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Lee model and mesic atoms

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An extension of the Lee model in quantum field theory is presented which has some of the features of mesic atoms. The model has three fermions V, N_1, and N_2, and two bosons θ_1 and θ_2, which interact according to $V \rightleftarrows N_\alpha + \theta_\alpha$, $\alpha = 1,2$. There is also a pair interaction between an N_α and θ_α particle which can produce atomic bound states whose energies are shifted and broadened by the interaction $V \leftrightarrows N_\alpha + \theta_\alpha$, $\alpha = 1,2$. The T matrix for $N_\beta - \theta_\alpha$ scattering is obtained in closed form, and its bound state and decaying or resonant state poles are investigated.

[NUCLEAR REACTIONS Model quantum field theory for mesic atoms.]

I. INTRODUCTION

The Lee model[1] was introduced to aid in the problem of understanding the renormalization procedure in quantum field theory. The model consists of two baryons N and V and a boson θ which interact by the elementary process $V \rightleftarrows N + \theta$. All divergent quantities can be removed by a mass renormalization and a coupling constant renormalization. The physical V-particle state and the N-θ scattering states can be obtained in closed form,[1] and the V-θ scattering problem, which allows for production ($V + \theta \rightarrow N + 2\theta$), is also solvable.[2]

The model has been used to illustrate the problem of ghost states in quantum field theory[3] and to test the definitions of the mass and lifetime of an unstable particle.[4] The book by Schweber[5] gives a complete treatment of the V and N-θ states, as well as extensive references to the literature.

The model presented here has a V particle, two N particles, and two θ particles which interact according to $V \rightleftarrows N_\alpha + \theta_\alpha$, $\alpha = 1,2$. There is also a pair interaction between N_α and θ_α ($\alpha = 1,2$) given by a static, Hermitian potential. As in the original Lee model,[1] only the V-particle is dressed by the interaction. We shall see that under certain circumstances the model describes something similar to a mesic atom, in that there can be atomic bound states produced by the pair interaction between an N_α and a θ_α particle, whose energies are shifted and broadened by the strong interaction $V \rightleftarrows N_\alpha + \theta_\alpha$. Thus, the model will be useful in studying strong interaction effects in mesic atoms.

Apparently, all methods for determining these effects attempt to relate them to information on meson-nucleon and meson-nucleus scattering. A simple formula due to Deser *et al.*[6] relates the energy shift and width of the mesic-atom state to the meson-nucleus scattering lengths. The uncertainties in this relation are still a subject of interest.[7] Deloff[8,9] has developed more sophisticated relations between the complex level shifts and the meson-nucleus scattering amplitudes by analytically continuing the S matrix below threshold.

The complex energies of a mesic atom can also be related to scattering information through the intermediary of a complex optical potential. A Kisslinger-type potential[10] used in a Klein-Gordon equation with an electromagnetic potential has been relatively successful[11-13] in accounting for level shifts and widths in pionic atoms. This approach is not as successful for kaonic atoms,[14,15] where the dynamics involves coupled channels ($K^- + N \rightarrow K^- + N$, $\bar{K}^0 + N$, $\Sigma + \pi$, $\Lambda + \pi$) and subthreshold resonances [$\Lambda(1405)$ and $\Sigma(1385)$].[16,17] In particular, it appears that in a nucleus the $\Lambda(1405)$ resonance can lead to a reversal of the sign of the shift produced by a simple optical potential.[15]

In the model presented here the mesons (the θ's) interact only with very simple objects (the V and the N's); therefore, the model cannot be used to assess those aspects of the theory of mesic atoms which involve complicated nuclear structure. It can be used to investigate the validity of the formula of Deser *et al*,[6] as well as the relations developed by Deloff.[8,9] The model can also be used to study the complications that arise as a result of coupled channels and subthreshold resonances.

One possible interpretation of the model is to think of N_1 and the dressed V particle as an n, the θ_1 as a π^0, and the N_2 and θ_2 as a p and π^-, respectively, thereby mimicking the simplest pimesic atom. Even this simplest of atoms has its atomic states broadened by the strong interaction, since the mass of the π^--p system is greater than the mass of the π^0-n system by 3.3 MeV. Another possible correspondence is N_1, θ_1, N_2, θ_2, V and Σ, π, p, K^-, $\Lambda(1405)$. The parameters of the model can be chosen so that the dressed V particle is unstable. This makes it possible to study the effect of a subthreshold resonance on an atomic state.

