PHYSICAL REVIEW C VOLUME 25, NUMBER 4 APRIL 1982

Unitary pole approximations and expansions in few-body systems

A. Casel, H. Haberzettl, and W. Sandhas
Physikalisches Institut der Universität Bonn, D-5300 Bonn 1, West Germany
(Received 10 August 1981)

The unitary pole approximations or expansions of the two-body subsystem operators are well known, and particularly efficient and practical, methods to reduce the three-body problem to an effective two-body theory. In the present investigation we develop generalizations of these approximation techniques to the subsystem amplitudes of problems with higher particle numbers. They are based on the expansion of effective potentials which, in contrast to the genuine two-body interactions, are now energy dependent. Despite this feature our generalizations require only energy independent form factors, thus preserving one of the essential advantages of the genuine two-body approach. The application of these techniques to the four-body case is discussed in detail.

[NUCLEAR REACTIONS Generalizations of unitary pole approximation and expansion. Application to four-body problem.]

I. INTRODUCTION

Practical approaches to solve the N-body problem ($N \geq 3$) are mostly based on methods which allow us to describe the original problem by an effective two-body theory. This can be achieved by successively introducing separable (pole) expansions for the various subsystem amplitudes contained in the kernels of N-body integral equations.[1] For this purpose efficient and technically manageable expansion or approximation methods are required, not only for the genuine two-body transition operators but also for the *effective two-body amplitudes,* which replace the subsystem amplitudes for higher particle numbers in this reduction procedure. It is the aim of the following investigations to develop such methods by extending well-known two-body approaches.

The simplest, and also best-known, application of the successive reduction scheme mentioned above concerns the *three-body* equations: They are reduced to effective two-body equations in one single step by employing separable expansions for the genuine two-body subsystem transition operators in their kernels.[2] A particularly important role in this context is played by the unitary pole expansion (UPE),[3] because it is comparatively easy to handle in numerical investigations, and moreover, in many practical three-body calculations its leading term, the unitary pole approximation (UPA) has proven to yield rather accurate results already, despite its simplicity.[4] As compared to the leading terms of other approaches (e.g., the Hilbert-Schmidt expansion), the UPA represents a physically motivated cluster picture which could serve as a guideline to develop models for general composite particle collisions.

These and other advantages, which will be discussed below, therefore make it desirable to develop a generalization of the two-body UPE, and hence also of the UPA, to problems with higher particle numbers. The first instance of application of such generalizations, and our principal reason for undertaking this investigation, is the treatment of the three-body input in four-body equations.

The pole expansions for the genuine two-body amplitude are usually obtained from separable expansions of the potential via the two-body Lippmann-Schwinger (LS) equations. Since the other subsystem amplitudes of the N-body problem, after applying the reduction procedure described in the beginning, are also determined by (effective) two-body equations of LS structure, generalizations of two-body expansion methods may be based on analogous expansions of the corresponding *effective potentials.* These potentials, however, are now *energy dependent.* No problems arise from this fact when generalizing the Hilbert-Schmidt method

since it works thoughout with energy dependent Sturmian functions and expansion coefficients. For the UPE or UPA, which uses *energy independent* expansion functions, that is not so; its usual derivation in the genuine two-body case hinges essentially on the energy independence of the potentials.[3] Nevertheless, in what follows we will show that a natural generalization of these methods can be found by taking into account the energy dependence of the potential via energy dependent coupling constants (in the UPA) or expansion coefficients (in the UPE), but keeping the form factors energy independent. With the latter feature, one of the main advantages of the usual two-body UPA or UPE, which makes them so easy to handle in numerical investigations, is preserved.

The paper is organized as follows: In view of the effective two-body structure achieved in each step of the reduction scheme and also in order to make our investigation independent of the number of particles, we consider in Sec. II a model two-body problem which, in contrast to the genuine two-body case, has an energy dependent potential. (In order to accommodate all features of effective two-body equations we also have to allow for a modified free Green's function, but this is of lesser importance for our investigations.) Assuming the existence of one bound state in our model, we derive the corresponding pole behavior and show that the energy dependence of the potential does indeed lead to a modification of the well-known usual two-body result. This modification implies that the conventional UPA procedure is not applicable in this case, and in Sec. III we therefore discuss various possibilities of extending it. Our final choice of a generalized UPA is obtained by requiring that the full energy dependent effective interaction be identical to its UPA part in the subspace spanned by the bound state wave function. Starting from the Hilbert-Schmidt method, we then systematically derive in Sec. IV an expansion which uses only energy independent expansion functions throughout and hence is the desired generalization of the UPE. Its leading term is indeed seen to be the UPA, which was found in Sec. III. As an example we furthermore apply our expansion to the three-body input of four-body equations in Sec. V.

II. POLE BEHAVIOR OF TRANSITION AMPLITUDES

As explained in the Introduction, we are interested in the pole behavior of effective two-body transition amplitudes describing composite particle collisions. To this end, we study first a model two-body scattering problem characterized by a transition operator $\mathscr{T}(z)$ which satisfies the Lippmann-Schwinger (LS) equation

$$\mathscr{T}(z)=\mathscr{V}(z)+\mathscr{V}(z)\mathscr{G}_0(z)\mathscr{T}(z)\ . \tag{2.1}$$

Here, in contrast to what is conventionally assumed in the genuine two-body case, the potential $\mathscr{V}(z)$ may depend on the energy parameter z,

$$\frac{d\mathscr{V}(z)}{dz}\neq 0\ , \tag{2.2}$$

and for the free Green's function $\mathscr{G}_0(z)$, we allow

$$\frac{d\mathscr{G}_0{}^{-1}(z)}{dz}\neq 1\ . \tag{2.3}$$

Moreover, we require $\mathscr{V}(z)$ and $\mathscr{G}_0(z)$ to be analytic operators in z with cuts only on the real axis above some energy E_c and that their adjoints are given by replacing z by z^*.

As we shall see, this model already contains all essential features of the composite particle problem considered later on (Sec. V). What remains to be done is to give the resulting expressions a matrix interpretation, thus incorporating the fact that we deal with a coupled set of effective two-body equations.

To extract the pole behavior of $\mathscr{T}(z)$, let us proceed in complete analogy to the genuine two-body theory, i.e., let us write the free Green's function in the form

$$\mathscr{G}_0(z)=[z-\mathscr{H}_0(z)]^{-1}\ , \tag{2.4}$$

where now, however, the "free Hamiltonian" $\mathscr{H}_0(z)$ has to be z dependent according to the property (2.3). Introducing in addition a "total Green's function" via

$$\begin{aligned}\mathscr{G}(z)&=[\mathscr{G}_0{}^{-1}(z)-\mathscr{V}(z)]^{-1}=[z-\mathscr{H}_0(z)-\mathscr{V}(z)]^{-1}\\&=[z-\mathscr{H}(z)]^{-1}\ ,\end{aligned} \tag{2.5}$$

we obtain in the usual way the representation

$$\mathscr{T}(z)=\mathscr{V}(z)+\mathscr{V}(z)\mathscr{G}(z)\mathscr{V}(z) \tag{2.6}$$

for the solution of Eq. (2.1).

