Trinucleon asymptotic normalization constants including Coulomb effects

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Exact theoretical expressions for calculating the trinucleon *S*- and *D*-wave asymptotic normalization constants, with and without Coulomb effects, are presented. Configuration-space Faddeev-type equations are used to generate the trinucleon wave functions, and integral relations for the asymptotic norms are derived within this framework. The definition of the asymptotic norms in the presence of the Coulomb interaction is emphasized. Numerical calculations are carried out for the *s*-wave *NN* interaction models of Malfliet and Tjon and the tensor force model of Reid. Comparison with previously published results is made. The first estimate of Coulomb effects for the *D*-wave asymptotic norm is given. All theoretical values are carefully compared with experiment and suggestions are made for improving the experimental situation. We find that Coulomb effects increase the ^{3}He *S*-wave asymptotic norm by less than 1% relative to that of ^{3}H, that Coulomb effects decrease the ^{3}He *D*-wave asymptotic norm by approximately 8% relative to that of ^{3}H, and that the distorted-wave Born approximation *D*-state parameter, D_2, is only 1% smaller in magnitude for ^{3}He than for ^{3}H due to compensating Coulomb effects.

[NUCLEAR STRUCTURE ^{3}H and ^{3}He, asymptotic normalization constants, Coulomb effects, Faddeev calculations.]

I. INTRODUCTION

Asymptotic normalization constants for the trinucleon bound states have received considerable attention in the last ten years both theoretically and experimentally. It has been proposed that the ^{3}H and ^{3}He asymptotic normalization constants be accorded the same status as other trinucleon properties such as the binding energy and the charge radius.[1] The underlying hope is that these quantities will provide a means for discriminating between trinucleon wave functions generated from various "realistic" models of the *NN* interaction. As matters stand at present, these goals have yet to be achieved, essentially because the experimental determination of these parameters is not complete, and theoretical predictions of these quantities have been limited principally to *NN* interaction models[2] without including Coulomb effects. The primary purpose of this paper is to present a complete theoretical picture of the ^{3}H and ^{3}He *S*- and *D*-wave asymptotic normalization constants with special emphasis on the treatment of Coulomb effects; numerical results for the Malfliet-Tjon (*s* wave) and the Reid-soft-core (partial-wave-local) potential models[2] are discussed.

Physically, an asymptotic normalization constant echoes the internal dynamics present in the wave function through overall normalization. Asymptotic normalization constants are defined such that their value is unity when the effective nuclear interaction in the asymptotic channel of interest is a zero-range interaction. This last statement applies whether there is a Coulomb interaction present or not. The effect of the Coulomb interaction is to make the zero-range-comparison wave function an exponentially decreasing Whittaker function rather than a simple exponential. Since the zero-range limit is never achieved, due to the boundary condition on the wave function at the origin, asymptotic normalization constants differ from unity. In fact, the *deuteron* S-wave asymptotic normalization constant is greater than one and is determined uniquely by its binding energy and the triplet effective range, while the D-wave asymptotic normalization is less than one and appears to follow from dispersion theory with only one-pion exchange and the deuteron binding energy as input.[3] At present, there is no such fundamental understanding of the trinucleon asymptotic normalization constants that might shed light on their proper status in nuclear physics. Currently, we must be content to accept their basic definition and the relationships that can be derived between asymptotic normalization constants and vertex constants which arise in dispersion theory.[4] In actuality, it is the latter circumstance that permits extraction of asymptotic normalization constants from data.

