Pion production and absorption in field theory

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We derive a coupled-channel description of pion production and absorption by the many-nucleon system from the field theoretic definition of the reaction matrix. Our result is manifestly consistent with nucleon antisymmetry, pion crossing, and unitarity and is exact, up to the neglect of terms involving at least $2N-2$ antinucleons. Our equation is approximately soluble in the limit of nonrelativistic nucleons, and in that limit we obtain explicit, Hamiltonian independent forms for the one-body operator and for leading terms in the two-body operator for use in distorted-wave Born approximation calculations.

[NUCLEAR REACTIONS Scattering theory, pion production, and absorption by many-nucleon systems.]

I. INTRODUCTION

In this paper, we examine the general problem of pion production and absorption by the many nucleon system in relativistic quantum field theory. Our goal is to obtain an exact, nonperturbative result to serve as a framework for the development or examination of various approximations. Low equation considerations provide the basis for this inquiry.

In two previous papers[1,2] we applied the Low equation to the $\pi^2\mathrm{H}\to NN$ reaction. Although the Low equation is unquestionably consistent with pion crossing and nucleon antisymmetry, we found that the exact manner in which these features are embedded in the equation is a matter of some subtlety which requires a careful treatment of the seagull terms. The result was a coupled, linear, inhomogeneous integral equation for the reaction matrix. The generalization of the ideas developed in these papers to the many-nucleon system is nontrivial. In particular, in the many-nucleon system, we encounter three formal difficulties:

(1) A straightforward extension of the channel coupling scheme of the one- and two-nucleon systems is not possible. For example, consider the direct term in the Low equation for pion production

$$(2\pi)^3\sum_n \frac{\delta^{(3)}(\vec{p}_n-\vec{p}_i)}{E_i-E_n+i\epsilon}\,{}_{\text{out}}\langle \mathcal{N}_1\ldots\mathcal{N}_N\,|\,j_\pi(0)\,|\,n\rangle \times\langle n\,|\,J_{N+1}(0)^\dagger\,|\,\mathcal{N}_{N+2}\ldots\mathcal{N}_{2N}\rangle_{\text{in}}\,, \tag{1}$$

[$j_\pi(0)$ and $J_{N+1}(0)^\dagger$ are the source functions for the pion and an incident nucleon, respectively]. In this expression, we may choose as intermediate states any complete set which spans the physical Hilbert space. For $N>2$, it is not possible to choose these states such that both matrix elements can be directly interpreted in terms of off-mass-shell scattering processes.

(2) Truncating the Low equation destroys its consistency with detailed balancing and pion crossing. That is, time reversing the truncated equations does not lead to the same result as crossing the external pion line.

(3) It is not clear, even for $N=2$, how to make the connection with nonrelativistic potential theory unambiguously. Although Banerjee *et al.*[3] do make this connection, their treatment of the seagull terms is flawed and they are unable to obtain the pion rescattering contribution. The alternative treatment of the seagulls proposed in Ref. 2 leads to a form in which both an off-energy-shell and an off-mass-

shell nucleon-nucleon scattering amplitude appear. Thus it is not clear how to define a nucleon-nucleon potential uniquely.

The present work obviates these difficulties. In Sec. II we generalize the channel coupling scheme which is the central idea of the Low equation for one- and two-nucleon systems, to the many-nucleon system. We obtain an exact result[4] in which nucleon antisymmetry, and consistency with pion crossing and unitarity are manifest, and the equation itself is form invariant under time reversal. In Sec. III we make the connection with nonrelativistic potential theory and obtain a Hamiltonian-independent two-nucleon transition operator for use in DWBA calculations.

II. DERIVATION OF THE BASIC FORMALISM

Owing to the self-conjugate nature of the pion field, the physical reaction matrices for pion production and pion absorption are simply related to the matrix element of the pion current between N-nucleon states. Applying the standard Lehmann-Symanzik-Zimmermann (LSZ) reduction techniques, we write the fully connected piece of this matrix element in the form

$$
\begin{aligned}
&{}_{\text{out}}\langle \mathcal{N}(\vec{p}_1)\ldots\mathcal{N}(\vec{p}_N)\,|\,j_\pi(0)\,|\,\mathcal{N}(\vec{p}_{N+1})\ldots\mathcal{N}(\vec{p}_{2N})\rangle_{\text{in}}^{c}\\
&= \int dx_1\ldots dx_{2N}\exp\left[i\left[\sum_{n=1}^{N}-\sum_{n=N+1}^{2N}\right]p_n\cdot x_n\right]\langle 0\,|\,\mathscr{D}_1\ldots\mathscr{D}_{2N}T(\psi_1\ldots\psi_N j_\pi\psi_{N+1}\ldots\psi_{2N})\,|\,0\rangle\,,\\
&= \text{"seagull terms"}\\
&\quad+\int dx_1\ldots dx_{2N}\exp\left[i\left[\sum_{n=1}^{N}-\sum_{n=N+1}^{2N}\right]p_n\cdot x_n\right]\langle 0\,|\,T(J_1\ldots J_N j_\pi J_{N+1}\ldots J_{2N})\,|\,0\rangle\,,
\end{aligned}
\tag{2}
$$

where we have introduced the notation

$$
\psi_n=\begin{cases}\psi(x_n) & n\le N\\ \psi(x_n)^\dagger & n\ge N+1\end{cases},
$$

$$
\mathscr{D}_n=\begin{cases}\bar{u}(\vec{p}_n)(-i\gamma\cdot\partial+m) & n\le N\\ [\bar{u}(p_n)(-i\gamma\cdot\partial+m)]^\dagger & n\ge N+1\end{cases},
$$

$$
j_\pi=j_\pi(0)\,,
$$

and where in the second step, we have taken the differential operators $\mathscr{D}_n$ inside the T product and retained all nonvanishing equal-time commutators and anticommutators (seagull terms) which result from the time differentiations. $J_n=\mathscr{D}_n\psi_n$ denotes the nucleon source function.