The detailed application of the model to the study of mesic atoms and its extension to more complicated "nuclei" will be presented in future publications. Here we shall obtain a closed form expression for the N_β-θ_α T matrix and investigate its bound state and resonance or decaying state poles. We shall see that in this sector the model is equivalent to a system with two coupled channels, where the strong interaction is described by energy dependent separable potentials. We shall also find that if the channel coupling is negligible, i.e., the widths are small, the strong interaction ($V \rightleftarrows N_\alpha + \theta_\alpha$) shifts the energies of the atomic states so that they interlace the energies obtained with no strong interaction.

In Sec. II the Hamiltonian is given and its simplest consequences are stated. The N_β-θ_α T matrix is derived in Sec. III and the equivalent coupled channel problem is presented. The renormalization of the model and the poles of the T matrix are examined in Sec. IV. A brief discussion is given in Sec. V. Units in which $\hbar = c = 1$ are used throughout.

II. THE HAMILTONIAN

The model describes three fermions V, N_1, and N_2 with masses m_0, m_1, and m_2, and two bosons θ_1 and θ_2 with masses μ_1 and μ_2. We shall see that the bare mass m_0 is renormalized to a physical mass m, while for the other masses there is no renormalization. The Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 , \\ \mathcal{H}_0 &= m_0 V^\dagger V \\ &\quad + \sum_{\alpha=1}^{2} m_\alpha N_\alpha^\dagger N_\alpha + \int d^3k \sum_{\alpha=1}^{2} a_\alpha^\dagger(\vec{k}) a_\alpha(\vec{k}) \omega_\alpha(k) , \\ \mathcal{H}_1 &= \int d^3k\, d^3q \sum_{\alpha=1}^{2} N_\alpha^\dagger a_\alpha^\dagger(\vec{k}) U_\alpha(\vec{k},\vec{q}) N_\alpha a_\alpha(\vec{q}) , \\ \mathcal{H}_2 &= \int d^3k \sum_{\alpha=1}^{2} g_{\alpha 0} u_\alpha(k) [V^\dagger N_\alpha a_\alpha(\vec{k}) \\ &\quad + a_\alpha^\dagger(\vec{k}) N_\alpha^\dagger V] . \end{aligned} \tag{1}$$

The operators $V^\dagger, N_\alpha^\dagger$ and V, N_α are the creation and annihilation operators, respectively, for the corresponding particles, and obey the usual anticommutation rules for fermions. The operators $a_\alpha^\dagger(\vec{k})$ and $a_\alpha(\vec{k})$ are the creation and annihilation operators, respectively, for a θ_α particle of momentum $\vec{k}$, and obey the usual commutation rules for bosons. Recoil effects for the fermions have been neglected, and it has been assumed that there is never more than one V or N_α particle present. The energy of the free bosons is given by

$$\omega_\alpha(k) = (k^2 + \mu_\alpha^2)^{1/2} . \tag{2}$$

The term $\mathcal{H}_1$ in the Hamiltonian describes the interaction of a θ_α particle with an N_α particle by means of the Hermitian potential $U_\alpha(\vec{k},\vec{q}) = U_\alpha^*(\vec{q},\vec{k})$. The last term in $\mathcal{H}$ describes the processes

$$V \rightleftarrows N_\alpha + \theta_\alpha , \quad \alpha = 1,2 , \tag{3}$$

with bare coupling constants $g_{\alpha 0}$ and cutoff functions $u_\alpha(k)$.

The bare vacuum characterized by

$$a_\alpha(\vec{k}) | 0 \rangle = N_\alpha | 0 \rangle = V | 0 \rangle = 0 \tag{4}$$

is also the physical vacuum, i.e.,

$$H | 0 \rangle = 0 . \tag{5}$$

The single particle states

$$\begin{aligned} | \theta_{\alpha \vec{k}} \rangle &= a_\alpha^\dagger(\vec{k}) | 0 \rangle , \\ | N_\alpha \rangle &= N_\alpha^\dagger | 0 \rangle \end{aligned} \tag{6}$$

satisfy

$$\begin{aligned} H | \theta_{\alpha \vec{k}} \rangle &= \omega_\alpha(k) | \theta_{\alpha \vec{k}} \rangle , \\ H | N_\alpha \rangle &= m_\alpha | N_\alpha \rangle , \end{aligned} \tag{7}$$

which shows that the θ_α and N_α are not dressed by the interaction. However,

$$|V\rangle = V^\dagger |0\rangle \tag{8}$$

is not an eigenstate of H, and hence there is a distinction between a bare and physical V particle.