For simplicity we assume that in our model only one bound state $|\Psi_B\rangle$ of energy $B<E_C$ exists, given by the eigenvalue equation

$$[B-\mathscr{H}_0(B)-\mathscr{V}(B)]\,|\,\Psi_B\rangle=0\ . \tag{2.7}$$

In order to study the pole of $\mathscr{G}(z)$, corresponding to this bound state, we introduce a further resolvent

$$\mathscr{G}_B(z)=[z-\mathscr{H}_0(B)-\mathscr{V}(B)]^{-1}\ , \tag{2.8}$$

which is related to $\mathscr{G}(z)$ by means of a resolvent equation

$$\mathscr{G}(z)=\mathscr{G}_B(z)+\mathscr{G}_B(z)[\mathscr{H}_0(z)+\mathscr{V}(z)-\mathscr{H}_0(B)-\mathscr{V}(B)]\mathscr{G}(z)\ . \tag{2.9}$$

From (2.7) and (2.8) we infer, as in the genuine two-body case, the pole behavior of (2.8) in the neighborhood of $z=B$, viz.,

$$\mathscr{G}_B(z\sim B)\sim\frac{|\,\Psi_B\rangle}{||\Psi_B||}\,\frac{1}{z-B}\,\frac{\langle\Psi_B\,|}{||\Psi_B||}\ . \tag{2.10}$$

It should be mentioned that $|\,\Psi_B\rangle$ has not been normalized to unity, since other normalization conditions will be used in the following, and consequently $||\Psi_B||^2$ has to occur in the denominator of (2.10). Inserting now this representation in (2.9), we obtain

$$\mathscr{G}(z\sim B)\sim|\,\Psi_B\rangle\frac{1}{(z-B)\langle\Psi_B\,|\left[1-\dfrac{d\mathscr{H}_0(B)}{dB}-\dfrac{d\mathscr{V}(B)}{dB}\right]|\,\Psi_B\rangle}\langle\Psi_B\,|\ . \tag{2.11}$$

As compared to (2.10), i.e., to the pole behavior known from the genuine two-body problem, the residue of this pole is modified by the additional terms in the denominator originating from a nonvanishing derivative of $\mathscr{H}_0(z)+\mathscr{V}(z)$ at $z=B$.

Rewriting this denominator with the help of (2.4), and bearing in mind the representation (2.6), we finally obtain for the pole behavior of the solution of (2.1)

$$\mathscr{T}(z\sim B)\sim\mathscr{V}(B)\,|\,\Psi_B\rangle\frac{1}{(z-B)\langle\Psi_B\,|\left[\dfrac{d\mathscr{G}_0{}^{-1}(B)}{dB}-\dfrac{d\mathscr{V}(B)}{dB}\right]|\,\Psi_B\rangle}\langle\Psi_B\,|\,\mathscr{V}(B)\ . \tag{2.12}$$

As in the genuine two-body case it is convenient to introduce "form factors"

$$|\,\Gamma\rangle=\mathscr{V}(B)\,|\,\Psi_B\rangle\ . \tag{2.13}$$

Equation (2.7) then guarantees also in our model the validity of the conventional relation

$$\mathscr{G}_0(B)\,|\,\Gamma\rangle=\mathscr{G}_0(B)\mathscr{V}(B)\,|\,\Psi_B\rangle=|\,\Psi_B\rangle\ , \tag{2.14}$$

and moreover, taking into account that, from $\mathscr{G}_0\mathscr{G}_0{}^{-1}=1$, we have

$$d\mathscr{G}_0(B)/dB=-\mathscr{G}_0(B)[d\mathscr{G}_0{}^{-1}(B)/dB]\mathscr{G}_0(B)\ , \tag{2.15}$$

we finally end up with the pole representation

$$\mathscr{T}(z\sim B)\sim|\,\Gamma\rangle\frac{-1}{(z-B)\langle\Gamma\,|\left[\dfrac{d\mathscr{G}_0(B)}{dB}+\mathscr{G}_0(B)\dfrac{d\mathscr{V}(B)}{dB}\mathscr{G}_0(B)\right]|\,\Gamma\rangle}\langle\Gamma\,|\ . \tag{2.16}$$

Let us compare this result again with the genuine two-body case. There the z independence of $\mathscr{V}$ and the fact that $d\mathscr{G}_0(z)/dz$ equals $-\mathscr{G}_0{}^{-2}(z)$ means that the scalar product in the denominator of (2.16) goes over into $||\Psi_B||^2$. The properties (2.2) and (2.3) lead to a modification of this factor, i.e., the *residue* of the pole is altered. Of course, only terms *linear* in $(z-B)$ in the denominator contribute to the residue, and consequently only *first* derivatives of $\mathscr{H}_0(z)$ and $\mathscr{V}(z)$ had to be expected, and indeed appear, in these modifications.

III. POLE APPROXIMATIONS

The unitary pole approximation (UPA) of two-body transition operators in three-body Faddeev-type equations is conventionally based on replacing the original interaction by an appropriately chosen separable potential. Despite the simplicity of the resulting expressions, this procedure has proven to be very successful in many practical three-body calculations.[4] As a first attempt to employ this concept also for our model two-body problem, we may replace analogously the z-dependent potential in Eq. (2.1) by

$$\mathscr{V}^s = -|\Gamma\rangle\lambda\langle\Gamma| \ , \tag{3.1}$$

with the form factor $|\Gamma\rangle$ given by Eq. (2.13). The corresponding solution of Eq. (2.1) is then obtained as

$$\mathscr{T}^s(z) = |\Gamma\rangle \frac{-1}{\lambda^{-1} + \langle\Gamma|\mathscr{G}_0(z)|\Gamma\rangle} \langle\Gamma| \ , \tag{3.2}$$

which of course has the form well known from the conventional two-body approach [cf. Eqs. (5.4) and (5.6)].

If the coupling parameter λ is chosen according to

$$\lambda^{-1} = -\langle\Gamma|\mathscr{G}_0(B)|\Gamma\rangle \ , \tag{3.3}$$

$\mathscr{T}^s(z)$ gets a pole at $z=B$, i.e., at the *same position* as the original transition operator $\mathscr{T}(z)$. The corresponding *residue* of $\mathscr{T}(z)$, however, is not reproduced by $\mathscr{T}^s(z)$. Indeed, expanding $\mathscr{G}_0(z)$ around $z=B$, we find

$$\mathscr{T}^s(z\sim B) \sim |\Gamma\rangle \frac{-1}{(z-B)\langle\Gamma|\dfrac{d\mathscr{G}_0(B)}{dB}|\Gamma\rangle} \langle\Gamma| \ , \tag{3.4}$$

which differs from (2.16) for potentials $\mathscr{V}(z)$ with a nonvanishing derivative at $z=B$. In other words, the separable potential (3.1) with the energy independent choice (3.3) fails to provide the desired pole behavior (2.16), a shortcoming which is not surprising since nothing of the original z dependence of $\mathscr{V}(z)$ has been built into this ansatz.

In order to take into account this dependence, we may replace in (3.1) the form factors or the coupling parameter by energy dependent expressions. In view of the fact that it is one of the major advantages of the usual UPA to work with energy independent form factors, we prefer the latter possibility. That is, we replace the constant λ in (3.1) by a function $\Lambda(z)$,

$$\mathscr{V}^s(z) = -|\Gamma\rangle\Lambda(z)\langle\Gamma| \ , \tag{3.5}$$

and consequently (3.2) goes over into

$$\mathscr{T}^s(z) = |\Gamma\rangle \frac{-1}{\Lambda^{-1}(z) + \langle\Gamma|\mathscr{G}_0(z)|\Gamma\rangle} \langle\Gamma| \ . \tag{3.6}$$

The pole behavior (2.16) is now most easily achieved by choosing $\Lambda(z)$ according to

$$\Lambda^{-1}(z) = \lambda^{-1} + (z-B)\langle\Gamma|\mathscr{G}_0(B)\frac{d\mathscr{V}(B)}{dB}\mathscr{G}_0(B)|\Gamma\rangle \ , \tag{3.7}$$

with λ^{-1} still given by (3.3). The corresponding transition operator then reads

$$\mathscr{T}^s(z) = |\Gamma\rangle \frac{-1}{\lambda^{-1} + \langle\Gamma|\mathscr{G}_0(z)|\Gamma\rangle + (z-B)\langle\Gamma|\mathscr{G}_0(B)\dfrac{d\mathscr{V}(B)}{dB}\mathscr{G}_0(B)|\Gamma\rangle} \langle\Gamma| \ . \tag{3.8}$$

This choice evidently represents the *minimal* extension of the conventional UPA because it corresponds to simply adding the missing term [cf. Eqs. (3.4) and (2.16)] in the denominator of Eq. (3.2).

