past within the context of the generator coordinate method.[28]

Despite our proof of "equivalence" of the two models, we are convinced of the importance of the development of the IBM alternative. The reasons for this opinion are reviewed in Sec. IX. Finally we remark that there is no difficulty in extending the

arguments of this paper to IBM II, which distinguishes neutrons and protons.[4]

## II. DEFINITION OF THE IBM MODEL

We introduce (together with their Hermitian conjugates) the operator $s^\dagger$ which creates a monopole boson and the operators $d^\dagger_\mu$, $\mu=-2,\ldots 2$ which create quadrupole bosons with magnetic quantum number $\mu$. In the microscopic formulation, which will not concern us in this paper, $s^\dagger$ and $d^\dagger_\mu$ are the fundamental elements of a mapping from the vector space created by fermion pairs of angular momentum zero or two onto the boson space of the IBM I model.[29–33] The 36 bilinear operators $s^\dagger s$, $s^\dagger d_\mu$, $d^\dagger_\mu s$, and $d^\dagger_\mu d_\nu$ are the generators of the Lie algebra U(6). Under the restriction to a fixed number, $N$, of bosons, where

$$N = s^\dagger s + \sum_\mu d^\dagger_\mu d_\mu = s^\dagger s + d^\dagger \cdot d = \hat{n}_s + \hat{n}_d \ , \tag{2.1}$$

the algebra becomes that of SU(6).

The definition of IBM I proposed by its authors was in terms of the structure of the model Hamiltonian, $H_{\rm IBM}$, which was assumed to be the most general polynomial of degree two in the generators of SU(6) invariant under rotations. The principal transition operators were equally assumed to be the simplest conceivable polynomials, for example, linear combinations of the generators themselves, with the appropriate tensorial character. Thus the quadrupole operator, up to a scale, is given as

$$Q_\mu = s^\dagger \tilde{d}_\mu + d^\dagger_\mu s + \chi (d^\dagger d)^{(2)}_\mu \ , \tag{2.2}$$

$$\tilde{d}_\mu = (-1)^\mu d_{-\mu} \ . \tag{2.3}$$

Each choice of the Hamiltonian and other operators leaves invariant the symmetric representation of SU(6) specified by a definite (integral) value of $N$. The carrier space of such a representation is a subspace of the six-dimensional oscillator space spanned by the $s$ and $d$ bosons, namely the space of states for a fixed $N$.

The simplest way of characterizing this space is in terms of the states

$$|[n_\mu], n_s\rangle = \prod_{\mu=-2}^{2} (n_\mu!)^{-1/2} (b^\dagger_\mu)^{n_\mu} \times (n_s!)^{-1/2} (s^\dagger)^{n_s} |0\rangle \ , \tag{2.4}$$

subject to the restriction

$$N = \sum_\mu n_\mu + n_s \equiv n_d + n_s = \text{constant} \ . \tag{2.5}$$

The physical interpretation is that $|0\rangle$, the "vacuum state," represents an inert core—perhaps the nearest closed shell nucleus—and $N$, the number of bosons, is half the number of fermions relative to that core. Thus $N$ characterizes the nucleus under study. Finally, the technical problem posed by the model requires the diagonalization of $H_{\rm IBM}$ in the basis (2.4) to determine the physical states.

Although we accept the essential features of the definition given above, we wish to broaden the definition somewhat, retaining the same degrees of freedom, but enlarging the class of permissible operators. Thus we define IBM I as follows: For the nucleus characterized by $N$, there is a class of eigenstates which can be modeled by suitable linear combinations of the states (2.4), carrying, in particular, a definite value of the total angular momentum. With this definition we mean to imply that the Hamiltonian can be any well-behaved rotationally invariant function of the generators and is not restricted to a simple polynomial in the generators. We gain a corresponding generality in the definition of other operators. In short, we are interested here in the definition of the model in the most general mathematical sense, rather than in the sense which properly motivates a physical discussion—whether a simplified form is a useful physical approximation.

What we have gained by this elaboration is the ability to make a precise mathematical statement about the possible equivalence of two models: Another model is equivalent to IBM I if and only if the vector space over which it is defined is equivalent to the vector space (2.4) and (2.5). The other model we have in mind is, of course, the Bohr-Mottelson model (BMM). We shall frame what we consider the essential question: Is there a formulation of the BMM which is equivalent to the IBM? The ramifications of this question will now be considered.