In the following, we assume the equal-time commutation and anticommutation relations (ETC's)

$$
\begin{aligned}
&[\phi_\pi(\vec{x},0),\dot{\phi}_\pi(\vec{y},0)]_-=Z_1\delta^{(3)}(\vec{x}-\vec{y})\,,\\
&[\psi_\alpha(\vec{x},0),\psi_\beta(y,0)^\dagger]_+=Z_2\delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y})\,,\\
&[\phi_\pi(\vec{x},0),\psi(\vec{y},0)]_-=[\phi_\pi(\vec{x},0),\psi(\vec{y},0)^\dagger]_-\\
&\qquad=0\,,
\end{aligned}
$$

etc., to be valid, with Z_1,Z_2 c numbers and where the knonecker delta concerns the discrete quantum numbers of the nucleon. We further assume that, in addition to the above, only the ETC's

$$
\begin{aligned}
&[J(\vec{x},0),\psi(\vec{y},0)^\dagger]_+\,,\\
&[\psi(\vec{x},0),j_\pi(0)]_-\,,\\
&[[\vec{\psi}(x,0),j_\pi(0)]_-,\psi(\vec{y},0)^\dagger]_+\,,
\end{aligned}
$$

and their Hermitian conjugates are nonvanishing. That is, we neglect terms which would require the introduction of canonical field operators with baryon number two in the interaction Lagrangian.[5] In this paper, we consider only the final term in Eq. (2) in detail, and content ourselves with a sketch of the analysis of the seagull terms.

We wish to rewrite Eq. (2) such that: (1) the contribution from the initial and final interactions are clearly exhibited, and (2) the initial interaction is treated equivalently to the final interaction (viz., form invariance under time reversal).[6] To appreciate what the separation of the initial interaction terms from the T product in Eq. (2) entails, consider replacing the "in" label on the ket $|\mathcal{N}_{N+2}\ldots\mathcal{N}_{2N}\rangle_{\text{in}}$ in Eq. (1) by "out" and using a complete set of "in" states in the sum. Thus modified, Eq. (1) is a suitable candidate for the initial interaction term. Application of the reduction technique to the external nucleons then leads naturally to a separation of this term into products involving only fully connected matrix elements, i.e., products of the form[7]

$$\langle 0 \,|\, T(J_1 \ldots j_\pi \ldots J_j) \,|\, n \rangle_{\text{in}\,\text{in}}\langle n \,|\, \bar{T}(J_{j+1} \ldots J_{2N}) \,|\, 0 \rangle \, .$$

Similar considerations hold for the final interaction. This strongly suggests looking for ways to break up the T product in Eq. (2) into a sum of terms, each of which involves the product of a time-ordered sequence of operators with an antitime ordered sequence, being careful to maintain a symmetry between the initial and final interaction terms. We demonstrate a technique for doing this below.

We begin with the simple operator identity

$$\begin{aligned} T(J_1 \ldots J_N j_\pi J_{N+1} \ldots J_{2N}) = \sum_P (-1)^P [& j_\pi T(J_1 \ldots J_{2N})\theta_- + T(J_1)\theta_+ j_\pi T(J_2 \ldots J_{2N})\theta_- \\ & + T(J_1 J_2)\theta_+ j_\pi T(J_3 \ldots J_{2N})\theta_- + \ldots \\ & + T(J_1 \ldots J_N)\theta_+ j_\pi T(J_{N+1} \ldots J_{2N})\theta_- \\ & + \ldots + T(J_1 \ldots J_{2N})\theta_+ j_\pi] \, . \end{aligned} \tag{3}$$

The $\theta_-(\theta_+)$ in this expression are defined such that they restrict the time arguments of the preceding T product to less (greater) than zero, e.g.,

$$T(J_1 \ldots J_{2N})\theta_- = T(J_1 \ldots J_{2N})\theta_{01}\theta_{02} \ldots \theta_{0,2N} \, ,$$

[here we introduce the obvious notation $\theta_{0n} = \theta(-x_n)$, $\theta_{n0} = \theta(x_n)$, $\theta_{mn} = \theta(x_m - x_n)$] and $\sum_P (-1)^P$ denotes the sum over all distinct permutations of the nucleon operators with the sign appropriate to the given permutation.[8]

The first $N+1$ terms in this identity contribute to the analog of the direct term in the Low equation for pion production, and we deal with these terms first. Associated with each of these terms is a T product of the form

$$T(J_i \ldots J_j)\theta_- = \sum_P (-1)^P [J_i \ldots J_j \theta_{0i} \ldots \theta_{0j} \theta_{i,i+1} \ldots \theta_{j-1,j}] \, , \tag{4}$$

where for simplicity, we have assumed the operators to be consecutively labeled, although this is clearly inessential. Our object initially is to express Eq. (4) as a sum over the product of T and $\bar{T}$ products by successive replacements of the form $\theta_{n,n+1} = 1 - \theta_{n+1,n}$, e.g.,

$$\begin{aligned} & [J_i \ldots J_j \theta_{i,i+1} \ldots \theta_{j-1,j}\theta_-] \\ & \quad = [J_i \ldots J_{j-1}\theta_{i,i+1} \ldots \theta_{j-2,j-1}\theta_-] J_j [1 - \theta_{j,j-1}] \begin{Bmatrix} 1 \\ \theta_{0j} \end{Bmatrix} \\ & \quad = [J_i \ldots J_{j-1}\theta_{i,i+1} \ldots \theta_{j-2,j-1}\theta_-] J_j \begin{Bmatrix} 1 \\ \theta_{0j} \end{Bmatrix} \\ & \qquad - [J_i \ldots J_{j-2}\theta_{i,i+1} \ldots \theta_{j-3,j-2}\theta_-] J_{j-1} J_j \theta_{j,j-1} \begin{Bmatrix} 1 \\ \theta_{0j} \end{Bmatrix} \begin{Bmatrix} 1 \\ \theta_{0,j-1} \end{Bmatrix} [1 - \theta_{j-1,j-2}] \, , \end{aligned}$$

etc. The curly braces indicate that at a given step of this process, where say we replace $\theta_{n-1,n}$ by $1 - \theta_{n,n-1}$, we have the option of either retaining the θ_{0n} associated with the newly isolated operator J_n or replacing it with unity, as the entire expression is multiplied by θ_{0i}. Since the direct term corresponds to an initial interaction among the incident nucleons followed by interaction with the external pion, we shall retain the θ_{0n} whenever it is associated with the operator of an incoming nucleon (i.e., $n \geq N+1$) and replace it by unity otherwise. This choice, which ensures that the initial interaction is ordered before the pionic interaction, when applied to each term in the T product in Eq. (4), leads to

$$T(J_i\ldots J_j)\theta_- = \sum_P (-1)^P \sum_{k=0}^{j-i} (-1)^k [T(J_i\ldots J_{j-k-1})\theta_-][\bar{T}(J_{j-k}\ldots J_j)\theta_{-(j-k)}\ldots\Theta_{-j}]$$
$$+(-1)^{j-i+1}\bar{T}(J_i\ldots J_j)\theta_+ , \quad (5)$$

where $\Theta_{-n}=\theta(-x_n)$ for $n \geq N+1$ and is unity otherwise. The final term in this expression occurs only if $j \leq N$, that is, all current operators on the left-hand side of this equation are associated with outgoing nucleons. In the following, we neglect this term as it contributes only if there are at least $2N$ antinucleons associated with the intermediate state sums, and it is surely negligible.[9]