One of the decisive features of the UPA in the genuine two-particle case is not only that it exactly reproduces the original transition operator at the pole but also that it represents a reasonable approximation in its neighborhood.[4] This is due to the fact that the z dependence of the problem originates in this case only from the free Green's function, which is fully taken into account when solving the LS equation for a separable approximation of the potential. In the present model, with its more complicated z dependence of $\mathscr{G}_0(z)$ [cf. Eq. (2.3)], the same argumentation would hold for (3.2), if $\mathscr{V}$ were z independent. Indeed, the denominator of (3.2), with λ given by (3.3), is determined by the whole Taylor series,

$$\mathscr{G}_0(z)-\mathscr{G}_0(B)=\sum_{n=1}^{\infty}\frac{(z-B)^n}{n!}\frac{d^n\mathscr{G}_0(B)}{dB^n}\ , \tag{3.9}$$

whereas its contribution to the residue stems only from the first term linear in $(z-B)$, according to Eq. (3.4). In the same sense we expect that taking into account only the first term of the expansion

$$\mathscr{V}(z)-\mathscr{V}(B)=\sum_{n=1}^{\infty}\frac{(z-B)^n}{n!}\frac{d^n\mathscr{V}(B)}{dB^n}\ , \tag{3.10}$$

as done in the minimal extension (3.8), is too restricted. This suggests that we use instead of (3.7) the choice

$$\Lambda^{-1}(z)=\lambda^{-1}+\langle\Gamma\,|\,\mathscr{G}_0(B)[\mathscr{V}(z)-\mathscr{V}(B)]\mathscr{G}_0(B)\,|\,\Gamma\rangle\ , \tag{3.11}$$

where $(z-B)[d\mathscr{V}(B)/dB]$ is replaced by the whole series (3.10). Consequently, as a possible generalization of (3.2), which contains more of the analytic behavior of $\mathscr{V}(z)$, we have

$$\mathscr{T}^s(z)=|\,\Gamma\rangle\frac{-1}{\lambda^{-1}+\langle\Gamma\,|\,\mathscr{G}_0(z)\,|\,\Gamma\rangle+\langle\Gamma\,|\,\mathscr{G}_0(B)[\mathscr{V}(z)-\mathscr{V}(B)]\mathscr{G}_0(B)\,|\,\Gamma\rangle}\langle\Gamma\,|\ . \tag{3.12}$$

So far our choices of $\Lambda(z)$ have been suggested by directly comparing the analytic behavior of the amplitudes $\mathscr{T}^s(z)$ of (3.6) and $\mathscr{T}(z)$. Alternatively, we may fix $\mathscr{V}^s(z)$ [and hence, $\Lambda(z)$] of Eq. (3.5) from the very beginning by adjusting it to the original $\mathscr{V}(z)$ in an appropriate way. We do this by requiring that $\mathscr{V}^s(z)$ be identical to $\mathscr{V}(z)$ in the subspace spanned by $\mathscr{G}_0(B)\,|\,\Gamma\rangle=|\,\Psi_B\,\rangle$, i.e.,

$$\langle\Gamma\,|\,\mathscr{G}_0(B)\mathscr{V}(z)\mathscr{G}_0(B)\,|\,\Gamma\rangle=\langle\Gamma\,|\,\mathscr{G}_0(B)\mathscr{V}^s(z)\mathscr{G}_0(B)\,|\,\Gamma\rangle\ . \tag{3.13}$$

This concept provides the choice

$$\Lambda(z)=-\frac{\langle\Gamma\,|\,\mathscr{G}_0(B)\mathscr{V}(z)\mathscr{G}_0(B)\,|\,\Gamma\rangle}{\langle\Gamma\,|\,\mathscr{G}_0(B)\,|\,\Gamma\rangle\langle\Gamma\,|\,\mathscr{G}_0(B)\,|\,\Gamma\rangle}\ , \tag{3.14}$$

which agrees with Eq. (3.3) at $z=B$. The latter fact implies that this $\Lambda(z)$ ensures the correct pole position for $\mathscr{T}^s(z)$, and, expanding $\Lambda(z)$ around $z=B$, we furthermore see that the correct residue (2.16) is also reproduced. [It should be clear that in the genuine two-body case this concept does not lead to a different choice of coupling parameters; there we have $\mathscr{V}(z)\equiv\mathscr{V}(B)$ and hence, $\Lambda(z)$ of Eq. (3.14) becomes energy independent and reduces to the choice (3.3) familiar from the two-body case.[5]]

Since this last approach not only guarantees the correct pole behavior but also takes into account correctly *all* of the contribution of $\mathscr{V}(z)$ in the relevant bound state subspace, we consider (3.6) with the choice (3.14), i.e.,

$$\mathscr{T}^{\mathrm{UPA}}(z)=|\,\Gamma\rangle\frac{1}{\dfrac{\langle\Gamma\,|\,\mathscr{G}_0(B)\,|\,\Gamma\rangle\langle\mathscr{G}_0(B)\,|\,\Gamma\rangle}{\langle\Gamma\,|\,\mathscr{G}_0(B)\mathscr{V}(z)\mathscr{G}_0(B)\,|\,\Gamma\rangle}-\langle\Gamma\,|\,\mathscr{G}_0(z)\,|\,\Gamma\rangle}\langle\Gamma\,|\ , \tag{3.15}$$

to be the natural generalization of the conventional UPA. This is corroborated by the fact that $\mathscr{T}^{\mathrm{UPA}}(z)$ is also obtainable from a generalization of the two-body unitary pole expansion (UPE) to the case of effective two-body equations, as will be shown in the next section.

It should be noted that all of the above considerations are valid independently of the normalization of $|\,\Gamma\rangle$. This is *a priori* clear, since in ansatz (3.1) any change in the normalization of $|\,\Gamma\rangle$ is absorbed in the coupling parameter λ. Moreover, taking into account defintion (3.3), Eqs. (3.4), (3.12), and (3.15) show explicitly that the approximations of $\mathscr{T}(z)$ do not depend on this normalization. Hence, without lack of generality, the normalization of $|\,\Gamma\rangle$ may be fixed according to

$$\lambda^{-1}=-\langle\Gamma\,|\,\mathscr{G}_0(B)\,|\,\Gamma\rangle=1\ . \tag{3.16}$$

This is the choice used in the following section.

IV. HILBERT-SCHMIDT METHOD AND GENERALIZED UNITARY POLE EXPANSION

In the genuine two-body problem an appropriate tool for solving the LS equation and studying properties of its solution is given by the Hilbert-

Schmidt method.[6] Moreover, this approach leads to the unitary pole expansion (UPE),[3,4] which is distinguished by its simplicity and efficiency in three-body calculations. Despite the fact that the application of the Hilbert-Schmidt method to the case of energy-dependent potentials is well known, it shall be repeated here in a way which will allow us to immediately also extend the UPE to this case.