## III. REDEFINITION OF THE BOHR-MOTTELSON MODEL

We start with a concise definition of the BMM, as it is conventionally understood. In this statement we shall omit completely the hydrodynamical baggage with which the model was weighed down in its initial incarnation, baggage that has long since been discarded by its authors.[5] We frame the definition in terms of a quadrupole boson described by its creation and annihilation operators $b^\dagger_\mu, b_\mu$: (i) In contrast to the operator $d^\dagger_\mu$, the application of the

operator $b_\mu^\dagger$ to a state does not change the number of fermions. (ii) The Hamiltonian $H_{\rm BM}$ of this model may be any well-behaved, rotationally invariant function of the $b_\mu^\dagger, b_\mu$, the transition operators any well-behaved tensors of appropriate rank. (iii) The vector space is the Hilbert space of the five dimensional oscillator, i.e., the set of states

$$|[n]\rangle = \prod_{\mu=-2}^{2} (n_\mu!)^{-1/2} (b_\mu^\dagger)^{n_\mu} |0\rangle_N , \tag{3.1}$$

where the designation $|0\rangle_N$ implies that the Hilbert space is constructed independently for each nucleus. Although we intend ultimately to connect the subscript $N$ to the number introduced in the previous section, for the moment its purpose is to distinguish different nuclei. In the usual definition, no bound is set on $n_b$, the number of quadrupole bosons, although if the diagonalization is actually done in the basis (3.1), there is always a practical bound.

With an eye to our eventual goal, we apply Occam's razor to the previous definition. The intrusion, theoretically, of an infinite Hilbert space has no experimental foundation. Furthermore, if the BMM, as the IBM, is to be related to a shell model, the $b$ operators, in a number conserving description, would be the essential elements in the mapping of fermion-number conserving multipole operators[34] to the boson description. This *suggests,* but does not imply, that we sharpen the previous definition in two ways which tie it more closely to a shell model picture and (not incidentally) render it equivalent to IBM I: (i) Assume that the reference state for the nucleus $N$ can be related to a fixed reference state $|0\rangle$ by the formula

$$|0\rangle_N = (N!)^{-1/2} (s^\dagger)^N |0\rangle , \tag{3.2}$$

where $s^\dagger$ will be identified with the $s$ boson of the IBM. (ii) The value of $n_b = n_d$ must lie in the interval

$$0 < n_d < N . \tag{3.3}$$

We are thus led to consider a modified BMM basis represented by the vectors

$$|[n], N\rangle = \prod_{\mu=-2}^{2} (n_\mu!)^{-1/2} (b_\mu^\dagger)^{n_\mu} \times (N!)^{-1/2} (s^\dagger)^N |0\rangle . \tag{3.4}$$

It is almost evident that the bases (2.4) and (3.4) must be equivalent. [The idea of replacing the more nebulous set (3.1) by the finite set (3.4) was actually stimulated by a physical conception which somewhat less explicitly has informed the authors work for some time.[35]] It is simply that if one wishes to construct a correlated $s$-$d$ shell model basis, there are indeed two options. One is the IBM choice of creating variable numbers of $s$ and $d$ bosons restricted to a sum of $N$. The other route is the one we have taken in (3.4): First create a reference state of $N$ bosons by creating $N$ Cooper pairs, i.e., $N$ $s$ bosons, after which we are no longer free to change the nucleon number. Next we use the $b^\dagger$ excitation to create the requisite number, $n_b$, of quadrupole excitations subject to the restriction (3.3).

Finally our definition of the BMM bears the same relation to the basis (3.4) that our definition of the IBM bore to the basis (2.4), namely that the states we wish to describe can be expanded in that basis. Since the Hamiltonian $H_{\rm BM}$ depends only on the $b$ operator, the $s$ boson appearing in the set (3.4) plays only a spectator role in this formulation. To the best of our knowledge this formulation is numerically consistent with any application of the BMM extant.