Equation (5) can now be used to separate antitime ordered sequences of nucleon operator clusters from the first $N+1$ terms in Eq. (3) in the desired fashion. To illustrate this process, we note that the $k=0$ part of the sum in Eq. (5) leads to

$$\sum_P (-1)^P [j_\pi T(J_1\ldots J_{2N-1})\theta_- + \ldots + T(J_1\ldots J_N)\theta_+ j_\pi T(J_{N+1}\ldots J_{2N-1})\theta_-]J_{2N}\Theta_{-2N}$$
$$= \sum_P (-1)^P T(J_1\ldots J_N j_\pi J_{N+1}\ldots J_{2N-1})J_{2N}\Theta_{-2N}$$
$$- \sum_P (-1)^P [\,T(J_1\ldots J_{N+1})\theta_+ j_\pi T(J_{N+2}\ldots J_{2N-1})\theta_-$$
$$+ \ldots + T(J_1\ldots J_{2N-1})\theta_+ j_\pi]J_{2N}\Theta_{-2N} . \quad (6)$$

Proceeding in a similar fashion with the remaining terms in the k sum, we obtain[10]

$$\sum_P (-1)^P \sum_{n=0}^{N} T(J_1\ldots J_n)\theta_+ j_\pi T(J_{n+1}\ldots J_{2N})\theta_-$$
$$= \sum_P (-1)^P \sum_{k=0}^{2N-1} (-1)^k T(J_1\ldots J_N j_\pi J_{N+1}\ldots J_{2N-k-1})[\bar{T}(J_{2N-k}\ldots J_{2N})\Theta_{-(2N-k)}\ldots\Theta_{-2N}]$$
$$- \sum_P (-1)^P \sum_{k=0}^{N-2} \sum_{j=N+1}^{2N-k-1} (-1)^k [T(J_1\ldots J_j)\theta_+ j_\pi T(J_{j+1}\ldots J_{2N-k-1})\theta_-]$$
$$\times [\bar{T}(J_{2N-k}\ldots J_{2N})\Theta_{-(2N-k)}\ldots\Theta_{-2N}] . \quad (7)$$

The first term in Eq. (7) contributes to the direct term (i.e., the initial interaction term) which has the general form that an interaction bubble with $k+1$ external nucleon lines, at least one of which must be an incident nucleon line,[11] occurs before a second interaction bubble involving the pion and the remaining external nucleon lines (see Fig. 1). The second term is a doublecounting term which we shall deal with later.

To obtain the cross term, that is, the direct term in the crossed-pion channel, in which the pionic interaction occurs before the final interactions among the nucleons, we require the identity

$$T(J_i\ldots J_j)\theta_+ = \sum_P (-1)^P \sum_{k=0}^{j-1} (-1)^k [\bar{T}(J_i\ldots J_{k+i})\Theta_i\ldots\Theta_{k+i}][T(J_{k+i+1}\ldots J_j)\theta_+]$$
$$+(-1)^{j-i+1}\bar{T}(J_i\ldots J_j)\theta_- , \quad (8)$$

where $\Theta_n=\theta(x_n)$ for $n \leq N$ and unity otherwise. While the derivation of this expression is similar to that of Eq. (5), we now retain θ_{n0} whenever it is associated with an outgoing current operator (i.e., $n \leq N$) to order the final interactions after the pionic interaction. Applying this expression to the $N+1$ final terms in Eq. (3), we obtain

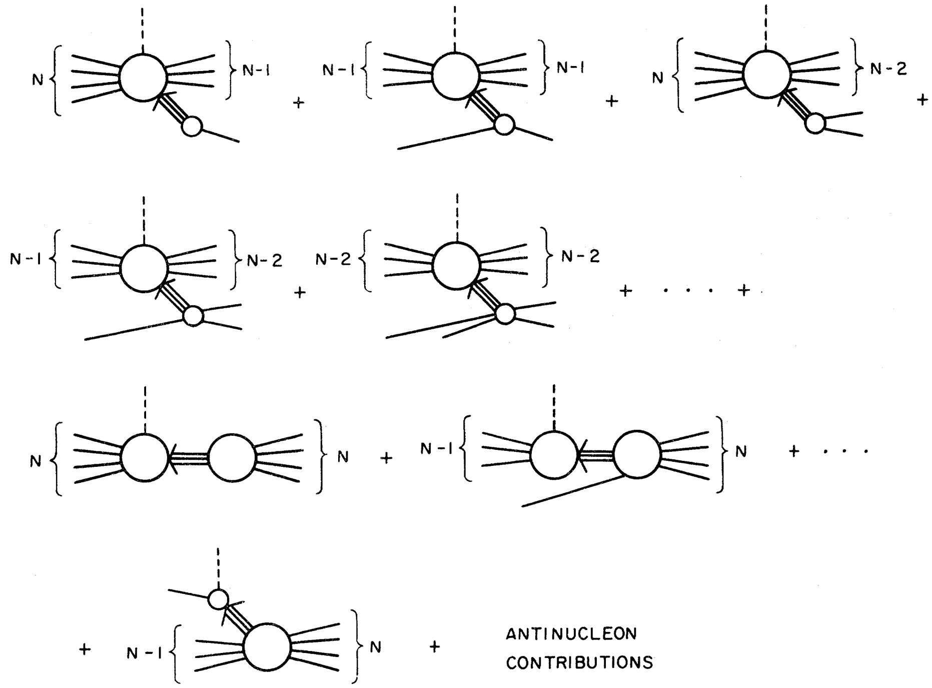

FIG. 1. Contributions to the direct term. Dashed lines represent pions, solid lines nucleons, and three closely spaced lines capped by an arrowhead represent an intermediate state sum. All interaction bubbles are fully connected in that they possess no disconnected lines. Those interaction bubbles with an external pion line are off-the-pion-mass-shell—all others are off shell in the incident nucleon variables.