It is straightforward to show that the operators

$$B = V^\dagger V + \sum_{\alpha=1}^{2} N_\alpha^\dagger N_\alpha \ , \qquad Q_\alpha = N_\alpha^\dagger N_\alpha - \int d^3k\, a_\alpha^\dagger(\vec{k}) a_\alpha(\vec{k}) \tag{9}$$

commute with $\mathscr{H}$. The existence of these operators is essentially the reason the model is tractable. From now on we shall only be concerned with those states for which the eigenvalues of the operators B, Q_1, and Q_2 are 1, 0, and 0, respectively. Such states are superpositions of the states $|V\rangle$ and

$$|N_\alpha,\theta_{\alpha\vec{k}}\rangle = N_\alpha^\dagger a_\alpha^\dagger(\vec{k})|0\rangle, \quad \alpha=1,2 \ . \tag{10}$$

In solving the model in this subspace it is most convenient to work with the T matrix for N_β-θ_α scattering, since this contains all the information on scattering, bound states, and resonances.

III. THE T MATRIX FOR N_β-θ_α SCATTERING

We begin by solving the equation

$$H|N_\alpha,\theta_{\alpha\vec{k}}\rangle_\pm = [m_\alpha + \omega_\alpha(k)]\,|N_\alpha,\theta_{\alpha\vec{k}}\rangle_\pm \tag{11}$$

for the in (+) and out (−) scattering states. Following Schweber's[5] analysis of N-θ scattering in the Lee model, it is straighforward to show that

$$|N_\alpha,\theta_{\alpha\vec{k}}\rangle_\pm = |N_\alpha,\theta_{\alpha\vec{k}}\rangle + \Gamma[E_\alpha(k)\pm i\epsilon] \times [g_{\alpha 0} u_\alpha(k)|V\rangle + \int d^3q\, |N_\alpha,\theta_{\alpha\vec{q}}\rangle U_\alpha(\vec{q},\vec{k})] \ , \tag{12}$$

where

$$E_\alpha(k) = m_\alpha + \omega_\alpha(k) \tag{13}$$

and

$$\Gamma(z) = (z-H)^{-1} \ . \tag{14}$$

From (12) it follows that the S-matrix elements are given by

$$_-\langle N_\beta,\theta_{\beta\vec{p}}|N_\alpha,\theta_{\alpha\vec{k}}\rangle_+ = \delta_{\beta\alpha}\delta^3(\vec{p}-\vec{k}) - 2\pi i\,\delta[E_\beta(p)-E_\alpha(k)] \times T_{\beta\alpha}(\vec{p},\vec{k};z) \ , \tag{15}$$

$$z = E_\beta(p) + i\epsilon = E_\alpha(k) + i\epsilon \ ,$$

with the T-matrix elements given by

$$T_{\beta\alpha}(\vec{p},\vec{k};z) = \delta_{\beta\alpha} U_\alpha(\vec{p},\vec{k}) + \left[u_\beta(p) g_{\beta 0}\langle V| + \int d^3l\, U_\beta(\vec{p},\vec{l})\langle N_\beta,\theta_{\beta\vec{l}}| \right] \times\Gamma(z)\left[|V\rangle g_{\alpha 0} u_\alpha(k) + \int d^3q\, |N_\alpha,\theta_{\alpha\vec{q}}\rangle U_\alpha(\vec{q},\vec{k}) \right] \ . \tag{16}$$

We see that in order to determine the T matrix we must evaluate the matrix elements of $\Gamma(z)$ in the subspace of the bare states which are eigenstates of B, Q_1, and Q_2 with eigenvalues 1, 0, and 0, respectively.

From (5), (8), (10), and (14), it follows that

$$z\langle V|\Gamma(z)|\psi\rangle - \langle 0|[V,H]\Gamma(z)|\psi\rangle = \langle V|\psi\rangle \ , \qquad z\langle N_\alpha,\theta_{\alpha\vec{k}}|\Gamma(z)|\psi\rangle - \langle 0|[N_\alpha a_\alpha(\vec{k}),H]\Gamma(z)|\psi\rangle = \langle N_\alpha,\theta_{\alpha\vec{k}}|\psi\rangle \ , \tag{17}$$

where $|\psi\rangle$ is any state in the subspace of interest. Working out the commutators leads to the equations

$$(z-m_0)\langle V|\Gamma(z)|\psi\rangle - \sum_{\alpha=1}^{2} g_{\alpha 0}\int d^3q\, u_\alpha(q)\langle N_\alpha,\theta_{\alpha\vec{q}}|\Gamma(z)|\psi\rangle = \langle V|\psi\rangle \ , \tag{18}$$

$$[z-m_\alpha-\omega_\alpha(k)]\langle N_\alpha,\theta_{\alpha\vec{k}}|\Gamma(z)|\psi\rangle - \int d^3q\, U_\alpha(\vec{k},\vec{q})\langle N_\alpha,\theta_{\alpha\vec{q}}|\Gamma(z)|\psi\rangle = g_{\alpha 0} u_\alpha(k)\langle V|\Gamma(z)|\psi\rangle + \langle N_\alpha,\theta_{\alpha\vec{k}}|\psi\rangle \ . \tag{19}$$