## IV. PROOF OF EQUIVALENCE BY MEANS OF THE HOLSTEIN-PRIMAKOFF TRANSFORMATION

The IBM I formulation is a special illustration of the general result that the generators of the algebra U($N$) [or SU($n$)] may be realized (the Schwinger representation) as bilinear, "number conserving" operators constructed from $n$ bosons $a_\lambda^\dagger$, namely $a_\lambda^\dagger a_\nu$, $\lambda, \nu = 1 \ldots n$. We shall refer to this as a linear realization. The Holstein-Primakoff realization is an alternative nonlinear realization in terms of $(n-1)$ bosons. Thus, for SU(6), to which we restrict further discussion, the two realizations are connected by the correspondence[36]

$$d_\mu^\dagger d_\nu = b_\mu^\dagger b_\nu , \tag{4.1}$$

$$d_\mu^\dagger s = (s^\dagger d_\mu)^\dagger = b_\mu^\dagger [N - \sum b_\lambda^\dagger b_\lambda]^{1/2} , \tag{4.2}$$

$$s^\dagger s = N - \sum_\mu d_\mu^\dagger d_\nu = N - \sum_\mu b_\mu^\dagger b_\mu . \tag{4.3}$$

The implication of our notation is that whereas the Schwinger realization is associated with the IBM formulation and the basis (2.4), the nonlinear HP realization expressed in terms of the five bosons $b_\mu^\dagger$ is associated with the BMM and the basis (3.4).

The most important observation of this paper[37] is

that the relations (4.1)—(4.3) map the two bases into one another. The easiest way to see this is to notice that (4.2) and (4.3) together permit us to write

$$b_\mu^\dagger = d_\mu^\dagger s (N - \sum_\lambda d_\lambda^\dagger d_\lambda)^{-1/2} . \tag{4.4}$$

Now consider the simplest example of the mapping of states utilizing (4.4), namely

$$\begin{aligned}(N!)^{-1/2} b_\mu^\dagger (s^\dagger)^N |0\rangle &= [(N-1)!]^{-1/2} N^{-1} \\ &\times d_\mu^\dagger s (s^\dagger)^N |0\rangle \\ &= [(N-1)!]^{-1/2} \\ &\times d_\mu^\dagger (s^\dagger)^{N-1} |0\rangle . \end{aligned} \tag{4.5}$$

Repetition of this elementary calculation then bears witness that (4.4) effects a one-to-one transformation of the basis states (3.4) into those of (2.4) (and conversely). By the same token, the substitutions (4.1)—(4.3) will transform $H_{\rm IBM}$ into the equivalent $H_{\rm BM}$ and conversely, as it will equally transform other operators of interest.

The conventional definition of IBM I, we recall, requires $H_{\rm IBM\,I}$ to be at most quadratic in the SU(6) generators. Under (4.1)—(4.3) the corresponding $H_{\rm BM}$ will be nonpolynomial. Conversely, a form of $H_{\rm BM}$ which is polynomial in the $b_\mu^\dagger, b_\mu$ will become a nonpolynomial operator under (4.4).

## V. TRANSFORMATION OF A SPECIFIC HAMILTONIAN

There remains some interest in looking at the effect of the mapping between models for a specific Hamiltonian. All the qualitative information which can be extracted from such a study can be obtained from an example such as the Hamiltonian

$$H_{\rm IBM} = \epsilon \sum_\mu d_\mu^\dagger d_\mu - \kappa \sum_\mu Q_\mu Q_{-\mu} (-1)^\mu , \tag{5.1}$$

where $Q_\mu$ is the operator defined by (2.2) and (2.3). The operator (5.1) contains three parameters, $\epsilon$ describing the relative excitation energy of a $d$ pair relative to an $s$ pair, $\kappa$ measuring the overall strength of the quadrupole-quadrupole force, and $\chi$ determining the relative importance of the two kinds of basic quadrupole that make up the expression (2.2).

The first step in the transition to the Bohr Hamiltonian is the substitution of the mapping (4.1)—(4.3). Without exposing the detailed form of the resulting expressions, we realize that $H_{\rm BM}$ contains those square root operators which occur in (4.2), namely $(N - \sum b_\mu^\dagger b_\mu)^{1/2}$. For diagonalization in the basis (3.4), this is of no consequence. The use of this elementary basis would be natural within the framework of the Lanczos method.[38] The square root would also cause no trouble in another basis in which the number of $b$ bosons is a good quantum number. Such a basis is the seniority basis,[10] which is the natural basis for studying the SU(5) or vibrational limit of the theory. Of course, the seniority basis has been widely used[6] for diagonalizing the $H_{\rm BM}$ quite independently of any special limit. If the vibrational limit is indeed relevant, one has $\langle b^\dagger \cdot b \rangle \ll N$ for all low-lying states, and it becomes permissible to expand the radicals in powers of $(b^\dagger \cdot b / N)$.