^{3}H and ^{3}He asymptotic normalization constants have been obtained from experiment by several different means: (1) forward dispersion relation analyses (FDR); (2) partial-wave dispersion relation analyses (PWDR); (3) FDR with Coulomb corrections (FDRC); and (4) fits to tensor analyzing powers for $(\vec{d},{}^3\mathrm{H})$ and $(\vec{d},{}^3\mathrm{He})$ reactions. In Table I, we list what are considered to be the most reliable and latest values. Several points should be noted. Firstly, there does not yet exist a consensus for the value of $C_S({}^3\mathrm{H}\to n+d)$. The source of this problem may lie in the intrinsic difficulty of neutron experiments. Secondly, the value for $C_S^C({}^3\mathrm{He}\to p+d)$ does appear to be well determined. The two values leading to the weighted average are FDRC analyses of two different reactions. Thirdly, it is not yet clear whether $C_S \cong C_S^C$ as one might intuitively expect. Fourthly, the measured value of D_2, which is approximately related to the negative ratio of C_D to C_S, clearly indicates that C_D is positive relative to C_S; the same holds for C_D^C relative to C_S^C. Finally, the reader should note that $\beta^2 D_2$ [with β to be defined in Eq. (6)] is consistent with the ratio G_2/G_0. Specifically for $^3\mathrm{H}(\beta^2=0.2012$

TABLE I. Experimental values for the ^{3}H and ^{3}He S- and D-wave asymptotic normalization constants.

Quantity extracted	Value	Method	Reference
$C_S{}^2({}^3\mathrm{H}\to nd)$	2.6 ± 0.3	FDR	5
$C_S{}^2({}^3\mathrm{H}\to nd)$	3.3 ± 0.1	PWDR (E_t and a_2 fixed at experimental values)	6
$(C_S^C)^2({}^3\mathrm{He}\to pd)$	3.24 ± 0.19[a] weighted average of $\begin{Bmatrix}3.3\pm0.4\\3.19\pm0.24\end{Bmatrix}$	FDRC	7,5
$D_2\cong-\dfrac{C_D}{\beta^2C_S}\;({}^3\mathrm{H}\to nd)$	-0.279 ± 0.012 fm^2	$(\vec{d},{}^3\mathrm{H})$	8
$D_2^C\cong-\dfrac{C_D^C}{\beta^2C_S^C}f(\kappa)({}^3\mathrm{He}\to pd)$	-0.339	$(\vec{d},{}^3\mathrm{He})$	8
	-0.37		9
	-0.22		10
$\dfrac{G_2}{G_0}=\dfrac{C_D}{C_S}({}^3\mathrm{H}\to nd)$	0.048 ± 0.007	$(\vec{d},{}^3\mathrm{H})$	11

[a]These values have been adjusted to correct for the fact that the Coulomb zero-range comparison function is not normalized to unity in Refs. 5 and 7. See Ref. 12 for details.

fm^{-2}) one finds $\beta^2 D_2 = -0.0561 \pm 0.0024$. To summarize, the experimental situation is not complete and requires some further effort. Nevertheless, already adequate data exist to challenge the theorist.

In Table II, we summarize the theoretical situation with respect to work involving realistic models of the NN interaction and Faddeev calculations. Scanning the table, we see several interesting points. Perhaps the most striking is that the Reid-soft-core results for $C_S(^3\text{H} \rightarrow n+d)$ are mutually consistent and agree with the PWDR result in Table I, whereas the one-boson-exchange model (OBE) yields a smaller result consistent with the FDR value. Next, we see for the only calculation which includes the Coulomb interaction that $C_S \cong C_S^C$. Moreover, a quick estimate of D_2 from the C_S and C_D values given and the theoretical model values of β^2 indicates consistency with experiment. Lastly, it is evident that further Coulomb work is needed.

With the above background at hand, the objectives of this paper are threefold: (1) To give a careful presentation of the formalism for calculating the trinucleon S- and D-wave asymptotic normalization constants in configuration space, from integral relations, and with or without the Coulomb interaction; (2) to present complete results for the ^{3}H and ^{3}He asymptotic normalization constants for the Malfliet-Tjon (MT) and Reid-soft-core (RSC) models of the NN interaction; and (3) to carry out a complete comparison with experiment.

To achieve the stated objectives, the text is laid out as follows: Sec. II contains the formalism, Sec. III comprises our numerical results, the comparison with experiment is made in Sec. IV, and Sec. V closes the main body with a brief discussion and our conclusions. Two appendices follow Sec. V, the first of which demonstrates the equivalence of various decompositions of the Schrödinger equation containing the Coulomb interaction in computing asymptotic norms, and the second gives a useful formula for calculating the Coulomb factor in the Whittaker norm. Those interested only in numerical results and comparison with experiment should skip to Sec. III.