$$\sum_P (-1)^P \sum_{n=N}^{2N} T(J_1 \ldots J_n)\theta_+ j_\pi T(J_{n+1} \ldots J_{2N})\theta_-$$

$$= \sum_P (-1)^P \sum_{k=0}^{2N-1} (-1)^k [\bar{T}(J_1 \ldots J_{k+1})\Theta_1 \ldots \Theta_{k+1}] T(J_{k+2} \ldots J_N j_\pi J_{N+1} \ldots J_{2N})$$

$$- \sum_P (-1)^P \sum_{k=0}^{N-2} \sum_{j=k+1}^{N-1} (-1)^k [\bar{T}(J_1 \ldots J_{k+1})\Theta_1 \ldots \Theta_{k+1}]$$

$$\times [T(J_{k+2} \ldots J_j)\theta_+ j_\pi T(J_{j+1} \ldots j_{2N})\theta_-] , \tag{9}$$

where here the first term contributes to a (pion) crossed term, which has the general form of a final interaction bubble involving one outgoing and k arbitrary external nucleon lines, preceded by a pionic interaction bubble involving the remaining external nucleon lines. The second piece of Eq. (9) is another doublecounting term. In the context of doublecounting, we note that we must also subtract

$$\sum_P (-1)^P T(J_1 \ldots J_N)\theta_+ j_\pi T(J_{N+1} \ldots J_{2N})\theta_-$$

from our final expression since this term has been included in both Eqs. (7) and (9).

The direct and crossed terms in Eqs. (7) and (9) are in the desired form—however, the doublecounting pieces may be written much more concisely. In the Appendix we show that the doublecounting terms may be combined to give

$$-\sum_P(-1)^P\sum_{l=0}^{2N-2}\sum_{k=0}^{2N-l-2}(-1)^{k+l}[\bar{T}(J_1\cdots J_{l+1})\Theta_1\cdots\Theta_{l+1}][T(J_{l+2}\cdots j_\pi\cdots J_{2N-k-1})]$$
$$\times[\bar{T}(J_{2N-k}\cdots J_{2N})\Theta_{-(2N-k)}\cdots\Theta_{-2N}]\,. \tag{10}$$

Graphically, a term in Eq. (10) has the general form of three interaction bubbles; the first with $k+1$ external nucleon lines, the final with $l+1$ external nucleon lines, and a middle or intermediate bubble with the external pion line and the remaining external nucleon linees. This arrangement is subject to the restriction that the first bubble must have at least one incident nucleon line and the last bubble, at least one outgoing nucleon line—otherwise the graph vanishes owing to energy considerations. This completes our manipulations.

We now claim that as a result of the foregoing analysis, we can write

$$\mathscr{D}_1\cdots\mathscr{D}_{2N}T(\psi_1\cdots\psi_N j_\pi\psi_{N+1}\cdots\psi_{2N})$$
$$=\sum_P(-1)^P\sum_{k=0}^{2N-1}(-1)^k[\mathscr{D}_1\cdots\mathscr{D}_{2N-k-1}T(\psi_1\cdots j_\pi\cdots\psi_{2N-k-1})]$$
$$\times[\mathscr{D}_{2N-k}\cdots\mathscr{D}_{2N}\bar{T}(\psi_{2N-k}\cdots\psi_{2N})]\Theta_{-(2N-k)}\cdots\Theta_{-2N}$$
$$+\sum_P(-1)^P\sum_{k=0}^{2N-1}(-1)^k[\mathscr{D}_1\cdots\mathscr{D}_{k+1}\bar{T}(\psi_1\cdots\psi_{k+1})]\Theta_1\cdots\Theta_{k+1}$$
$$\times[\mathscr{D}_{k+2}\cdots\mathscr{D}_{2N}T(\psi_{k+2}\cdots j_\pi\cdots\psi_{2N})]$$
$$-\sum_P(-1)^P\sum_{l=0}^{2N-2}\sum_{k=0}^{2N-l-2}(-1)^{k+l}[\mathscr{D}_1\cdots\mathscr{D}_{l+1}\bar{T}(\psi_1\cdots\psi_{l+1})]\Theta_1\cdots\Theta_{l+1}$$
$$\times[\mathscr{D}_{l+2}\cdots\mathscr{D}_{2N-k-1}T(\psi_{l+2}\cdots j_\pi\cdots\psi_{2N-k-1})]$$
$$\times[\mathscr{D}_{2N-k}\cdots\mathscr{D}_{2N}\bar{T}(\psi_{2N-k}\cdots\psi_{2N})]\Theta_{(2N-k)}\cdots\Theta_{-2N}\,. \tag{11}$$

To complete the derivation of Eq. (11) we sketch the proof that the complement of seagull terms in Eq. (2) are precisely those required by the right-hand side of Eq. (11). First, consider those seagulls involving the T product of $[\psi_n,j_\pi]_-$ or $[[\psi_n,j_\pi]_-,\psi_m]_+$ with either $2N-1$ (for the former ETC) or $2N-2$ (the latter) nucleon source functions, e.g.,

$$T(J_1\cdots J_{N-1}[\psi_N,j_\pi]_-J_{N+1}\cdots J_{2N})\,.$$

While little is known of these ETC's in general, it can be shown that Lorentz covariance, coupled with the (assumed) vanishing of the ETC of ψ with ϕ_π and $\dot{\phi}_\pi$, requires both of the ETC's $[\psi_n,j_\pi]_-$ and $[[\psi_n,j_\pi]_-,\psi_m]_+$ to be local operators (i.e., no Schwinger terms). We note that the preceding analysis (but with different numbers of nucleon operators) applies directly to these T products as well. For those seagulls involving one or more ETC of the form $[J_i,\psi_j]$, we note that our manipulation with regard to θ_{ij} (whose derivative gives rise to this ETC) has been to make the replacement $\theta_{ij}\rightarrow 1-\theta_{ji}$, which has the same singularity. Thus both sides of Eq. (11) have precisely the same complement of these ETC's. It can be shown that the remaining seagulls in Eq. (2), all of which involve Z_2, do not contribute by virtue of the assumed c-number nature of Z_2.