Our interest in studying $H_{\rm BM}$, however, is not to discover alternative means for exact diagonalization, but rather in relating to the various special limits associated with geometrical pictures. Toward this end, it is convenient to change coordinates, first to canonical coordinates, $x_\mu, p_\mu$, according to the equations

$$b_\mu^\dagger = 2^{-1/2}(x_\mu - i p_\mu^\dagger) , \tag{5.2a}$$

$$\tilde{b}_\mu = 2^{-1/2}(x_\mu + i p_\mu^\dagger) , \tag{5.2b}$$

$$\tilde{x}_\mu = x_\mu^\dagger , \quad \tilde{p}_\mu = p_\mu^\dagger , \tag{5.3}$$

$$[x_\mu, p_\nu] = [x_\mu^\dagger, p_\mu^\dagger] = i\delta_{\mu\nu} . \tag{5.4}$$

By assigning characteristic magnitudes to the matrix elements of these variables, at least three regimes may be distinguished: (In all cases we take $N \gg 1$.) (i) The vibrational limit for low-lying energy states: The matrix elements of $x_\mu$ and $p_\mu$ as well as of their commutator are all of order unity, typical for an harmonic oscillator. As stated above, we may then expand in powers of $(b_\mu^\dagger b_\mu / N)$, i.e., of $(x_\mu \tilde{x}_\mu / N)$ and $(p_\mu \tilde{p}_\mu / N)$. (ii) The rotational limit for low-lying energy states: The largest matrix elements of $x_\mu$ are of order $N^{1/2}$, those of $p_\mu$ of order $N^{-1/2}$, whereas the commutator has the required intermediate value. In this limit, we may carry out the adiabatic approximation, expanding in powers of $(p_\mu \tilde{p}_\mu / N)$. This is the case we shall consider in detail below, working to the second order in the momenta, the conventional adiabatic approximation. (iii) The semiclassical limit. In this limit both $x_\mu$ and $p_\mu$ are $0(N^{1/2})$, but there is a large cancellation in the commutator reducing the latter to the size required.

Further detailed considerations will be confined

to regime (ii), since in the "popular mind" it is this regime that is most closely associated with the original ideas of Bohr and Mottelson. From our point of view this limit is interesting because the associated approximate Hamiltonian is the natural object upon which to base a discussion of phase transitions of the nuclear ground state configuration. The study of the semiclassical limit will not be undertaken here because it is more of mathematical than of physical interest, since with growing excitation energy, we must, increasingly, take into account the coupling to other degrees of freedom of the nuclear system.

We therefore substitute (5.2) into $H_{\rm BM}$ and expand in powers of the kinetic energy. To exhibit most clearly the relative orders of magnitude of different terms for a deformed nucleus, it is convenient to rescale coordinates and momenta,

$$x_\mu \to \sqrt{N} x_\mu, \quad p_\mu \to N^{-1/2} p_\mu \ , \tag{5.5}$$

and to introduce scalar shape variables $\beta,\gamma$, defined in terms of the new variables by the equations

$$x_\mu \tilde{x}_\mu \equiv x^2 = \beta^2 \ , \tag{5.6}$$

$$[(x \otimes x)^{(2)} \otimes x]^{(0)} \equiv -(\tfrac{2}{7})^{1/2} \beta^3 \cos 3\gamma \ . \tag{5.7}$$

These should be viewed as mathematically acceptable but physically tentative definitions of these parameters (see below).

We choose, furthermore, to work with the dimensionless Hamiltonian

$$H = (H_{\rm BM}/N\epsilon) = t + v \tag{5.8}$$

and the dimensionless coupling strength

$$F = (\kappa N/\epsilon) \ . \tag{5.9}$$

For $v$, we retain the first two terms in an expansion in $N^{-1}$, namely

$$v = v_0 + N^{-1} v_1 \ , \tag{5.10}$$

and for $t$ only the leading term (adiabatic approximation)