The Hilbert-Schmidt method is based on the definition of Sturmian function $|\Gamma_n(E)\rangle$ for $E \leq E_c$ by the eigenvalue equation

$$\mathscr{V}(E)\mathscr{G}_0(E)|\Gamma_n(E)\rangle=\eta_n(E)|\Gamma_n(E)\rangle\ , \tag{4.1}$$

which may be cast into the form

$$\{[-\mathscr{G}_0(E)]^{1/2}\mathscr{V}(E)[-\mathscr{G}_0(E)]^{1/2}\}[-\mathscr{G}_0(E)]^{1/2}|\Gamma_n(E)\rangle=-\eta_n(E)[-\mathscr{G}_0(E)]^{1/2}|\Gamma_n(E)\rangle\ . \tag{4.2}$$

The properties of $\mathscr{V}(z)$ and $\mathscr{G}_0(z)$ proposed in Sec. II imply that $[-\mathscr{G}_0(E)]^{1/2}\mathscr{V}(E)\ [-\mathscr{G}_0(E)]^{1/2}$ is self-adjoint in this energy region. As a further condition we assume also that the Schmidt-norm of this operator be finite for $E<E_C$. The eigenstates $[-\mathscr{G}_0(E)]^{1/2}|\Gamma_n(E)\rangle$ in (4.2) form a complete orthogonal set which, when normalized according to

$$\langle\Gamma_n(E)|\mathscr{G}_0(E)|\Gamma_m(E)\rangle=-\delta_{nm}\ , \tag{4.3}$$

provide a decomposition of the identity

$$1=\sum_n[-\mathscr{G}_0(E)]^{1/2}|\Gamma_n(E)\rangle\langle\Gamma_n(E)|[-\mathscr{G}_0(E)]^{1/2}\ , \tag{4.4}$$

from which we obtain the more convenient representations

$$1=-\sum_n\mathscr{G}_0(E)|\Gamma_n(E)\rangle\langle\Gamma_n(E)|\ , \tag{4.5}$$

$$1=-\sum_n|\Gamma_n(E)\rangle\langle\Gamma_n(E)|\mathscr{G}_0(E)\ . \tag{4.6}$$

Multiplying $\mathscr{V}(E)$ from the right by (4.5) and from the left by (4.6), we find

$$\mathscr{V}(E)=\sum_{m,n}|\Gamma_m(E)\rangle\langle\Gamma_m(E)|\mathscr{G}_0(E)\mathscr{V}(E)\mathscr{G}_0(E)|\Gamma_n(E)\rangle\langle\Gamma_n(E)|\ , \tag{4.7}$$

which because of Eqs. (4.1) and (4.3) reduces to

$$\mathscr{V}(E)=-\sum_n|\Gamma_n(E)\rangle\eta_n(E)\langle\Gamma_n(E)|\ , \tag{4.8}$$

with

$$\eta_n(E)=-\langle\Gamma_n(E)|\mathscr{G}_0(E)\mathscr{V}(E)\mathscr{G}_0(E)|\Gamma_n(E)\rangle\ . \tag{4.9}$$

Consequently, after inserting this expansion into (2.1), we obtain

$$\mathscr{T}(E)=\sum_n|\Gamma_n(E)\rangle\frac{-\eta_n(E)}{1-\eta_n(E)}\langle\Gamma_n(E)|\ . \tag{4.10}$$

This, of course, is the well-known generalization of the two-body Hilbert-Schmidt expansion.

It is illustrataive to see how the pole behavior, derived in Sec. II, Eq. (2.16), is verified by this expansion. According to Eq. (2.7), one of the eigenvalues in (4.1), say $\eta_1(E)$, will then become unity,

$$\eta_1(B)=1\ , \tag{4.11}$$

and consequently the first of the eigenvalue equations (4.1) reduces to the homogeneous LS equation

$$\mathscr{V}(B)\mathscr{G}_0(B)|\Gamma_1(B)\rangle=|\Gamma_1(B)\rangle\ , \tag{4.12}$$

where, because of Eqs. (2.13) and (2.14), we have, of course,

$$|\Gamma_1(B)\rangle=\mathscr{V}(B)|\Psi(B)\rangle=|\Gamma\rangle\ . \tag{4.13}$$

Now, due to (4.11) the first term in the sum (4.10) has a pole at $E=B$. Hence, expanding $\eta_1(E)$ around $E=B$, we get for $\mathscr{T}(E)$ in the neighborhood of this pole

$$\mathscr{T}(E\sim B)\sim |\Gamma_1(B)\rangle\tau(E)\langle\Gamma_1(B)| \ , \tag{4.14}$$

with the pole behavior exhibited by the effective propagator

$$\tau(E)=\frac{1}{(E-B)\dfrac{d\eta_1(B)}{dB}} \ . \tag{4.15}$$

By differentiating (4.3) and (4.9), the inverse residue $d\eta_1(B)/dB$ of this pole is then found to be

$$\frac{d}{dB}\eta_1(B)=-\langle\Gamma_1(B)|\left[\frac{d\mathscr{G}_0(B)}{dB}+\mathscr{G}_0(B)\frac{d\mathscr{V}(B)}{dB}\mathscr{G}_0(B)\right]|\Gamma_1(B)\rangle \ , \tag{4.16}$$

and consequently Eq. (4.14) assumes the explicit form

$$\mathscr{T}(E\sim B)\sim |\Gamma_1(B)\rangle\frac{-1}{(E-B)\langle\Gamma_1(B)|\dfrac{d\mathscr{G}_0(B)}{dB}+\mathscr{G}_0(B)\dfrac{d\mathscr{V}(B)}{dB}\mathscr{G}_0(B)|\Gamma_1(B)\rangle}\langle\Gamma_1(B)| \ . \tag{4.17}$$

Because of Eq. (4.13), this result agrees with (2.16). As compared to the previous approaches, this derivation shows that the pole behavior can be understood as a consequence of the analytic behavior of the eigenvalue $\eta_1(E)$ around $E=B$, a fact which lends additional support to choosing a z-dependent coupling parameter in Eq. (3.5) of the previous section.

Because of the energy dependence of the Sturmian functions $|\Gamma_n(E)\rangle$, the Hilbert-Schmidt expansion is very cumbersome. In particular, this method requires us to solve Eq. (4.1) numerically again and again for a wide range of energies. And, moreover, the analytic continuation of the whole expansion to complex z is not straightforward. In the genuine two-body case this difficulty is avoided by using the *unitary pole expansion* (UPE) instead. It is usually introduced by noting that for *energy independent* potentials $\mathscr{V}$, the energy E in the right-hand side of the Hilbert-Schmidt expansion is only an arbitrary parameter and, therefore, can be fixed at will.[3,4] Choosing in particular $E=B$, i.e.,

$$\mathscr{V}=-\sum_n |\Gamma_n(B)\rangle\eta_n(B)\langle\Gamma_n(B)| \ , \tag{4.18}$$

leads to the UPE of $\mathscr{T}(z)$. However, since in our model $\mathscr{V}$ is energy dependent, $\mathscr{V}=\mathscr{V}(E)$, Eq. (4.18) is no longer valid for $E\neq B$. A modification of this procedure is therefore required.