Equation (19) can be solved for $\langle N_\alpha,\theta_{\alpha\vec{k}}|\Gamma(z)|\psi\rangle$ in terms of the quantities on the right hand side by using standard Green's function techniques. Substituting the resulting expression for $\langle N_\alpha,\theta_{\alpha\vec{k}} \mid \Gamma(z) \mid \psi\rangle$ into (18) leads to an algebraic equation for $\langle V \mid \Gamma(z) \mid \psi\rangle$, which is easily solved. In writing out the results it is convenient to use a Dirac notation. We define

$$\langle \vec{p} \mid u_\alpha\rangle = u_\alpha(p) , \tag{20}$$

$$\langle \vec{p} \mid H_\alpha \mid \vec{k}\rangle = [m_\alpha+\omega_\alpha(k)]\delta^3(\vec{p}-\vec{k})+U_\alpha(\vec{p},\vec{k}) , \tag{21}$$

$$\langle \vec{p} \mid U_\alpha \mid \vec{k}\rangle = U_\alpha(\vec{p},\vec{k}) , \tag{22}$$

$$G_\alpha(z)=(z-H_\alpha)^{-1} , \tag{23}$$

and assume

$$\int \mid \vec{k}\rangle d^3k \langle \vec{k} \mid = 1 . \tag{24}$$

We find for the various matrix elements of $\Gamma(z)$ the expressions

$$\begin{aligned}
&\langle V \mid \Gamma(z) \mid V\rangle = d^{-1}(z) ,\\
&\langle V \mid \Gamma(z) \mid N_\alpha,\theta_{\alpha\vec{q}}\rangle = d^{-1}(z) g_{\alpha 0}\langle u_\alpha \mid G_\alpha(z) \mid \vec{q}\rangle ,\\
&\langle N_\beta,\theta_{\beta\vec{l}} \mid \Gamma(z) \mid V\rangle = \langle \vec{l} \mid G_\beta(z) \mid u_\beta\rangle g_{\beta 0} d^{-1}(z) ,\\
&\langle N_\beta,\theta_{\beta\vec{l}} \mid \Gamma(z) \mid N_\alpha,\theta_{\alpha\vec{q}}\rangle = \left\langle \vec{l} \,\middle|\, \left[\delta_{\beta\alpha}G_\alpha(z)+G_\beta(z)\mid u_\beta\rangle \frac{g_{\beta 0}g_{\alpha 0}}{d(z)}\langle u_\alpha \mid G_\alpha(z)\right] \middle|\, \vec{q}\right\rangle ,
\end{aligned} \tag{25}$$

where

$$d(z)=z-m_0-\sum_{\alpha=1}^{2} g_{\alpha 0}{}^2\langle u_\alpha \mid G_\alpha(z) \mid u_\alpha\rangle . \tag{26}$$

Substituting these results into (16), we find

$$T_{\beta\alpha}(\vec{p},\vec{k};z)=\langle \vec{p} \mid T_{\beta\alpha}(z) \mid \vec{k}\rangle , \tag{27}$$

with

$$T_{\beta\alpha}(z)=\delta_{\beta\alpha}t_\alpha(z)+[1+U_\beta G_\beta(z)] \times t_{\beta\alpha}(z)[1+G_\alpha(z)U_\alpha] , \tag{28}$$

where

$$t_\alpha(z)=U_\alpha+U_\alpha G_\alpha(z)U_\alpha \tag{29}$$

and

$$t_{\beta\alpha}(z)=\mid u_\beta\rangle \frac{g_{\beta 0}g_{\alpha 0}}{d(z)}\langle u_\alpha \mid . \tag{30}$$