$$t = N^{-2} t_0 \ . \tag{5.11}$$

The results of the relatively straightforward algebra are

$$v_0 = (\tfrac{1}{2} - 2F)\beta^2 + F(2/\sqrt{7})\chi(1 - \tfrac{1}{2}\beta^2)^{1/2}\beta^3 \cos 3\gamma + F[1 - (\chi^2/14)]\beta^4 \ , \tag{5.12}$$

$$v_1 = -6F\beta^2 + (3/\sqrt{7})F\chi[1 - \tfrac{1}{2}\beta^2]^{-1/2}\beta^3 \cos\gamma \ , \tag{5.13}$$

$$\begin{aligned} t_0 = {} & \tfrac{1}{2}\underline{p}^2 + \tfrac{1}{2}F\{\underline{p}^2, \beta^2\} - F\chi(1/8\sqrt{7})\{\beta^2 \cos 3\gamma, \{[1 - \tfrac{1}{2}\beta^2]^{-1/2}, \underline{p}^2\}\} \\ & - (1/2\sqrt{2})F\chi\{[1 - \tfrac{1}{2}\beta^2]^{1/2}, \{x_\mu, \{p \otimes p\}^{(2)}_\mu\}\} - \tfrac{1}{4}F\chi^2\{(x \otimes x)^{(2)}_\mu, (p \otimes p)^{(2)}_\mu\} \ , \end{aligned} \tag{5.14}$$

where $\underline{p}^2 = p_\mu \tilde{p}_\mu$. In the limit $N \to \infty$, all that remains of $h$ is the first term, $v_0$, which defines the *potential energy surface* and provides the usual basis for the discussion of nuclear shapes. This conclusion depends on $v_0$, $v_1$, $t_0$, etc., being of order unity (see below).

## VI. STUDY OF THE POTENTIAL ENERGY SURFACE

The dominance of $v_0$ in the limit $N \to \infty$ holds provided the rescaled variables (5.5) are of order unity, which is characteristic of the deformed regime. The fact that we shall also use this function to characterize the spherical regime as well requires further discussion. For in this case the order of magnitude of the rescaled variables, $x \sim N^{-1/2}$, $p \sim N^{1/2}$ dictates that potential and kinetic energies are of the same order. However, in this regime the equilibrium value of $\beta$ is zero and the classical minimum energy per particle is zero because both $t$ and $v$ are zero (additive constants in $h$ having been dropped). It follows that the computation of a spherical minimum for $v$ is sufficient to establish the phase. For the deformed phase, it also yields the leading term in the energy, namely the deformation energy per particle. It is necessary to bear in mind, however, that especially in the vibrational and transition regions, the kinetic energy will play an essential role in determining the level structure.

Our next task is to search for the critical points of $v_0$. Before pursuing this task, we notice that $v_0$ becomes complex for $\beta > \sqrt{2}$. This is not a defect of the theory since it can be traced to the condition $b^\dagger b < N$ inherent in the HP mapping (4.1)–(4.3). Of course we are accustomed, in the geometrical versions of the collective model, to a variable $\beta$

which has access to the entire real axis. On the other hand, if we look for a foundation of the collective model in a finite dimensional shell model, we might expect the "natural" definition of $\beta$ to contain a restriction of the kind found.

We have introduced the word natural with irony because it requires only a simple point transformation to convert scales. This transformation is also necessary to compare with previous work. We therefore introduce the transformation

$$\beta \equiv \sqrt{2}\bar{\beta}/(1+\bar{\beta}^2)^{1/2} \tag{6.1}$$

under which $v_0(\beta,\gamma)$ becomes (dropping the bar)

$$v_0 = \frac{(\frac{1}{2}-2F)2\beta^2}{(1+\beta^2)} + \frac{2F\chi(\frac{2}{7})^{1/2}\beta^3\cos 3\gamma}{(1+\beta^2)^2} + F[1-(\chi^2/14)]\frac{4\beta^4}{(1+\beta^2)^2} . \tag{6.2}$$

In this form, the fact that the collective Hamiltonian will support only a finite number of bound states is expressed by the property that

$$\lim_{\beta\to\infty} v_0 = 1-(\tfrac{2}{7})\chi^2 , \tag{6.3}$$

which also puts an upper bound on interesting values of $\chi^2$. Equation (6.2) agrees with corresponding expressions in Refs. 11–17. For further discussion of the potential energy surface, we refer to these works and to Ref. 22.

The next step in a complete study is the transformation of the kinetic energy to intrinsic coordinates. This has been carried out, but since the calculation is standard and contains nothing of special interest to our study, it will be omitted from the present paper.