To this end, we make use of the fact that the expansions (4.5) and (4.6) of the unit operator contain the energy as an arbitrary parameter, which consequently may be fixed at $E=B$:

$$1=-\sum_n \mathscr{G}_0(B)|\Gamma_n(B)\rangle\langle\Gamma_n(B)| \ , \tag{4.19}$$

$$1=-\sum_n |\Gamma_n(B)\rangle\langle\Gamma_n(B)|\mathscr{G}_0(B) \ . \tag{4.20}$$

Multiplying now $\mathscr{V}(z)$ by these decompositions, we get instead of Eq. (4.7)

$$\mathscr{V}(z)=\sum_{m,n} |\Gamma_m(B)\rangle\langle\Gamma_m(B)|\mathscr{G}_0(B)\mathscr{V}(z)\mathscr{G}_0(B)|\Gamma_n(B)\rangle\langle\Gamma_n(B)| \ ; \tag{4.21}$$

in other words, instead of (4.8) and (4.9), we have

$$\mathscr{V}(z)=-\sum_{m,n} |\Gamma_m(B)\rangle\eta_{mn}(B,z)\langle\Gamma_n(B)| \ , \tag{4.22}$$

with

$$\eta_{mn}(B,z)=-\langle\Gamma_m(B)|\mathscr{G}_0(B)\mathscr{V}(z)\mathscr{G}_0(B)|\Gamma_n(B)\rangle \ . \tag{4.23}$$

Inserting this expansion in Eq. (2.1) leads to

$$\mathscr{T}(z)=-\sum_{m,n} |\Gamma_m(B)\rangle\tau_{mn}(B,z)\langle\Gamma_n(B)| \ , \tag{4.24}$$

where

$$[\tau^{-1}(B,z)]_{mn}=[\eta^{-1}(B,z)]_{mn}+\langle\Gamma_m(B)|\mathscr{G}_0(z)|\Gamma_n(B)\rangle \tag{4.25}$$

and $\eta^{-1}(B,z)$ is the inverse of the matrix defined by the elements (4.23).

This treatment represents the most natural generalization of the conventional two-body UPE, as is exemplified by the following features:

(i) For z-independent potentials, i.e., $\mathscr{V}(B) \equiv \mathscr{V}(z)$, we have by means of Eqs. (4.1) and (4.3)

$$\eta_{mn}(B,B)=\delta_{mn}\eta_n(B) , \tag{4.26}$$

that is, (4.21) contains (4.18), and thus the conventional UPE, as a special case.

(ii) The characteristic advantage of the two-body UPE of being able to work exclusively with energy-independent form factors is preserved. Only the effective propagator matrix $\tau_{mn}(B,z)$ is more complicated, but nonetheless well manageable in practice.

Our generalized UPE was based on choosing the arbitrary parameter E in the unit operator expansions (4.5) and (4.6) to be equal to the binding energy B. This choice has the consequence that, as in the two-body case, the first term of the expansion (4.22) represents just the UPA ansatz for the potential introduced in the previous section [see (3.5), with (3.14), (3.16), and (4.13)]. Correspondingly, the bound state pole of $\mathscr{T}(z)$ is contained solely in the first term of (4.24). It should be emphasized, however, that our approach is applicable also to situations where no bound state exists. In these cases we may choose B to be some other value which appears appropriate to the problem under consideration. (In the corresponding situation of the genuine two-body case one usually takes B to be the position of the lowest branch point.) Again, if the UPE is dominated by one term, then this term defines a generalized UPA also for this nonpolar case.

A further, decisive advantage of the UPE is that its analytic structure is explicitly given via Eq. (4.25) for arbitrary complex z, which in particular means that our expansions remain valid *automatically beyond the scattering threshold.* This is in contrast to the Hilbert-Schmidt expansion where the corresponding continuation represents a nontrivial step.

Let us finally compare our generalized UPE with the energy dependent pole expansion (EDPE) proposed by Sofianos *et al.*,[7] which represents an alternative way of avoiding the undesirable features of the Hilbert-Schmidt expansion. This comparison is immediately accomplished with the help of the unit operator representations (4.19) and (4.20), already exploited extensively in the above considerations. Indeed, rewriting Eq. (4.25) as

$$[\tau^{-1}(B,z)]_{mn}=\sum_{i,j}[\eta^{-1}(B,z)]_{mi}[\Delta^{-1}(B,z)]_{ij}[\eta^{-1}(B,z)]_{jn} , \tag{4.27}$$

where

$$[\Delta^{-1}(B,z)]_{ij}=\eta_{ij}(B,z)+\langle\Gamma_i(B)\,|\,\mathscr{G}_0(B)\mathscr{V}(z)\mathscr{G}_0(z)\mathscr{V}(z)\mathscr{G}_0(B)\,|\,\Gamma_j(B)\rangle , \tag{4.28}$$

we get instead of Eq. (4.24)

$$\mathscr{T}(z)=-\sum_{n,n}\sum_{i,j}|\,\Gamma_m(B)\rangle\eta_{mi}(B,z)\Delta_{ij}(B,z)\eta_{jn}(B,z)\langle\Gamma_n(B)\,| , \tag{4.29}$$

which immediately leads to the EDPE

$$\mathscr{T}(z)=-\sum_{i,j}\mathscr{V}(z)\mathscr{G}_0(B)\,|\,\Gamma_i(B)\rangle\Delta_{ij}(B,z)\langle\Gamma_j(B)\,|\,\mathscr{G}_0(B)\mathscr{V}(z) . \tag{4.30}$$

This expansion, originally introduced to treat the three-body input in the four-body formalism of Ref. 8, was obtained as an extension of a two-body expansion scheme by Adhikari and Sloan.[9] In contrast to the generalized UPE, the EDPE works with energy-dependent form factors

$$|\,\tilde{\Gamma}_i(B,z)\rangle=\mathscr{V}(z)\mathscr{G}_0(B)\,|\,\Gamma_i(B)\rangle , \tag{4.31}$$

but, unlike those of the Hilbert-Schmidt expansion, they have an analytical behavior which is *explicitly* given via $\mathscr{V}(z)$ and, therefore, the aforementioned shortcomings of the Hilbert-Schmidt expansion, avoided in the generalized UPE, do not occur for the EDPE either. Moreover, in four-body binding energy calculations the EDPE has been found to converge more rapidly than the Hilbert-Schmidt expansion.[10] Let us add that for the genuine two-body case the EDPE, like the generalized UPE, reduces to the conventional UPE.

V. APPLICATION TO THREE-BODY AMPLITUDES

In the preceding sections we have worked out generalizations of the conventional UPA and UPE to a model two-body problem which differed from the genuine two-body problem in the properties (2.2) and (2.3) of the potential and free Green's function, respectively. These generalizations are immediately applicable to any composite particle collision process formulated as an effective two-body problem. As the simplest example we consider here in detail the UPA or UPE for the three-body input in four-body equations.

The most appropriate starting point for this purpose is the three-body formalism by Alt, Grassberger, and Sandhas (AGS) (Ref. 2) which, in order to make the present paper self-contained, will be briefly recapitulated. The three-particle transition operators $U_{\beta\alpha}(z)$ of this theory are defined as

$$U_{\beta\alpha}(z)=G_\beta^{-1}(z)G(z)G_\alpha^{-1}(z)-\delta_{\beta\alpha}G_\alpha^{-1}(z) . \quad (5.1)$$

The labels β and α denote two-particle subsystems of the three particles, and $G_\beta(z)=(z-H_\beta)^{-1}=(z-H_0-V_\beta)^{-1}$ is the resolvent of channel β (i.e., the full resolvent of subsystem β), with H_0 being the three-body kinetic energy operator and V_β the interaction of the two particles internal to β. $G_\alpha(z)$ is defined similarly, and $G(z)=(z-H)^{-1}$ is the full three-body resolvent of the total Hamiltonian $H=H_0+\sum_\gamma V_\gamma$. The operators $U_{\beta\alpha}(z)$ satisfy the coupled system of equations

$$U_{\beta\alpha}(z)=\bar{\delta}_{\beta\alpha}G_0^{-1}(z)+\sum_\gamma \bar{\delta}_{\beta\gamma}T_\gamma(z)G_0(z)U_{\gamma\alpha}(z) , \quad (5.2)$$

where $\bar{\delta}_{\beta\gamma}=1-\delta_{\beta\gamma}$; the $T_\gamma(z)$ are the two-body transition operators and $G_0(z)=(z-H_0)^{-1}$ is the usual free Green's function associated with H_0.