Introducing Eq. (11) into Eq. (2) and inserting complete sets of intermediate states between square brackets, after some effort we obtain

$$
{}_{\text{out}}\langle \mathcal{N}_1 \ldots \mathcal{N}_N \,|\, j_\pi(0) \,|\, \mathcal{N}_{N+1} \ldots \mathcal{N}_{2N} \rangle^c_{\text{in}}
$$

$$
\begin{aligned}
&= T(\text{DIS}) - (2\pi)^3 \sum_n \frac{\delta^{(3)}(\vec{p}_i - \vec{p}_n)}{E_i - E_n + i\epsilon}\, {}_{\text{out}}\langle \mathcal{N}_1 \ldots \mathcal{N}_N \,|\, j_\pi(0) \,|\, n\rangle_{\text{in}}\, {}_{\text{in}}\langle n \,|\, T_N^{(-)}(p_{N+1}, \ldots, p_{2N})^\dagger \,|\, 0\rangle \\
&\quad - (2\pi)^3 \sum_m \frac{\delta^{(3)}(\vec{p}_f - \vec{p}_m)}{E_f - E_m + i\epsilon} \langle 0 \,|\, T_N^{(+)}(p_N, \ldots, p_1)^\dagger \,|\, m\rangle_{\text{out}}\, {}_{\text{out}}\langle m \,|\, j_\pi(0) \,|\, \mathcal{N}_{N+1} \ldots \mathcal{N}_{2N}\rangle_{\text{in}} \\
&\quad - (2\pi)^6 \sum_{m,n} \frac{\delta^{(3)}(\vec{p}_f - \vec{p}_m)}{E_f - E_m + i\epsilon} \frac{\delta^{(3)}(\vec{p}_i - \vec{p}_n)}{E_i - E_n + i\epsilon} \langle 0 \,|\, T_N^{(+)}(p_N, \ldots, p_1)^\dagger \,|\, m\rangle_{\text{out}} \\
&\quad \times {}_{\text{out}}\langle m \,|\, j_\pi(0) \,|\, n\rangle_{\text{in}}\, {}_{\text{in}}\langle n \,|\, T_n^{(-)}(p_{N+1}, \ldots, p_{2N})^\dagger \,|\, 0\rangle \,, \qquad (12)
\end{aligned}
$$

where $T(\text{DIS})$ is a piece which cancels any contributions containing one or more completely disconnected nucleon lines which arise from the terms involving the intermediate state sums, $p_f = p_1 + \ldots p_N$, $p_i = p_{N+1} + \ldots + p_{2N}$, and where

$$
\begin{aligned}
&(2\pi)^3 \frac{\delta^{(3)}(\vec{p}_1 + \ldots + \vec{p}_N - \vec{p}_n)}{E_1 + \ldots + E_N - E_n - i\epsilon}\, {}_{\text{out}}\langle n \,|\, T_N^{(+)}(p_N, \ldots, p_1) \,|\, 0\rangle \\
&= \sum_{j=1}^{N} \sum_P (-1)^P (i)^j \int dx_1 \ldots dx_j\, e^{i \sum_{k=1}^{j} p_k \cdot x_k}\, {}_{\text{out}}\langle n \,|\, a(p_N)^\dagger_{\text{out}} \ldots a(p_{j+1})^\dagger_{\text{out}} \\
&\qquad \times T(\psi_j^\dagger \ldots \psi_1^\dagger) \mathcal{D}_j^\dagger \ldots \mathcal{D}_1^\dagger \,|\, 0\rangle \qquad (13)
\end{aligned}
$$

defines a cluster decomposition of the N-nucleon$\rightarrow$(general state n) process into contributions with differing numbers of disconnected nucleon lines. A similar expression holds for $T^{(-)}$. This equation is shown schematically in Fig. 2.

Equation (12) is a linear, coupled integral equation for the off-the-pion-mass-shell, N-nucleon reaction matrix. This equation is form invariant under time reversal and consequently has a manifest consistency with pion crossing—pion production and pion absorption are described in a completely eqivalent fashion. Moreover, this feature can easily be retained by appropriately truncating the intermediate state sums. All matrix elements are directly interpretable in terms of off-shell scattering processes and the equation itself is manifestly antisymmetric in the nucleon variables. Additionally, we note that we have escaped the traditional anathema of the Low equation—the modeling of the seagull terms. All dynamical information relating to the pion-nucleon interaction enters our theory in the form of the elementary pion-nucleon amplitudes. The manner in which these amplitudes enter our formalism will become more obvious in Sec. III.

III. CONNECTION WITH POTENTIAL THEORY

To illustrate the physical content of our theory and make the connection with more standard approaches, we consider the N-nucleon intermediate

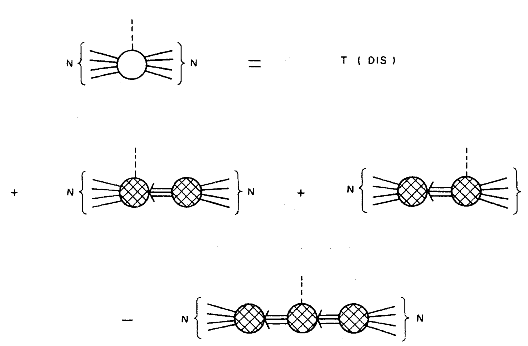

FIG. 2. The diagrammatic representation of Eq. (12). In this diagram, the cross-hatched bubbles are allowed to have disconnected nucleon lines. Figure 1 represents a separation of the interaction bubbles in the direct term (the first term on the second line) into connected contributions.

state contributions to Eq. (12) in the limit of nonrelativistic nucleons. Initially, we ignore the presence of bound states, to simplify our notation. To classify the connectedness structure of this equation we introduce the partition a_n of the N nucleons into one n-nucleon cluster ($1 \leq n \leq N$) and N-n one-nucleon clusters. For a given n, the set $\{a_n\} = a_n, b_n, \ldots$ has $N!/n!(N-n)!$ distinct members.[12] Let $|\psi_\alpha(a_n)^{(\pm)})$ be the corresponding scattering state, with one interacting n-particle subsystem (the members of the n-nucleon cluster in a_n) and N-n spectator (noninteracting) nucleons. This state satisfies

$$(H_0 + U_{a_n})\,|\psi_\alpha(a_n)^{(\pm)}) \equiv E_\alpha\,|\psi_\alpha(a_n)^{(\pm)}) \,, \tag{14}$$

where U_{a_n} represents the sum over the potential interactions internal to the n-nucleon cluster. The corresponding Moller wave operators are

$$\Omega(a_n)^{(\pm)} = \sum_\alpha |\psi_\alpha^{(\pm)})(\psi_\alpha(a_n)^{(\pm)}| \,,$$

$$\omega(a_n)^{(\pm)} = \sum_\alpha |\psi_\alpha(a_n)^{(\pm)})(\chi_\alpha| \,, \tag{15}$$

where $|\psi_\alpha) = |\psi_\alpha(a_N))$ and $|\chi_\alpha) = |\psi_\alpha(a_1))$ are the interacting and plane wave n-nucleon states, respectively. Then in obvious notation we define the cluster decomposition

$$i(2\pi)^3\delta^{(3)}(\vec{p}_f + \vec{Q} - \vec{p}_1)_{\text{out}}\langle \mathcal{N}_1 \ldots \mathcal{N}_N \,|\, j_\pi(0) \,|\, \mathcal{N}_{N+1} \ldots \mathcal{N}_{2N} \rangle_{\text{in}}$$

$$= \sum_{N=1}^{N} \sum_{an} \sum_{bn} (\psi_\beta(b_n)^{(-)} \,|\, \mathcal{J}(b_n, a_n) \,|\, \psi_\alpha(a_n)^{(+)}) \,. \tag{16}$$