## VII. CALCULATION OF THE POTENTIAL ENERGY SURFACE FROM A COHERENT STATE

As stated above, the potential energy surface (6.2) has been obtained independently by a number of authors. All have used the same method, distinct from ours. Here we recall briefly the elements of the method and then mark its limitations.

The method utilizes the coherent state

$$|N;\beta,\gamma\rangle = \frac{1}{\sqrt{N!(1+\beta^2)^N}}(B^\dagger)^N|0\rangle , \tag{7.1}$$

where

$$B^\dagger = s^\dagger + \beta[\cos\gamma d_0^\dagger + \sqrt{1/2}\sin\gamma(d_2^\dagger + d_{-2}^\dagger)] . \tag{7.2}$$

This state is normalized. For distinct values of $\beta,\gamma$ we have the overlap

$$\langle N;\beta'\gamma'|N;\beta,\gamma\rangle = [1+\beta\beta'\cos(\gamma-\gamma')]^N/(1+\beta^2)^{N/2}(1+\beta'^2)^{N/2} \overset{N\to\infty}{\to} \exp[N\beta\beta'\cos(\gamma-\gamma') - \tfrac{1}{2}N(\beta^2+\beta'^2)] . \tag{7.3}$$

Formulas for calculating the matrix elements of any boson number conserving operator between pairs of states of the set (7.1) have been given by Ginocchio and Kirson.[16] They are derived by repeated application of such simple formulas as

$$s\,|N;\beta,\gamma\rangle = [N/(1+\beta^2)]^{1/2}\,|N-1;\beta,\gamma\rangle , \tag{7.4}$$

$$d_0\,|N;\beta,\gamma\rangle = \beta\cos\gamma[N/(1+\beta^2)]^{1/2} \times |N-1;\beta,\gamma\rangle , \tag{7.5}$$

$$d_2^\dagger\,|N;\beta,\gamma\rangle = \frac{1}{\sqrt{2}}\beta\sin\gamma[N/(1+\beta^2)]^{1/2} \times |N-1;\beta,\gamma\rangle . \tag{7.6}$$

Examination of the results shows that

$$\frac{\langle N;\beta,\gamma|H^2|N;\beta\gamma\rangle - [\langle N;\beta,\gamma|H|N;\beta,\gamma\rangle]^2}{|\langle N;\beta,\gamma|H|N;\beta,\gamma\rangle]^2} = O\left[\frac{1}{N}\right]^{N\to\infty} \to 0 . \tag{7.7}$$

This implies that $|N;\beta,\gamma\rangle$ becomes an eigenstate of $H$ in this limit. A straightforward calculation of the expectation value of $H$, Eq. (5.1), in this state then yields, when rescaled, the potential energy surface (6.2), not surprisingly, since the same limit ($N\to\infty$) produced the latter as the surviving part of the energy.

The elegance of the route taken to the potential energy surface through the use of the coherent state is balanced by the fact that it represents a cul de sac as far as determination of the full Bohr Hamiltoni-

an is concerned. As shown by GK a full collective description can still be obtained with the help of the coherent state when one views it as an intrinsic state within the method of generator coordinates. We review this point of view below, especially in its relation to and contrast with our *method.*

## VIII. APPLICATION OF THE METHOD OF GENERATOR COORDINATES: RELATION TO THE DYSON MAPPING

In order to study the full eigenvalue problem

$$H\,|N,\nu\rangle = E_{N,\nu}\,|N,\nu\rangle\ , \tag{8.1}$$

where $|N,\nu\rangle$ is an exact eigenstate and $\nu$ stands for all quantum numbers other than the boson number, we introduce an unnormalized intrinsic state,

$$|N,\alpha\rangle = (N!)^{-1/2}(B^\dagger)^N\,|0\rangle\ , \tag{8.2}$$

$$B^\dagger = s^\dagger + \alpha^*_\mu d^\dagger_\mu \equiv s^\dagger + \alpha^*\cdot d^\dagger\ , \tag{8.3}$$

where in place of $\beta,\gamma$ (and the Euler angles), we find it more convenient to utilize laboratory coordinates $\alpha^*_\mu$, which may be complex numbers.