If we assume now, in accordance with the model of the preceding sections, that the system under consideration admits only one three-particle bound state $|\psi_B\rangle$ of energy B, then $G(z)$ has a pole at $z=B$. Hence, multiplying Eq. (5.2) by $(z=B)$ and taking the limit $z=B$, we immediately find with the help of the spectral decomposition of $G(z)$ in Eq. (5.1) that the homogeneous equation, corresponding to (5.2), is satisfied by the bound states:

$$G_\beta^{-1}(B)|\psi_B\rangle=\sum_\gamma \bar{\delta}_{\beta\gamma}T_\gamma(B)G_0(B)G_\gamma^{-1}(B)|\psi_B\rangle . \quad (5.3)$$

In order to be able to apply now the formalism developed in the previous sections, we first have to reduce (5.2) to an effective two-body equation. This can be achieved, e.g., by using the conventional UPE for $T_\gamma(z)$. For notational simplicity we restrict ourselves in the following to the UPA part of this expansion. *However, it should be emphasized that this restriction does not affect the general validity of the considerations presented below.* As said in Sec. III, the UPA is obtained by taking the two-body potential V_γ to be of separable form,

$$V_\gamma=-|g_\gamma\rangle\lambda_\gamma\langle g_\gamma| , \quad (5.4)$$

whence

$$T_\gamma(z)=|g_\gamma\rangle t_\gamma(z)\langle g_\gamma| , \quad (5.5)$$

with the two-body propagator

$$t_\gamma(z)=\frac{-1}{\lambda_\gamma^{-1}+\langle g_\gamma|G_0(z)|g_\gamma\rangle} . \quad (5.6)$$

Using Eq. (5.5) as input for Eq. (5.2) and multiplying then by $\langle g_\beta|G_0(z)$ and $G_0(z)|g_\alpha\rangle$ from the left and right, respectively, we arrive at an effective two-body matrix equation of LS form [cf. Eq. (2.1)],

$$\mathscr{T}(z)=\mathscr{V}(z)+\mathscr{V}(z)\mathscr{G}_0(z)\mathscr{T}(z) , \quad (5.7)$$

where the elements of the operator-valued matrices $\mathscr{G}_0$, $\mathscr{V}$, and $\mathscr{T}$ are given as

$$\mathscr{G}_{0\beta\alpha}(z)=\delta_{\beta\alpha}t_\alpha(z) , \quad (5.8)$$

$$\mathscr{V}_{\beta\alpha}(z)=\bar{\delta}_{\beta\alpha}\langle g_\beta|G_0(z)|g_\alpha\rangle , \quad (5.9)$$

$$\mathscr{T}_{\beta\alpha}(z)=\langle g_\beta|G_0(z)U_{\beta\alpha}(z)G_0(z)|g_\alpha\rangle . \quad (5.10)$$

This formulation of the three-body problem evidently shows the structure of the model studied in the previous sections. That is, in agreement with assumptions (2.2) and (2.3) the effective potential is z dependent, with

$$-\frac{d}{dz}\mathscr{V}_{\beta\alpha}(z)=\bar{\delta}_{\beta\alpha}\langle g_\beta|G_0(z)G_0(z)|g_\alpha\rangle , \quad (5.11)$$

and the derivative of $\mathscr{G}_0^{-1}(z)$,

$$\frac{d}{dz}\mathscr{G}_{0\beta\alpha}^{-1}(z)=\delta_{\beta\alpha}\langle g_\beta|G_0(z)G_0(z)|g_\alpha\rangle \quad (5.12)$$

differs in general from unity. Furthermore, the homogeneous version of Eq. (5.7), i.e.,

$$|\Gamma\rangle=\mathscr{V}(B)\mathscr{G}_0(B)|\Gamma\rangle , \quad (5.13)$$

is seen from Eq. (5.3) to assume the explicit form

$$|\Gamma_\beta\rangle = \sum_\gamma \bar{\delta}_{\beta\gamma}\langle g_\beta | G_0(B) | g_\gamma\rangle t_\gamma(B) | \Gamma_\gamma\rangle \ , \tag{5.14}$$

with the components of $|\Gamma\rangle$ given by

$$|\Gamma_\beta\rangle = \langle g_\beta | G_0(B) G_\beta^{-1}(B) | \psi_B\rangle \ . \tag{5.15}$$

This structural analogy allows us now to use all of the formulas, derived in the preceding sections, if only we bear in mind that operator multiplication now contains also a summation over two-body indices.

As one immediate consequence we find that the realization of $|\Psi_B\rangle$ of Sec. II in the present effective formalism is *not* given by the genuine three-body bound state wave function $|\psi_B\rangle$, but according to Eq. (2.14) by an effective wave function

$$|\Psi_B\rangle = \mathcal{G}_0(B) | \Gamma\rangle \ , \tag{5.16}$$

with components

$$\begin{aligned} |\Psi_{B\gamma}\rangle &= t_\gamma(B) | \Gamma_\gamma\rangle \\ &= -\lambda_\gamma \langle g_\gamma | \psi_B\rangle \ . \end{aligned} \tag{5.17}$$

The latter equality, which was obtained with the help of the operator identity $T_\gamma G_0 \equiv V_\gamma G_\gamma$, shows that $|\Psi_{B\gamma}\rangle$ is essentially given by the Faddeev component $V_\gamma |\psi_B\rangle$ of the genuine wave function.

The special behavior of the effective two-body amplitudes $\mathcal{T}_{\beta\alpha}(z)$ can now easily be obtained from the generally valid pole representation (2.12). In this context we note that the derivatives (5.11) and (5.12), occurring in the denominator of (2.12), differ only in the factors $\bar{\delta}_{\beta\alpha}$ and $\delta_{\beta\alpha}$, i.e., in the respective restrictions on the allowed two-body indices. In the sum, however, these restrictions are removed, and hence we have from Eq. (2.12)

$$\mathcal{T}_{\beta\alpha}(z\sim B) \sim |\Gamma_\beta\rangle \frac{1}{(z-B)\sum_{\gamma,\mu}\langle\Gamma_\gamma | t_\gamma(B)\langle g_\gamma | G_0(B) G_0(B) | g_\mu\rangle t_\mu(B) | \Gamma_\mu\rangle} \langle\Gamma_\alpha| \ . \tag{5.18}$$

With the defintion (5.15) of the form factors we furthermore find

$$\begin{aligned} \sum_\mu G_0(B) | g_\mu\rangle t_\mu(B) | \Gamma_\mu\rangle &= \sum_\mu G_0(B) | g_\mu\rangle t_\mu(B)\langle g_\mu | G_0(B) G_\mu^{-1}(B) | \psi_B\rangle \\ &= \sum_\mu G_0(B) T_\mu(B) G_0(B) G_\mu^{-1}(B) | \psi_B\rangle \\ &= G_0(B) \sum_\mu V_\mu | \psi_B\rangle \\ &= | \psi_B\rangle \ , \end{aligned} \tag{5.19}$$

and (5.18) thus reduces to

$$\mathcal{T}_{\beta\alpha}(z\sim B) \sim |\Gamma_\beta\rangle \frac{1}{(z-B)\langle\psi_B | \psi_B\rangle} \langle\Gamma_\alpha| \ , \tag{5.20}$$

or, written in a more detailed way, to

$$\mathcal{T}_{\beta\alpha}(z\sim B) \sim \langle g_\beta | G_0(B) G_\beta^{-1}(B) \frac{|\psi_B\rangle}{||\psi B||} \frac{1}{(z-B)} \frac{\langle\psi_B|}{||\psi_B||} G_\alpha^{-1}(B) G_0(B) | g_\alpha\rangle \ . \tag{5.21}$$

It is clear that the same result is obtained in a more direct way by using the fact that the spectral decomposition of $G(z)$ in the neighborhood of $z = B$ is given by

$$G(z\sim B) \sim \frac{|\psi_B\rangle}{||\psi_B||} \frac{1}{(z-B)} \frac{\langle\psi_B|}{||\psi_B||} \ . \tag{5.22}$$

[The occurrence of $||\psi_B||^2$ in the denominator is necessary here in order to ensure that the spectral decomposition of $G(z)$ is performed with orthonormal states $|\psi_B\rangle/||\psi_B||$. The state $|\psi_B\rangle$ itself need not be normalized to unity—at least not at this stage (see below).] Indeed, inserting (5.22) in the definition (5.1) and going over to the amplitudes (5.10), we immediately reproduce the pole behavior (5.21).