Equations (26)–(30) are the essential results of the model in the subspace associated with N_β-θ_α scattering. The structure of (28) is exactly the same as the expression for the T matrix in the two-potential formalism of Gell-Mann and Goldberger.[18,19] In fact, it is straightforward to show that $t_{\beta\alpha}(z)$ is the solution of the equation

$$t_{\beta\alpha}(z)=W_{\beta\alpha}(z)+\sum_\gamma W_{\beta\gamma}(z)G_\gamma(z)t_{\gamma\alpha}(z) , \tag{31}$$

with the energy dependent potential given by

$$W_{\beta\alpha}(z)=\mid u_\beta\rangle \frac{g_{\beta 0}g_{\alpha 0}}{z-m_0}\langle u_\alpha \mid . \tag{32}$$

Thus, $T_{\beta\alpha}(z)$ can be thought of as arising from the solution of a two channel problem with the Hamiltonian given by

$$H_{\beta\alpha}(z)=\delta_{\beta\alpha}H_\alpha+W_{\beta\alpha}(z) . \tag{33}$$

It is interesting to note that $W_{\beta\alpha}(z)$ is the Born approximation for the T matrix when $U_\alpha=0$, $\alpha=1,2$. Energy dependent potentials of this type have been used by Miller[20] to describe pion-nucleon and pion-nucleus scattering in a formalism based on the Chew-Low theory.

IV. BOUND STATES, DECAYING STATES, AND RESONANCES

Bound state energies as well as pole positions for decaying states and resonances are given by the zeros of the denominator function $d(z)$ defined by Eq. (26). In order to determine the properties of $d(z)$ it is useful to make eigenfunction expansions of the Green's functions $G_\alpha(z)$ which appear in (26) and are given by (23) and (21). We can write

$$G_\alpha(z)=\sum_n \frac{\mid \alpha n\rangle\langle \alpha n \mid}{z-E_{\alpha n}} + \int \frac{\mid \alpha\vec{k}\rangle_\pm d^3k\,{}_\pm\langle \alpha\vec{k} \mid}{z-E_\alpha(k)} , \tag{34}$$

where the bound states $|\alpha n\rangle$ and the in (+) and out (−) scattering states $|\alpha\vec{k}\rangle_\pm$ are solutions of

$$H_\alpha|\alpha n\rangle = E_{\alpha n}|\alpha n\rangle\ ,\qquad H_\alpha|\alpha\vec{k}\rangle_\pm = E_\alpha(k)|\alpha\vec{k}\rangle_\pm\ , \tag{35}$$

with

$$E_\alpha(k) = m_\alpha + \omega_\alpha(k)\ . \tag{36}$$

Inserting (34) into (26), we obtain

$$d(z) = z - m_0 - \sum_{\alpha=1}^{2} g_{\alpha 0}^2\left[\sum_n \frac{\rho_{\alpha n}}{z - E_{\alpha n}} + \int_{\mu_\alpha}^{\infty} \frac{d\omega\sqrt{\omega^2-\mu_\alpha^2}\,\rho_\alpha(\omega)}{z - m_\alpha - \omega}\right]\ , \tag{37}$$

where

$$\rho_{\alpha n} = |\langle \alpha n|u_\alpha\rangle|^2\ ,\qquad \rho_\alpha(\omega) = 4\pi\omega\,|{}_\pm\langle\alpha\vec{k}|u_\alpha\rangle|^2, k = \sqrt{\omega^2-\mu_\alpha^2}\ . \tag{38}$$

From (28), (29), and (34) it appears that $T_{\beta\alpha}(z)$ has the poles of $G_\beta(z)$ and $G_\alpha(z)$; however, by using (30) and (37) it is straightforward to show that these poles are illusory. Thus, all the poles of $T_{\beta\alpha}(z)$ arise from the zeros of $d(z)$.

According to (37), $d(z)$ has simple poles at $z = E_{\alpha n}$ and two right hand branch cuts beginning at $z = m_\alpha + \mu_\alpha$, $\alpha = 1,2$; also,

$$d^*(z) = d(z^*)\ . \tag{39}$$

Thus, except for the singularities mentioned, $d(z)$ is a real, analytic function of z. Using (26) and (23) it is straightforward to derive the relation

$$\mathrm{Im}\,d(z) = \mathrm{Im}(z)\left[1 + \sum_{\alpha=1}^{2} g_{\alpha 0}^2\langle u_\alpha|G_\alpha^\dagger(z)G_\alpha(z)|u_\alpha\rangle\right]\ , \tag{40}$$

which shows that $\mathrm{Im}\,d(z)$ has the same sign as $\mathrm{Im}(z)$, and furthermore, $d(z)$ cannot vanish unless $\mathrm{Im}(z) = 0$. All of this establishes the fact that $d(z)$ is a generalized R function, i.e., a function of the type that appears in the study of the Low equation for simple field theories.[21]