It is not difficult to convince ourselves that a representation of the form

$$|N,\nu\rangle = \prod_\mu \oint d\alpha^*_\mu f_{N,\nu}(\underline{\alpha})\,|N,\underline{\alpha}\rangle \tag{8.4}$$

is possible, where each contour is any circle enclosing the origin. The demonstration consists first, in noting that $|N,\nu\rangle$ may be expanded in terms of the (unnormalized) oscillator basis

$$|n_s,n_2,n_1,n_0,n_{-1},n_{-2}\rangle = (s^\dagger)^{n_s}(d^\dagger_2)^{n_2}\cdots(d^\dagger_{-2})^{n_{-2}}\,|0\rangle\ . \tag{8.5}$$

The latter, in turn, may be obtained from a multipole contour integral by repeated use of the formula

$$a^{n_1}b^{n-n_1} = \frac{n_1!(n-n_1)!}{n!}\frac{1}{2\pi i}\oint\frac{(a+bz)^n}{z^{n-n_1+1}}dz\ , \tag{8.6}$$

and $(a+bz)$ equals either the sum of the terms in $B^\dagger$, Eq. (8.3), in the first step, or a partial sum of such terms in the succeeding steps.

Thus, one is encouraged to think of the states $|N,\underline{\alpha}\rangle$ as a basis, to introduce a "wave function"

$$\psi_{N,\nu}(\underline{\alpha}) = \langle N,\underline{\alpha}\,|\,N,\nu\rangle\ , \tag{8.7}$$

and to study the Schrödinger equation [notice that in (8.7), $\alpha_\mu$ occurs, rather than $\alpha^*_\mu$]

$$\langle N,\underline{\alpha}\,|\,H\,|\,N,\underline{\alpha}\rangle = \mathscr{H}(\alpha_\mu,\partial/\partial\alpha_{\mu'})\psi_{N,\nu} = E_{N,\nu}\psi_{N,\nu}(\underline{\alpha})\ . \tag{8.8}$$

The "Hamiltonian" may be found by deriving

$$\langle N,\underline{\alpha}\,|\,s^\dagger = \langle N-1,\underline{\alpha}\,|\,\sqrt{N}\ , \tag{8.9}$$

$$\langle N,\underline{\alpha}\,|\,d^\dagger_\mu = \langle N-1,\underline{\alpha}\,|\,\sqrt{N}\,\alpha_\mu\ , \tag{8.10}$$

$$\langle N-1,\underline{\alpha}\,|\,d_\mu = N^{-1/2}(\partial/\partial\alpha_\mu)\langle N,\underline{\alpha}\,|\ , \tag{8.11}$$

$$\langle N-1,\underline{\alpha}\,|\,s = N^{1/2}[1-N^{-1}\alpha_\lambda(\partial/\partial\alpha_\lambda)]\,\langle N;\underline{\alpha}\,|\ . \tag{8.12}$$

For the generators of SU(6), these formulas yield the correspondence

$$s^\dagger d_\mu \to \partial/\partial\alpha_\mu\ , \tag{8.13}$$

$$d^\dagger_\mu s \to N\alpha_\mu[1-N^{-1}\alpha_\lambda(\partial/\partial\alpha_\lambda)]\ , \tag{8.14}$$

$$d^\dagger_\mu d_{\mu'} \to \alpha_\mu(\partial/\partial\alpha_{\mu'})\ , \tag{8.15}$$

$$s^\dagger s \to [N-\alpha_\lambda(\partial/\partial\alpha_\lambda)]\ . \tag{8.16}$$

The salient feature of this correspondence is that it does not maintain the Hermitian conjugation relationship of the operator $s^\dagger d_\mu$ and $d^\dagger_\mu s$. It follows that $\mathscr{H}(\alpha_\mu,\partial/\partial\alpha_{\mu'})$, which is obtained by substituting the above correspondence into the IBM Hamiltonian, is also formally non-Hermitian if it contains terms with an odd number of $d$ boson operators (as does the general IBM Hamiltonian studied by GK as well as the special one considered by us).