II. FORMALISM

Integral relations for calculation of the triton asymptotic normalization constant were first derived in Ref. 17 (see also Ref. 18). In this section, we generalize that work to include the Coulomb interaction and thus obtain integral relations for the asymptotic normalization of ^{3}He.[19] Our objectives are to carefully define the asymptotic normalization constant of ^{3}He—especially with respect to Coulomb factors that enter, to outline the derivation of the integral relations that follows from the definition, and to apply the procedure to the configuration-space, five-channel equations[20] obtaining integral relations for the S- and D-wave ^{3}He asymptotic normalization constants. For clarity, we divide this section into two parts: Subsection A contains a derivation of the "^{3}He" asymptotic-normalization integral relation where three equal mass, spinless particles interact by means of a pairwise s-wave short-range interaction, and two of which carry charge $e>0$. In this way we can emphasize the aspects arising from the Coulomb interaction without complications due to angular-momentum coupling. Subsection B contains the results for the configuration-space, five-channel, local-potential equations.

TABLE II. Theoretical values for the ^{3}H and ^{3}He S- and D-wave asymptotic normalization constants (Faddeev calculations).

Quantity calculated	Value	Model	B_3 (MeV)	Reference
$C_S(^3\text{H}\rightarrow nd)$	1.776 ± 0.003	RSC5	6.96	13
$C_D(^3\text{H}\rightarrow nd)$	0.065 ± 0.002	RSC5		
$C_S(^3\text{H}\rightarrow nd)$	1.612	OBE	7.38	14
$C_S(^3\text{H}\rightarrow nd)$	1.706	RSC3	6.40	15
$C_S^C(^3\text{He}\rightarrow pd)$	1.765	RSC3	5.78	
$C_S(^3\text{H}\rightarrow nd)$	1.76	RSC5	7.1	16
$C_D(^3\text{H}\rightarrow nd)$	0.065	RSC5		

A. Spin independent, s-wave derivation

Consider the case of three identical mass, spinless particles, where particle 3 is uncharged and particles 1 and 2 are positively charged. Assume that each pair (ij) experiences the same short-range s-wave interaction V_{ij} that, in the absence of any Coulomb interaction V^C, supports a single two-body bound state, but that in the presence of the Coulomb interaction is not strong enough to produce a two-body bound state. Also, assume that the three-body system has a single bound state with these interactions. This shall be our model of ^{3}He.

The three-body Hamiltonian can be written as

$$H=H_0+V+V_{12}^C \,, \tag{1}$$

$$H_0=\frac{\vec{p}_x^{\,2}}{2\mu_x}+\frac{\vec{p}_y^{\,2}}{2\mu_y}\equiv H_{0_x}+H_{0_y} \,, \tag{2}$$

$$V=\sum_{i<j=1}^{3} V_{ij} \,, \tag{3}$$

$$V_{12}^C=\frac{\alpha}{x_3} \,, \tag{4}$$

where $\vec{p}_x$ and $\vec{p}_y$ are the momenta conjugate to the Jacobi coordinates $\vec{x}_i=\vec{r}_j-\vec{r}_k$ and $\vec{y}_i=\vec{r}_i-\frac{1}{2}(\vec{r}_j+\vec{r}_k)$ with i, j, and k taken in cyclic permutation (see Fig. 1), $\mu_x=M/2$, $\mu_y=2M/3$, M is the mass of a nucleon, and α is the fine structure constant. The $J^\pi=0^+$ ground-state wave function is determined from

$$H\Psi(\overline{12},3)=E\Psi(\overline{12},3) \tag{5}$$

with

$$E\equiv-\frac{\gamma^2}{M}-\frac{3\beta^2}{4M}\equiv-B_d-B_{pd}\equiv-B_3 \,, \tag{6}$$