By inserting the identity

$$\mathrm{Im}\frac{1}{E \pm i\epsilon - m_\alpha - \omega} = \mp\pi\delta(E - m_\alpha - \omega) \tag{41}$$

into (37) we find

$$\mathrm{Im}\,d(E \pm i\epsilon) = \pm\pi\sum_{\alpha=1}^{2} g_{\alpha 0}^2\sqrt{(E - m_\alpha)^2 - \mu_\alpha^2}\times\rho_\alpha(E - m_\alpha)\theta(E - m_\alpha - \mu_\alpha)\ , \tag{42}$$

where E is a real energy and ϵ is a positive, infinitesimal parameter. Assuming for the sake of definiteness that $m_1 + \mu_1 < m_2 + \mu_2$, we see that $\mathrm{Im}\,d(E \pm i\epsilon)$ with $E > m_1 + \mu_1$ can vanish only if $\rho_1(E - m_1)$ has a zero for $m_1 + \mu_1 < E < m_2 + \mu_2$, or if $\rho_1(E - m_1)$ and $\rho_2(E - m_2)$ both vanish at some value of $E > m_2 + \mu_2$. Of course, in order for $d(E \pm i\epsilon)$ to vanish, $\mathrm{Re}\,d(E \pm i\epsilon)$ must vanish at the above mentioned zeros. Thus, except under very special circumstances $d(z)$ can vanish only for $z = E < m_1 + \mu_1$.

In order to proceed in as simple a way as possible, we shall assume from now on that H_1 has no bound states, and that H_2 produces bound states with energies $E_{2n} > m_1 + \mu_1$. Other possibilities can be treated along the same lines as those presented here. From (23) it follows that

$$-\frac{d}{dE}\sum_{\alpha=1}^{2} g_{\alpha 0}^2\langle u_\alpha|G_\alpha(E)|u_\alpha\rangle = \sum_{\alpha=1}^{2} g_{\alpha 0}^2\langle u_\alpha|G_\alpha^2(E)|u_\alpha\rangle > 0,\quad E < m_1 + \mu_1 \tag{43}$$

which when combined with the fact that $m_0 - E$ decreases linearly with E, shows that m_0 can always be chosen so that $d(E) = 0$ has a solution for $E < m_1 + \mu_1$. Assuming this has been done we identify this energy with the physical mass m of the V particle. By combining

$$d(m)=0, m<m_1+\mu_1 \tag{44}$$

with (26), we can write

$$d(z)=(z-m)\left[1+\sum_{\alpha=1}^{2} g_{\alpha 0}{}^2\langle u_\alpha \,|\, G_\alpha(m)G_\alpha(z)\,|\, u_\alpha\rangle\right], \tag{45}$$

thereby eliminating the bare mass m_0 from the expression for $d(z)$. In the spirit of the original Lee model[1,5] we define a renormalization constant Z_v by

$$Z_v=\left[1+\sum_{\alpha=1}^{2} g_{\alpha 0}{}^2\langle u_\alpha \,|\, G_\alpha{}^2(m)\,|\, u_\alpha\rangle\right]^{-1}, \tag{46}$$

where

$$0<Z_v<1 . \tag{47}$$

Following the treatment of the original Lee model[1,5] it is straightforward to show that Z_v is the probability of finding a bare V particle in the physical V particle. Solving for the "one" in (46) and substituting it into (45), we find

$$d(z)=Z_v{}^{-1}h(z) , \tag{48}$$

where

$$h(z)=(z-m)\left[1-(z-m)\sum_{\alpha=1}^{2} g_{\alpha}{}^2\langle u_\alpha \,|\, G_\alpha{}^2(m)G_\alpha(z)\,|\, u_\alpha\rangle\right], \tag{49}$$

with

$$g_\alpha=Z_v^{1/2}g_{\alpha 0} . \tag{50}$$

From (46) and (50) we find the alternate expression

$$Z_v=1-\sum_{\alpha=1}^{2} g_{\alpha}{}^2\langle u_\alpha \,|\, G_\alpha{}^2(m)\,|\, u_\alpha\rangle . \tag{51}$$

We shall assume that g_α and u_α are such that $Z_v>0$, i.e., that there are no "ghost" states.[3,5] In order to write the T matrix in terms of renormalized quantities, we combine (30), (48), and (50) to obtain

$$t_{\beta\alpha}(z)=|\,u_\beta\rangle \frac{g_\beta g_\alpha}{h(z)}\langle u_\alpha\,| . \tag{52}$$

Thus, as in the original Lee model,[1,5] the T matrix can be written completely in terms of renormalized quantities.