The relationship to the Holstein-Primakoff mapping can be established in a few steps. First we map

$$\alpha_\mu \to b^\dagger_\mu,\quad (\partial/\partial\alpha_\mu)\to b_\mu\ . \tag{8.17}$$

Thus (8.13)–(8.16) are now written

$$(s^\dagger d_\mu)_D = b_\mu\ , \tag{8.18}$$

$$(d^\dagger_\mu s)_D = b^\dagger_\mu[N-b^\dagger\cdot b]\ , \tag{8.19}$$

$$(d^\dagger_\mu d_{\mu'})_D = b^\dagger_\mu b_{\mu'}\ , \tag{8.20}$$

$$(s^\dagger s)_D = N-b^\dagger\cdot b\ . \tag{8.21}$$

The subscript $D$ reminds us that what we have here is a version of the Dyson mapping.[25,39,40] Notice what happens under this mapping to a formally Hermitian combination:

$$(d^\dagger_\mu d_{\mu'}d^\dagger_{\mu''}s + s^\dagger d_{\mu''}d^\dagger_{\mu'}d_\mu)_D = b^\dagger_\mu b_{\mu'}b^\dagger_{\mu''}[N-b^\dagger\cdot b] + b_{\mu''}b^\dagger_{\mu'}b_\mu\ , \tag{8.22}$$

which is not Hermitian because of the presence of the term quintic in the boson operators. This is the origin of the formal non-Hermiticity in the work of GK.

The Dyson and Holstein-Primakoff mappings are connected by a similarity transformation which can be stated in several forms. We shall give it in terms of an operator $S$, where

$$S(d_\mu^\dagger s)_D S^{-1}=(d_\mu^\dagger s)_{\rm HP} , \tag{8.23}$$

$$S(s^\dagger d_\mu)_D S^{-1}=(s^\dagger d_\mu)_{\rm HP}=(d_\mu^\dagger s)_{\rm HP} . \tag{8.24}$$

The solution of these conditions shows that $S$ is a function only of the operator $n_b=b^\dagger\cdot b$ and is defined by the recurrence relation

$$S(n_b+1)=[N-n_b]^{-1/2}S(n_b) . \tag{8.25}$$

Since the elementary orthonormal basis for the HP representation is known [Eq. (3.4)], the operator $S$ may be used to generate the basis for the Dyson representation whose states differ only by normalization constants from the former.

Thus we have shown the equivalence of our work to that of GK, which corresponds to the use of an orthogonal but non-normalized basis. An interesting alternative interpretation of the work of GK may be given.[26] It has been shown that the mapping (8.13)—(8.16) can be related to an orthonormal basis in the Bargmann Hilbert space.[27] This view of generator coordinates is not completely new.[28]

## IX. IF THE IBM AND THE BMM ARE EQUIVALENT, WHY STUDY THE IBM?

The major result of this paper is that the BMM can be put into a mold, without doing violence to its essential physical content, which renders it unitarily equivalent to IBM I. The ideas involved can clearly be extended to models which treat neutron and proton pairs individually. The mapping is such that a polynomial Hamiltonian in one form transforms into an irrational function in the other. The determination of which model is better served by a polynomial approximation has to be settled by the microscopic theory, we believe, since it appears that there will be little distinction in the quality of fit to experiment between polynomial models of the two types with the same number of parameters. What has proved most attractive about IBM, aside from the vigor with which it has been transformed into a useful and active tool for the analysis of experiment, has been first that it is the natural form in which dynamical symmetries emerge, and second, since it is formulated in terms of number changing operators, it naturally suggests that the analysis be carried out by smooth variation of the parameters of the Hamiltonian with nucleon number.

It is also clear why only the SU(5) symmetry[41] is manifest in the BMM model. For the latter the natural algebra is the semidirect product of the Weyl algebra of the canonical variables $b^\dagger, b$ with the U(5) of the products $b^\dagger b$. If we use a *polynomial* in these operators as Hamiltonian, it is impossible to realize the O(6) and SU(3) symmetries as exact symmetries, since the mapping (4.1)—(4.3) shows that the expression of these in terms of the $b$ boson is nonlinear and irrational. Nevertheless, the same basic physics as is contained in the IBM polynomials is also expressed by the BMM polynomial forms. [It should be noted that Janssen *et al.*,[2] who first realized the SU(6) symmetry in the HP mode, were led naturally also to a consideration of the subgroups, for instance SU(3), in the nonlinear form.[42]]

To conclude in a practical vein, our work implies that fine tuning of the IBM to bring it into more precise accord with experiment will necessarily lead to an enlargement of the framework of the model along the lines we have insisted upon, namely, the inclusion of higher order terms in the various operators.

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