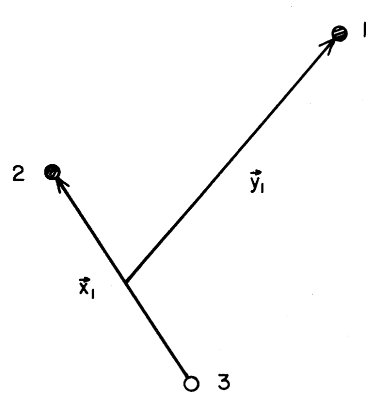

FIG. 1. Example of Jacobi coordinates as used in the text. The shaded particles are taken to be charged.

where the subscripts p and d represent "proton" and "deuteron," respectively. The overscore in Eq. (5) indicates that pair (12) is symmetric under interchange. The two-body binding energy for the bound state of a charged and uncharged particle is $B_d=\gamma^2/M$, whereas $B_{pd}=3\beta^2/4M$ represents the binding energy of the proton to the deuteron in ^{3}He. The ground-state wave function can be decomposed as

$$\Psi(\overline{12},3)=\chi(1,23)+\chi(2,31)+\eta(3,\overline{12}) \tag{7}$$

$$\equiv\chi_1+\chi_2+\eta \,, \tag{8}$$

where the components satisfy the coupled (Faddeev) equations

$$[E-H_0-V_{23}-V_{12}^C]\chi_1=V_{23}(\chi_2+\eta) \,, \tag{9a}$$

$$[E-H_0-V_{31}-V_{12}^C]\chi_2=V_{31}(\chi_1+\eta) \,, \tag{9b}$$

$$[E-H_0-V_{12}-V_{12}^C]\eta=V_{12}(\chi_1+\chi_2) \,. \tag{9c}$$

From Eqs. (9), it is clear that as $y_1\equiv|\vec{y}_1|\rightarrow\infty$ $(y_2\rightarrow\infty)$, $\chi_1(\chi_2)$ describes the asymptotic behavior of $\Psi(\overline{12},3)$ as charged particle 1(2) is removed from the bound pair 23 (31). Likewise, as $y_3\rightarrow\infty$, η describes the asymptotic behavior of $\Psi(\overline{12},3)$ as the neutral particle is removed from the unbound, charged pair. The operators on the left-hand side of Eqs. (9) tell us that in these respective limits, $\chi_1(\chi_2)$ behaves asymptotically as a decaying Whittaker function and η as a decaying exponential. Within this framework, the asymptotic normalization constant can be precisely defined.

In defining the asymptotic normalization constant for ^{3}He, we first treat the triton to remind the reader of details and to make it available for reference. The triton asymptotic normalization constant C_t is defined from Eqs. (9) with $V_{12}^C=0$:

$$\lim_{y_1\rightarrow\infty}\Psi(\vec{y}_1,\vec{x}_1)\rightarrow C_t N_{ZR}\frac{e^{-\beta y_1}}{y_1}Y_{00}(\hat{y}_1)\Phi_d(\vec{x}_1) \,, \tag{10}$$

where

$$[H_{0_{x_1}}+V(x_1)]\Phi_d(\vec{x}_1)=-B_d\Phi_d(\vec{x}_1) \,, \tag{11}$$

$$\int d^3x_1\Phi_d^\dagger(\vec{x}_1)\Phi_d(\vec{x}_1)=1 \,, \tag{12}$$

and

$$N_{ZR}=\sqrt{2\beta} \,, \tag{13}$$

obtained from

$$N_{ZR}{}^{-2}=\int_0^\infty dy_1 e^{-2\beta y_1} . \quad (14)$$

Analogously, when $V_{12}^C \neq 0$ in Eqs. (9), we define the ^{3}He asymptotic normalization constant, C_t^C (superscript indicating Coulomb), as follows:

$$\lim_{y_1\to\infty} \Psi(\vec{y}_1,\vec{x}_1)\to \lim_{y_1\to\infty} \chi(\vec{y}_1,\vec{x}_1) \to C_t^C N_W \frac{W_{-\kappa,1/2}(2\beta y_1)}{y_1} \times Y_{00}(\hat{y}_1)\Phi_d(\vec{x}_1) , \quad (15)$$

where $W_{-\kappa,1/2}(2\beta y_1)$ is the Whittaker function that goes logarithmically to a constant at the origin and decays exponentially for $y_1\to\infty$,