Under the assumptions we have made, $d(z)$ and $h(z)$ have only one zero on the physical sheet at $z=m$, the physical mass of the V particle. We now consider the possibility of zeros on other Riemann sheets. If they are close enough to the real axis, the real and imaginary parts of such zeros can be interpreted as giving the energy and width of decaying bound states and resonances. Let us first consider what happens if we let $g_{10}{}^2=g_1{}^2=0$. According to (26) and (48), the real zeros of $d(z)$ and $h(z)$ will be given by

$$m_0-E=-g_{20}{}^2\langle u_2\,|\, G_2(E)\,|\, u_2\rangle,\ E<m_2+\mu_2 . \tag{53}$$

Since the right hand side of (53) has simple poles at $z=E_{2n}$ and is monotonic increasing between successive poles, there will be one solution of (53) between each pair of adjacent poles as well as one with $E<E_{2,1}$. It is interesting to note that the same distribution of bound state poles occurs for a T matrix arising from a combination of an attractive Coulomb potential and an attractive, rank-one separable potential.[22] It is important to note that a solution of (53) that falls on one of the intervals $E_{2n}<E<E_{2,n+1}$ is trapped in the sense that no matter how the parameters $g_{20}{}^2$ and m_0 are varied it cannot travel outside its interval.

In general, as $g_1{}^2=Z_v g_{10}{}^2$ is turned on, the trapped zeros of $d(z)=Z_v{}^{-1}h(z)$ escape to another Riemann sheet, thus becoming decaying bound states or resonances. Fortunately, the problem of continuing $h(z)$ or $d(z)$ onto another Riemann sheet has essentially been solved by Levy[4] in his study of resonances in the original Lee model. Each of the two functions of z defined by the integrals in (37) has two Riemann sheets arising from square-root singularities at $z=m_\alpha+\mu_\alpha$. An easy way to see this is to define a function $A(z)$ by

$$h(z)=A(z)+\pi i g_1{}^2\rho_1(z-m_1)\times[(z-m_1)^2-\mu_1{}^2]^{1/2}, \quad (54)$$

where the square-root function is specified by

$$\begin{aligned} &z-m_1-\mu_1=|z-m_1-\mu_1|e^{i\theta_1},\quad 0<\theta_1<2\pi,\\ &z-m_1+\mu_1=|z-m_1+\mu_1|e^{i\theta_2},\quad -\pi<\theta_2<\pi,\\ &[(z-m_1)^2-\mu_1{}^2]^{1/2}=|(z-m_1)^2-\mu_1{}^2|^{1/2}\times e^{i(\theta_1+\theta_2)/2}. \end{aligned} \quad (55)$$

From (37), (41), (48), and (50) it follows that

$$A(E+i\epsilon)-A(E-i\epsilon)=0,\quad m_1+\mu_1<E<m_2+\mu_2, \quad (56)$$

thus, the singularity of $h(z)$ at $z=m_1+\mu_1$ is given explicitly by the square-root function in (54). In writing (56) we have assumed that $\rho_1(z-m_1)$ is analytic in the neighborhood of the interval indicated. Of course, the square-root singularity at $z=m_2+\mu_2$ can be made explicit in exactly the same way. Choosing the opposite sign for the square root in (54) gives the continuation of $h(z)$ onto another Riemann sheet, which we call $H(z)$, i.e.,

$$H(z)=A(z)-\pi i g_1{}^2\rho_1(z-m_1)\times[(z-m_1)^2-\mu_1{}^2]^{1/2}, \quad (57)$$

and we have

$$H(z)=h(z)-2\pi i g_1{}^2\rho_1(z-m_1)\times[(z-m_1)^2-\mu_1{}^2]^{1/2}. \quad (58)$$

From (39), (48), and (55), it follows that

$$H^*(z)=H(z^*) \quad (59)$$

in the region in which $\rho_1(z-m_1)$ is a real, analytic function of z. Using (55), it is straightforward to show that for $\mathrm{Re}(z)\geq m_1$ the real part of the square root in (58) is positive for $\mathrm{Im}(z)>0$ and negative for $\mathrm{Im}(z)<0$. Taking into account (40) and (48), it follows that $\mathrm{Im}H(z)$ can vanish for $\mathrm{Im}(z)\neq 0$, while $\mathrm{Im}h(z)$ cannot. Thus, it appears that the trapped zeros of $h(z)=Z_v d(z)$ that occur when $g_1{}^2=0$ move onto the Riemann sheet defined by (58). From (54)–(57), it follows that

$$h(E+i\epsilon)=H(E-i\epsilon),\quad m_1+\mu_1<E<m_2+\mu_2, \quad (60)$$

thus, the zeros of $H(z)$ just below the real z axis are close to the physical region, i.e., the upper rim of the right hand cut in $h(z)$. As a result of (59), a zero of $H(z)$ in the lower half plane will have a mate symmetrically located with respect to the real z axis; however, this mate is not close to the physical region.