$\mathcal{J}(b_n, a_n)$ is the n-connected pion transition operator which transforms the members of the n-nucleon cluster in a_n into the members of the n-nucleon cluster in b_n. By n connected we mean that the matrix element of the operator between interacting n-nucleon states possesses no disconnected nucleon lines. Finally, we associate the matrix elements of $T_N^{(+)}$ and $T_N^{(-)}$ with the prior and post forms of the elastic N-nucleon scattering matrices, respectively, e.g.,

$$-(2\pi)^3\delta^{(3)}(\vec{p}_f - \vec{p}_f{}')_{\text{out}}\langle \mathcal{N}_1' \ldots \mathcal{N}_N' \,|\, T_N^{(+)}(p_N, \ldots, p_1) \,|\, 0 \rangle = (\psi_{\beta'}^{(-)} \,|\, U \,|\, \chi_\beta) \,, \tag{17}$$

where U is the full N-nucleon potential.

With the above definitions, the N-nucleon approximation to Eq. (12) is

$$\begin{aligned}
(\psi_\beta^{(-)} \,|\, \mathcal{J}(b_N, a_N) \,|\, \psi_\alpha^{(+)})
&= \mathcal{T}(\text{DIS}) + \sum_{n=1}^{N} \sum_{a_n} \sum_{b_n} [\,(\chi_\beta \,|\, UG(E_\beta^+)\Omega(b_n)^{(-)}\mathcal{J}(b_n, a_n) \,|\, \psi_\alpha(a_n)^{(+)}) \\
&\qquad + (\psi_\beta(b_n)^{(-)} \,|\, \mathcal{J}(b_n, a_n)\Omega(a_n)^{(+)\dagger} G(E_\alpha^+) U \,|\, \chi_\alpha) \\
&\qquad - (\chi_\beta \,|\, UG(E_\beta^+)\Omega(b_n)^{(-)}\mathcal{J}(b_n, a_n)\Omega(a_n)^{(+)\dagger} G(E_\alpha^+) U \,|\, \chi_\alpha)] \,,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&= \mathcal{T}(\text{DIS}) + \sum_{n=1}^{N} \sum_{a_n} \sum_{b_n} (\chi_\beta \,|\, [\,\omega(b_n)^{(-)}\mathcal{J}(b_n, a_n)\omega(a_n)^{(+)\dagger} \\
&\qquad - \Omega(b_n)^{(-)}\mathcal{J}(b_n, a_n)\Omega(a_n)^{(+)\dagger}] \,|\, \chi_\alpha) \,,
\end{aligned} \tag{19}$$

where $G(E^+) = (E - H + i\epsilon)^{-1}$ is the fully interacting N-nucleon Green's function. The second equality is easily obtained by using

$$\Omega(a_n)^{(\pm)\dagger} G(E_\alpha^+) U \,|\, \chi_\alpha) = [\omega(a_n)^{(\pm)} - \Omega(a_n)^{(\pm)\dagger}] \,|\, \chi_\alpha)$$

and the definition of $\omega(a_n)^{(\pm)}$.

The content of Eq. (19) is clarified by two observations. First, since the $n = N$ contribution of the sum is

$$(\psi_\beta^{(-)}\,|\,\mathscr{J}(b_N,a_N)\,|\,\psi_\alpha^{(+)})-(\chi_\beta\,|\,\mathscr{J}(b_N,a_N)\,|\,\chi_\alpha)$$

[i.e., $\Omega(a_N)^{(\pm)}=1$ and $\omega(a_N)^{(\pm)}=\Omega(a_1)^{(\pm)}$], Eq. (19) reduces to an equation for the plane wave matrix elements of $\mathscr{J}(b_N,a_N)$. Second, we note that the $\mathscr{J}(b_{N-1},a_{N-1})$ contribution to $\mathscr{T}$(DIS) consists of one spectator nucleon and a connected piece which is identical to the right-hand side of Eq. (12) (but with $N-1$ total nucleons) in the no-pion approximation. By definition, this is

$$\sum_{a_{N-1}}\sum_{b_{N-1}}[(\psi_\beta(b_{N-1})^{(-)}\,|\,\mathscr{J}(b_{N-1},a_{N-1})\,|\,\psi_\alpha(a_{N-1})^{(+)}-(\chi_\beta\,|\,\mathscr{J}'(b_{N-1},a_{N-1})\,|\,\chi_\alpha)]\ ,$$

where $\mathscr{J}'(b_{N-1},a_{N-1})$ represents the $(N-1)$-connected higher intermediate state contributions to the right-hand side of the $(N-1)$-nucleon version of Eq. (12) (e.g., one-pion contributions). That is, we do not necessarily truncate the $(N-1)$-nucleon equation in the same fashion as the N-nucleon equation. More intuitively, $\mathscr{J}'(b_{N-1},a_{N-1})$ is that part of the $(N-1)$-nucleon process which cannot be accounted for by embedding $(N-j)$-nucleon processes $(j>1)$ in the $(N-1)$-nucleon system. We stress that by definition, this operator is $(N-1)$-connected. The analysis of the contributions of the remaining $\mathscr{J}(b_n,a_n)$ to $\mathscr{T}$(DIS) is identical. Thus, from the above considerations, the operator version of Eq. (22) is

$$\mathscr{J}(b_N,a_N)=\sum_{n=1}^{N-1}\sum_{a_n}\sum_{b_n}[\mathscr{J}'(b_n,a_n)-\Omega(b_n)^{(-)}\mathscr{J}'(b_n,a_n)\Omega(a_n)^{(+)^\dagger}]+\mathscr{J}'(b_N,a_N)\ , \tag{20}$$

where we have added the N-nucleon $\mathscr{J}'$ to the right-hand side to account for processes not considered in the original approximation. We note that the inclusion of bound states modifies Eq. (20) only by the extension of the definition of $\Omega(a_n)$ to include all bound state possibilities.