Although this latter derivation is much more straightforward than the one presented in detail

above, it does not show at all how the correct pole behavior is *guaranteed* within the framework of a *consistent* effective two-body formalism. The detailed derivation was given here in order to demonstrate that the essential point in obtaining (5.21) was the specific interplay of the derivatives (5.11) and (5.12).

Having established the fact that the application of the two-body model developed in Sec. II really does reproduce the correct pole behavior of the three-particle amplitude, we in the following give the explicit formulas of our generalized UPE in the three-body case.

The eigenvalue equation (4.1) now reads

$$\sum_{\gamma} \bar{\delta}_{\beta\gamma} \langle g_\beta \,|\, G_0(B) \,|\, g_\gamma \rangle t_\gamma(B) \,|\, \Gamma_\gamma^n \rangle = \eta_n(B) \,|\, \Gamma_\beta^n \rangle \ , \tag{5.23}$$

which, due to the matrix nature of the effective two-body treatment, is a system of coupled equations for the components $|\,\Gamma_\beta^n\rangle$ of $|\,\Gamma^n\rangle$. The $|\,\Gamma_\beta\rangle$ introduced by Eq. (5.14) are the ones that correspond to the eigenvalue $\eta_1(B)=1$ [cf. Eq. (4.12)]. According to Eq. (4.3), the Sturmian functions are normalized by

$$\sum_{\gamma} \langle \Gamma_\gamma^n \,|\, t_\gamma(B) \,|\, \Gamma_\gamma^m \rangle = -\delta_{nm} \ , \tag{5.24}$$

and in view of (5.15) this implies that, in general, the bound state $|\,\psi_B\rangle$ cannot be expected to being normalized to unity simultaneously [see Eq. (5.38)]. The propagator $\tau_{mn}(B,z)$ of the full UPE [cf. Eq. (4.24)]

$$\mathscr{T}_{\beta\alpha}(z) = -\sum_{m,n} |\,\Gamma_\beta^m\rangle \tau_{mn}(B,z) \langle \Gamma_\alpha^n \,| \ . \tag{5.25}$$

is determined by

$$[\tau^{-1}(B,z)]_{mn} = [\eta^{-1}(B,z)]_{mn} + \sum_{\gamma} \langle \Gamma_\gamma^m \,|\, t_\gamma(z) \,|\, \Gamma_\gamma^n \rangle \ , \tag{5.26}$$

with

$$\eta_{mn}(B,z) = -\sum_{\alpha,\beta} \bar{\delta}_{\beta\alpha} \langle \Gamma_\beta^m \,|\, t_\beta(B) \langle g_\beta \,|\, G_0(z) \,|\, g_\alpha \rangle t_\alpha(B) \,|\, \Gamma_\alpha^n \rangle \ . \tag{5.27}$$

For completeness we also give the UPA part of the expansion (5.25); it corresponds (see Sec. III) to approximating the effective two-body potential (5.9) by

$$\mathscr{V}_{\beta\alpha}^{\mathrm{UPA}}(z) = -\,|\,\Gamma_\beta\rangle \eta_{11}(B,z) \langle \Gamma_\alpha \,| \tag{5.28}$$

and reads

$$\mathscr{T}_{\beta\alpha}^{\mathrm{UPA}}(z) = |\,\Gamma_\beta\rangle \frac{-1}{\dfrac{1}{\eta_{11}(B,z)} + \sum_\gamma \langle \Gamma_\gamma \,|\, t_\gamma(z) \,|\, \Gamma_\gamma \rangle} \langle \Gamma_\alpha \,| \ . \tag{5.29}$$

Note in this context that the separable three-body amplitudes used in Ref. 11 are obtained from (5.29) by further approximating $\eta_{11}(B,z)$ by the constant $\eta_{11}(B,B)$, which according to the present investigation [cf. discussion following Eq. (3.4)] clearly does not reproduce the correct pole behavior given by Eq. (5.20). In the three-body case such an approximation neglects to a large extent the important exchange contributions to the amplitude originating solely from the effective potential $\bar{\delta}_{\beta\alpha}\langle g_\beta \,|\, G_0(z) \,|\, g_\alpha \rangle$ contained in $\eta_{11}(B,z)$. Moreover, as will be explained below, the wrong pole behavior leads to a wrong renormalization of the four-body amplitude.

In order to conclude this section, we recall how also the four-body problem is reduced to an effective two-body problem, if one uses the generalized UPE for the three-body amplitude $\mathscr{T}_{\beta\alpha}(z)$. Within the framework of the AGS theory four-particle reactions are described with the help of operators $U_{\beta\alpha}^{\sigma\rho}(z)$ satisfying[8]

$$U_{\beta\alpha}^{\sigma\rho}(z) = \bar{\delta}_{\sigma\rho}\delta_{\beta\alpha} G_0{}^{-1}(z) T_\alpha{}^{-1}(z) G_0{}^{-1}(z) + \sum_{\tau,\gamma} \bar{\delta}_{\sigma\tau} U_{\beta\gamma}^{\tau}(z) G_0(z) T_\gamma(z) G_0(z) U_{\gamma\alpha}^{\tau\rho}(z) \ , \tag{5.30}$$

where σ, ρ, and τ label the three-body subsystems and the $U_{\beta\gamma}^{\tau}(z)$ are the corresponding operators determined by Eq. (5.1). Because we are dealing here with a system of four particles, we had to introduce an additional index τ, labeling the three-body cluster in which $U_{\beta\gamma}^{\tau}$ acts and of which β and γ are two-particle subsystems. Also all other three-body quantities acquire now this index.