We now proceed to find an approximate expression for one of the zeros z_n of $H(z)$ in the lower half of the z plane under the assumption that it is close to E_{2n}, one of the eigenvalues of H_2. We introduce a function $F(z)$ which is analytic in the neighborhood of E_{2n} by

$$F(z)=\begin{matrix} D(z)h(z), & \mathrm{Im}(z)>0\\ D(z)H(z), & \mathrm{Im}(z)<0 \end{matrix} \quad (61)$$

where $D(z)$ is a function which is analytic in the neighborhood of E_{2n} and behaves similar to

$$D(z)\to z-E_{2n},\quad z\to E_{2n}. \quad (62)$$

From (34), (38), (49), and (58) it follows that

$$h(z),H(z)\to-\frac{g_2{}^2\rho_{2n}}{z-E_{2n}},\quad z\to E_{2n}, \quad (63)$$

so $D(z)$ removes the pole at $z=E_{2n}$. Using

$$F(z)=F(E_{2n})+F'(E_{2n})(z-E_{2n})+\dots \quad (64)$$

and assuming higher order terms are negligible we obtain

$$z_n-E_{2n}=-\frac{F(E_{2n})}{F'(E_{2n})}, \quad (65)$$

or, with a little manipulation,

$$z_n-E_{2n}=\frac{g_2{}^2\rho_{2n}}{h_n(E_{2n}+i\epsilon)-\Delta_n}, \quad (66)$$

where

$$h_n(z)=h(z)+\left[\frac{z-m}{E_{2n}-m}\right]^2\frac{g_2{}^2\rho_{2n}}{z-E_{2n}} \quad (67)$$

and

$$\Delta_n = \lim_{z \to E_{2n}} \frac{d}{dz} \left[D(z) \left[\frac{z-m}{E_{2n}-m} \right]^2 \frac{g_2{}^2 \rho_{2n}}{z-E_{2n}} \right] . \tag{68}$$

V. DISCUSSION

Equation (66) bears some resemblance to the results of Deser *et al.*[6] in that it is closely related to the matrix element [see Eqs. (38) and (52)]

$$\langle 2n \mid t_{22}(E+i\epsilon) \mid 2n \rangle = \frac{g_2{}^2 \rho_{2n}}{h(E+i\epsilon)} . \tag{69}$$

This will facilitate using the model presented here to test their formula relating the energy of a mesic atom to the meson-nucleus scattering length.

The method we have used to obtain (66) is essentially the same as the one used by Deloff[9] to locate the complex poles of the S matrix arising from a combination of a Coulomb potential and a complex optical potential. In his case the role of $D(z)$ is played by the Coulomb Jost function, and he finds the method to be very accurate. As he points out, (65) can be iterated (Newton-Raphson method) to obtain precise values for z_n when $|z-E_{2n}|$ becomes large. Deloff's work suggests that a good choice here for $D(z)$ is the Fredholm determinant[23] corresponding to the single particle Hamiltonian H_2, since this will take out all of the bound state poles in $h(z)$ and make $F(z)$ analytic in a large region. A much simpler choice is

$$D(z) = \left[\frac{E_{2n}-m}{z-m} \right]^2 (z-E_{2n}) , \tag{70}$$

and leads to $\Delta_n = 0$.

Precise numerical determinations of the complex energies z_n, and various approximations for them such as (66) will be the subject of a future investigation. In particular, the trajectories of the z_n that arise when the parameters in the strong interaction ($V \rightleftarrows N_\alpha + \theta_\alpha$) are varied, will be studied and compared with the results obtained by Krell,[24] Koch *et al.*,[25] and Kok[26] for short-range complex interactions. It will be of special interest to determine the trajectories when the V particle is unstable ($m > m_1 + \mu_1$) and has a complex energy close to the $N_2 - \theta_2$ bound state energies (E_{2n}) produced by the single particle Hamiltonian H_2, as this will give some insight into the effect of subthreshold resonances in kaonic atoms.[27]

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