This recursion relation gives a simple prescription for embedding the $(N-i)$-nucleon processes $(0<i<N-1)$ in the N-nucleon system such that there are no disconnected nucleon contributions. For example, when the $(N-1)$-nucleon transition operator contribution

$$\mathscr{J}'(b_{N-1},a_{N-1})-\Omega(b_{N-1})^{(-)}\mathscr{J}'(b_{N-1},a_{N-1})\Omega(a_{N-1})^{(+)^\dagger}$$

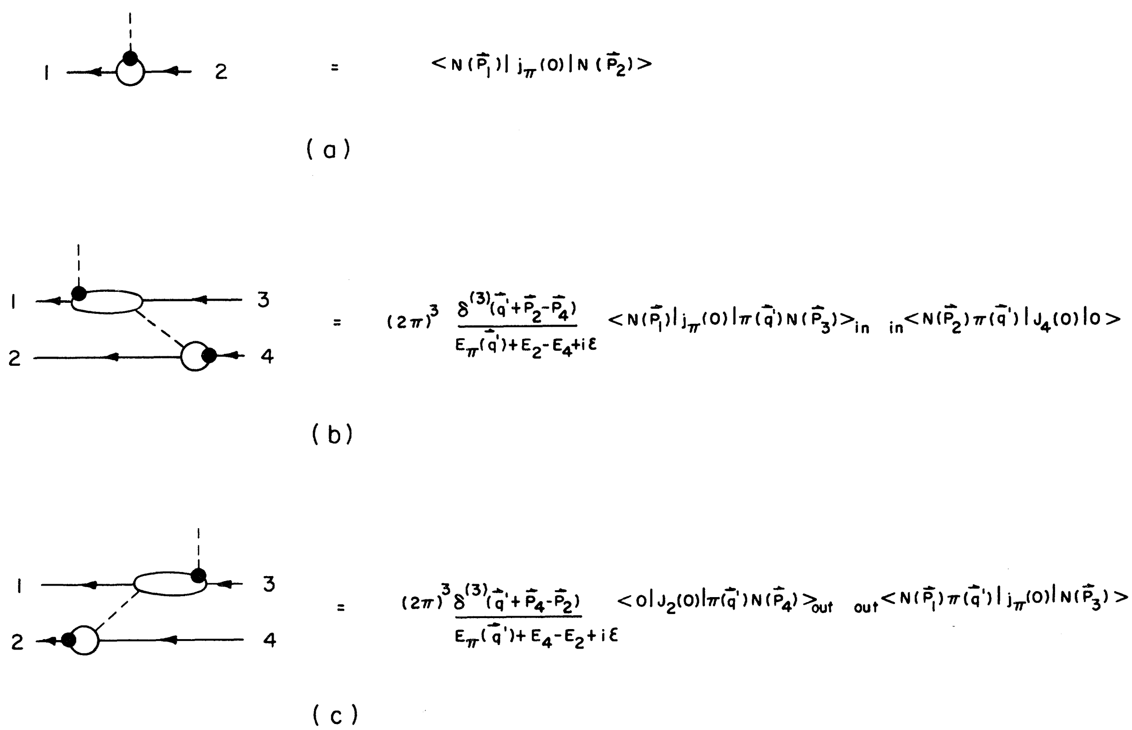

FIG. 3. (a) The one-body operator, (b) the backward (in time) rescattering graph, and (c) the forward rescattering graph. The pion rescattering graphs represent the leading contributions to the two-body operators.

is embedded in the interacting N-nucleon medium, the second term cancels the one-nucleon disconnected piece of the first. However, the real strength of the present approach is that the transition operators $\mathscr{J}'$ can be obtained explicitly, at least to leading order. The one-body operator is obtained trivially from the matrix element of j_π between one-nucleon states [see Fig. 3(a)]. The dominant contribution to the two-body operator follows from the one-pion contribution to the $N=2$ process and is easily determined to be the pion rescattering contribution.[13] The relevant diagrams are depicted in Figs. 3(b) and (c). We note that at this point we have recovered the canonical form of the two-nucleon mechanism in the distorted-wave Born approximation. However, there are some important differences between the present result and previous work. First, the diagrams in Figs. 3(b) and (c) are time-ordered diagrams rather than Feynman diagrams. Although the two types of diagrams have the same pion-pole singularity, calculationally they are quite different. For example, in the time-ordered diagram, the pion vertex is off-the-nucleon-mass-shell and thus it has both pseudoscalar and pseudovector pieces and its form factors have a resonance structure close to the physical region.[14] This is in marked contrast to the Feynman diagram which has a pure pseudoscalar vertex and a real form factor with a dipole type of behavior. Second, our results are independent of the choice of a model Lagrangian. And third, our formalism has no provision for, nor does it require, the subtraction of a pion exchange graph to prevent doublecounting. The exact relationship of doublecounting arguments based on perturbation theory[15] to the present work is unclear, but it is unequivocally true that no particular sequence of operator and theta-function products are doublecounted in our final expression.

The utility of the present work ultimately resides in providing improved fits to the data. Calculations for pion production by the two-nucleon system are presently under way and we shall report on the results in a future paper.