In the same way in which Eq. (5.7) was obtained

from (5.2) one finds in a first step with the separable approximation (5.5) of the two-body amplitude $T_\gamma(z)$ that Eq. (5.30) may be rewritten as

$$\mathscr{U}^{\sigma\rho}(z)=\bar{\delta}_{\sigma\rho}\mathscr{G}_0{}^{-1}(z)+\sum_\tau \bar{\delta}_{\sigma\tau}\mathscr{T}^\tau(z)\mathscr{G}_0(z)\mathscr{U}^{\tau\rho}(z)\ , \tag{5.31}$$

where the matrix elements of $\mathscr{G}_0$ and $\mathscr{T}^\tau$ are given by (5.8) and (5.10), respectively, and

$$\mathscr{U}^{\sigma\rho}_{\beta\alpha}(z)=\langle g_\beta \mid G_0(z)U^{\sigma\rho}_{\beta\alpha}(z)G_0(z)\mid g_\alpha\rangle\ . \tag{5.32}$$

Equation (5.31) is evidently an *effective three-body* equation of the form (5.2), where the role of the two-body amplitude T_γ in (5.2) is now played by *effective* two-body amplitude $\mathscr{T}^\tau$. Hence, proceeding in complete analogy to the genuine three-body case, we may replace $\mathscr{T}^\tau$ by our generalized UPE or, for notational simplicity, but its UPA part (5.29), which we write as

$$\mathscr{T}^\tau_{\beta\alpha}(z)=\mid\Gamma^\tau_\beta\rangle t^\tau(z)\langle\Gamma^\tau_\alpha\mid\ . \tag{5.33}$$

This second step then leads to the desired effective two-body equation of LS form

$$\underline{T}(z)=\underline{V}(z)+\underline{V}(z)\underline{G}_0(z)\underline{T}(z)\ , \tag{5.34}$$

with the matrix elements given by

$$\underline{G}_{0\sigma\rho}(z)=\delta_{\sigma\rho}t^\rho(z)\ , \tag{5.35}$$

$$\underline{V}_{\sigma\rho}(z)=\bar{\delta}_{\sigma\rho}\langle\Gamma^\sigma\mid\mathscr{G}_0(z)\mid\Gamma^\rho\rangle =\bar{\delta}_{\sigma\rho}\sum_\beta\langle\Gamma^\sigma_\beta\mid t_\beta(z)\mid\Gamma^\rho_\beta\rangle\ , \tag{5.36}$$

$$\underline{T}_{\sigma\rho}(z)=\langle\Gamma^\sigma\mid\mathscr{G}_0(z)\mathscr{U}^{\sigma\rho}(z)\mathscr{G}_0(z)\mid\Gamma^\rho\rangle =\sum_{\beta,\alpha}\langle\Gamma^\sigma_\beta\mid t_\beta(z)\langle g_\beta\mid G_0(z)U^{\sigma\rho}_{\beta\alpha}(z)G_0(z)\mid g_\alpha\rangle t_\alpha(z)\mid\Gamma^\rho_\alpha\rangle\ . \tag{5.37}$$

The form factors $\mid\Gamma^\rho_\alpha\rangle$, appearing in these expressions, were normalized according to (5.24), but this was done for convenience only because, as one immediately reads off from (5.18), the three-body pole behavior does not depend on this choice (recall also the discussion at the end of Sec. III). The four-body amplitude $\underline{T}_{\sigma\rho}(z)$ of Eq. (5.37), however, strongly depends on this choice. In particular, it will only be equal to the correct physical amplitude on the energy-shell if the initial and final channel wave functions $\mid\psi^\rho_B\rangle$ and $\langle\psi^\sigma_B\mid$, contained in $\underline{T}_{\sigma\rho}(z)$ via $\mid\Gamma^\rho_\alpha\rangle$ and $\langle\Gamma^\sigma_\beta\mid$ according to Eq. (5.15), are both normalized to unity. But Eq. (5.24) implies

$$\sum_{\beta,\alpha}\bar{\delta}_{\beta\alpha}\langle\psi^\tau_B\mid V_\beta G_0(B_\tau)V_\alpha\mid\psi^\tau_B\rangle=-1\ , \tag{5.38}$$

and this will in general of course *not* correspond to $||\psi^\tau_B||^2=1$. The amplitude $\underline{T}_{\sigma\rho}$, therefore, has to be renormalized. Let us briefly recapitulate how this is done; this will then also further explain why we have put so much emphasis on the pole behavior in the present work.

Evidently, the properly renormalized amplitude $\underline{T}^R_{\sigma\rho}(z)$ is given by

$$\underline{T}^R_{\sigma\rho}(z)=\sqrt{R_\sigma}\underline{T}_{\sigma\rho}(z)\sqrt{R_\rho}\ , \tag{5.39}$$

where

$$R_\rho=\frac{1}{||\psi^\rho_B||^2}\ , \tag{5.40}$$

and similarly for R_σ. Writing Eq. (5.34) in more detail, we see that the renormalized amplitude satisfies the modified equation

$$\underline{T}^R_{\sigma\rho}(z)=\sqrt{R_\sigma}\underline{V}_{\sigma\rho}(z)\sqrt{R_\rho} +\sum_\tau\sqrt{R_\sigma}\underline{V}_{\sigma\tau}(z)\sqrt{R_\tau}\frac{t^\tau(z)}{R_\tau}\underline{T}^R_{\tau\rho}(z)\ . \tag{5.41}$$

In order to obtain the renormalization constants R_τ, one could evaluate the matrix element in the denominator of Eq. (5.18) for all three-body clusters τ. However, the way in which they are usually calculated is to take the residues of the propagators $t^\tau(z)$. Now, both of these ways are equivalent *only* if one *knows* that $t^\tau(z)$ has the correct pole behavior, because we then have from Eq. (5.20) that

$$\lim_{z\to B_\tau}(z-B_\tau)t^\tau(z)=R_\tau\ . \tag{5.42}$$

But, for an arbitrary pole approximation ansatz of the general form (5.33) the usage of Eq. (5.42) as a prescription to calculate the renormalization constants is quite meaningless, unless one has verified

beforehand that it has the correct pole behavior (5.20). As we have mentioned already, the three-body propagator employed in Ref. 11 does not have the correct pole behavior and one finds that for the triton propagator the results, obtained from calculating the matrix element in the denominator of Eq. (5.18) and using Eq. (5.42), differ by roughly a factor of 2.

[1]P. Grassberger and W. Sandhas, Nucl. Phys. B2, 181 (1967); W. Sandhas, in *Few Body Dynamics,* edited by A. N. Mitra, I. Slaus, V. S. Bhasin, and V. K. Gupta (North-Holland, Amsterdam, 1976), and references contained therein.

[2]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. B2, 167 (1967).

[3]E. Harms, Phys. Rev. C 1, 1667 (1970). An interesting extension of the UPE, designed to treat the case of coupled $^3S_1$-$^3D_1$ channels in the two-nucleon problem, is given by I. R. Afnan and J. M. Read, Aust. J. Phys. 26, 725 (1973); in particular, we note that our arguments, leading to the choice (3.14) of the UPA potential strength, are similar to those used by Afnan and Read.

[4]See for example, J. S. Levinger, in *Springer Tracts in Modern Physics* (Springer, Berlin, 1974), Vol. 71, and references contained therein.

[5]A. Casel, H. Haberzettl, and W. Sandhas, in *Proceedings of the Ninth International Conference on the Few Body Problem,* edited by M. J. Moravcsik and F. S. Levin (University of Oregon, Eugene, 1980),Vol. I.

[6]S. Weinberg, Phys. Rev. 130, 776 (1963); 131, 440 (1963).

[7]S. Sofianos, N. J. McGurk, and H. Fiedeldey, Nucl. Phys. A318, 295 (1979).

[8]E. O. Alt, P. Grassberger, and W. Sandhas, Joint Institute for Nuclear Research Report No. E4-6688, 1972; W. Sandhas, in *Progress in Particle Physics,* edited by P. Urban [Acta Phys. Austriaca, Suppl. XIII, 679 (1974)]; Czech. J. Phys. B25, 251 (1975).

[9]S. K. Adhikari, Phys. Rev. C 10, 1623 (1974); S. K. Adhikari and I. H. Sloan, Nucl. Phys. A241, 429 (1975).

[10]S. Sofianos, H. Fiedeldey, and H. Haberzettl, Phys. Rev. C 22, 1772 (1980).

[11]A. C. Fonseca and P. E. Shanley, Phys. Rev. D 13, 2255 (1976); Phys. Rev. C 14, 1343 (1976); R. Perne and W. Sandhas, Phys. Rev. Lett. 39, 788 (1977).