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APPENDIX

To derive Eq. (10), we first use Eqs. (5) and (8) to rewrite the $N+1$ term in Eq. (3) in the form

$$\sum_P (-1)^P T(J_1 \ldots J_N)\theta_+ j_\pi T(J_{N+1} \ldots J_{2N})\theta_-$$
$$= \sum_P (-1)^P \sum_{k=0}^{N-1} (-1)^k [\overline{T}(J_1 \ldots J_{l+1})\Theta_1 \ldots \Theta_{l+1}]$$
$$\times [T(J_{l+2} \ldots J_N)\theta_+ j_\pi T(J_{N+1} \ldots J_{2N-k-1})\theta_-]$$
$$\times [\overline{T}(J_{2N-k} \ldots J_{2N})\Theta_{-(2N-k)} \ldots \Theta_{-2N}] \,. \tag{1'}$$

Consider next the doublecounting piece of Eq. (7). Applying Eq. (8), this becomes

$$-\sum_P (-1)^P \sum_{k=0}^{N-2} \sum_{j=N+1}^{2N-k-1} \sum_{l=0}^{j-1} (-1)^{k+l} [\overline{T}(J_1 \ldots J_{l+1})\Theta_1 \ldots \Theta_{l+1}]$$
$$\times [T(J_{l+2} \ldots J_j)\theta_+ j_\pi T(J_j \ldots J_{2N-k-1})\theta_-][\overline{T}(J_{2N-k} \ldots J_{2N})\Theta_{-(2N-k)} \ldots \Theta_{-2N}] \tag{2'}$$

[notice that the final term in Eq. (8) may be discarded here since $j \geq N+1$]. Separating the l sum into two pieces so that the j and l sums may be interchanged, i.e.,

$$\sum_{k=0}^{N-2} \sum_{j=N+1}^{2N-k-1} \left[\sum_{l=0}^{N-1} + \sum_{l=N}^{j-1} \right] = \sum_{k=0}^{N-2} \sum_{l=0}^{N-1} \sum_{j=N+1}^{2N-k-1} + \sum_{k=0}^{N-2} \sum_{l=N}^{2N-k-2} \sum_{j=l+1}^{2N-k-1} \tag{3'}$$

we see that the second l sum is, by definition,

$$-\sum_P(-1)^P\sum_{k=0}^{N-2}\sum_{l=N}^{2N-k-2}(-1)^{k+l}[\bar{T}(J_1\ldots J_{l+1})\Theta_1\ldots\Theta_{l+1}][T(J_{l+2}\ldots j_\pi\ldots J_{2N-k-1})]$$
$$\times[\bar{T}(J_{2N-k}\ldots J_{2N})\Theta_{-(2N-k)}\ldots\Theta_{-2N}]\ . \tag{4'}$$

In like fashion, using Eq. (5), the doublecounting piece of Eq. (9) may be written in the form

$$-\sum_P(-1)^P\sum_{l=0}^{N-2}\sum_{k=N}^{2N-l-2}(-1)^{k+l}[\bar{T}(J_1\ldots J_{l+1})\Theta_1\ldots\Theta_{l+1}][T(J_{l+2}\ldots j_\pi\ldots J_{2N-k-1})]$$
$$\times[\bar{T}(J_{2N-k}\ldots J_{2N})\Theta_{-(2N-k)}\ldots\Theta_{-2N}]$$
$$-\sum_P(-1)^P\sum_{k=0}^{N-1}\sum_{l=0}^{N-2}\sum_{j=l+1}^{N-1}(-1)^{k+l}[\bar{T}(J_1\ldots J_{l+1})\Theta_1\ldots\Theta_{l+1}]$$
$$\times[T(J_{l+2}\ldots J_j)\theta_+j_\pi T(J_{j+1}\ldots J_{2N-k-1})\theta_-]$$
$$\times[\bar{T}(J_{2N-k}\ldots J_{2N})\Theta_{-(2N-k)}\ldots\Theta_{-2N}]\ . \tag{5'}$$

It is now an elementary exercise to combine the first l sum in Eq. (2 ′) with Eq. (1 ′) and the second piece of Eq. (5 ′) to obtain

$$-\sum_P(-1)^P\sum_{l=0}^{N-1}\sum_{k=0}^{N-1}(-1)^{k+l}[\bar{T}(J_1\ldots J_{l+1})\Theta_1\ldots\Theta_{l+1}][T(J_{l+2}\ldots j_\pi\ldots J_{2N-k-1})]$$
$$\times[\bar{T}(J_{2N-k}\ldots J_{2N})\Theta_{-(2N-k)}\ldots\Theta_{-2N}]\ . \tag{6'}$$

Adding this result to Eq. (4 ′) and the first term in Eq. (5 ′) gives Eq. (10) in the main text.

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[1]Roger H. Hackman, Phys. Rev. C 10, 1893 (1979).

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[3]M. K. Banerjee, C. A. Levinson, M. D. Shuster, and D. A. Zollman, Phys. Rev. C 3, 509 (1971); 13, 2444 (1976).

[4]In our final result, we neglect terms whose lowest-order nonvanishing contribution involves intermediate states with a least $2N-2$ total antinucleons. However there is no difficulty, in principle, to including these terms.

[5]This does not mean we neglect dibaryon effects. Dibaryon resonances, if they exist, would be generated dynamically by the appropriate Low equations.

[6]By initial (final) interaction, we mean an interaction among the incident (outgoing) nucleons which occurs before (after) the interaction involving the external pion.

[7]Here, $\bar{T}$ denotes the antitime-ordered product as defined, for example, in J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), p. 272.

[8]This includes interchanging the $J_i, i\leq N$ and $J_j, j\geq N+1$. However, a rearrangement of the operators within a given T or $\bar{T}$ product is not to be regarded as a distinct permutation.

[9]This term may easily be incorporated in our final result, Eq. (12). However, its energy denominator will be of the order of $[2Nm_N]^2$.

[10]We adopt the convention that whenever the upper or lower limit on a sum exceeds a permissible value, the "relevant" pieces of the T product are to be replaced by unity; e.g., for $k=2N-1$, $T(J_1\ldots j_\pi\ldots J_{2N-k-1})\to T(j_\pi)=j_\pi$.

[11]If the initial bubble has no incident nucleon lines, then all Θ_{-i} are unity. Consequently, the time integrations associated with the $\bar{T}$ product [see Eq. (2)] are constrained only relative to one another and this term (after the insertion of intermediate states) must be proportional to an overall energy delta function of the form $\delta(E_n+E_i+\ldots+E_j)$, $i,\ldots,j\leq N$, which vanishes. A similar argument holds for the cross term and the doublecounting term.

[12]Thus, we regard $\{a_1\}$ as having N distinct members, each with a different nucleon as the member of the "interacting" one-nucleon cluster. This definition aids in the cluster decomposition, Eq. (16).

[13]This will be discussed more thoroughly in a paper which is currently in preparation.

[14]T. Mizutani and P. Rochus, Phys. Rev. C 19, 958 (1979).

[15]By this we mean approaches which analyze the dynamics in terms of admissable classes of Feynman diagrams. Such approaches of course, are not limited to perturbation theory. A recent example is provided by K. L. Kowalski, E. R. Siciliano, and R. M. Thaler, Phys. Rev. C 19, 1843 (1979); see also Kerson Huang and A. Arthur Weldon, Phys. Rev. D 11, 257 (1975).