$$\kappa=\frac{\mu_y\alpha}{\beta}=\frac{2M\alpha}{3\beta}>0 , \quad (16)$$

and

$$N_W=N_{ZR}\left[\frac{\Gamma(3+\kappa)\Gamma(2+\kappa)}{2\,{}_3F_2(\kappa,2,1+\kappa;3+\kappa,2+\kappa;1)}\right]^{1/2} , \quad (17)$$

obtained from

$$N_W{}^{-2}=\int_0^\infty dy_1[W_{-\kappa,1/2}(2\beta y_1)]^2 . \quad (18)$$

In Eq. (17), $\Gamma(z)$ is the gamma function and ${}_3F_2$ is a hypergeometric function. Clearly, when $\kappa\to 0$, $N_W\to N_{ZR}$, and

$$\lim_{y_1\to\infty} W_{-\kappa,1/2}(2\beta y_1)\to\frac{e^{-\beta y_1}}{(2\beta y_1)^\kappa}\underset{\kappa\to 0}{\to} e^{-\beta y_1} . \quad (19)$$

We caution the reader that it is not correct, nor is it a valid approximation, to use the asymptotic form of the Whittaker function in Eq. (15).[21]

Derivation of the integral relation for C_t^C follows from its definition in Eq. (15) and by use of Eq. (9a). We project from the left in Eq. (9a) with the bra $\langle\Phi_d\vec{y}_1|$ to write

$$-\langle\Phi_d\vec{y}_1|(B_{pd}+H_{0_y}+V_{12}^C)|\chi_1\rangle-\langle\Phi_d\vec{y}_1|(B_d+H_{0_x}+V_{23})|\chi_1\rangle=\langle\Phi_d\vec{y}_1|V_{23}|\chi_2+\eta\rangle , \quad (20)$$

but from Eq. (11) the second term on the left-hand side vanishes. Therefore,

$$-\langle\Phi_d\vec{y}_1|(B_{pd}+H_{0_y}+V_{12}^C)|\chi_1\rangle =\langle\Phi_d\vec{y}_1|V_{23}|\chi_2+\eta\rangle . \quad (21)$$

Proceeding further we define $\mathscr{Y}^C(y_1)=y_1\langle\Phi_d\vec{y}_1|\chi_1\rangle$ and subtract

$$\left\langle\Phi_d\vec{y}_1\left|\frac{\alpha}{y_1}\right|\chi_1\right\rangle=\frac{\alpha}{y_1}\langle\Phi_d\vec{y}_1|\chi_1\rangle \quad (22)$$

from each side of Eq. (21) to obtain

$$\left[\frac{d^2}{dy_1{}^2}-\beta^2-\frac{2\kappa\beta}{y_1}\right]\mathscr{Y}^C(y_1) =\frac{4M}{3}y_1\left[\langle\Phi_d\vec{y}_1|V_{23}|\chi_2+\eta\rangle +\left\langle\Phi_d\vec{y}_1\left|\frac{\alpha}{x_3}-\frac{\alpha}{y_1}\right|\chi_1\right\rangle\right] . \quad (23)$$

The last term on the right-hand side—a matrix element of the "polarization" potential—arises because the charge associated with the bound pair of particles does not reside at the pair's center-of-mass, but on the charged particle. We now convert Eq. (23) into an integral equation

$$\mathscr{Y}^C(y_1)=-\frac{4M}{3}\int_0^\infty y_1'dy_1'G(y_1,y_1') \times\left[\langle\Phi_d\vec{y}_1'|V_{23}|\chi_2+\eta\rangle +\left\langle\Phi_d\vec{y}_1'\left|\frac{\alpha}{x_3}-\frac{\alpha}{y_1}\right|\chi_1\right\rangle\right] \quad (24)$$

with Green's function

$$G(y_1,y_1') =\frac{\Gamma(1+\kappa)}{2\beta}M_{-\kappa,1/2}(2\beta y_{1<})W_{-\kappa,1/2}(2\beta y_{1>}) , \quad (25)$$

where $M_{-\kappa,1/2}(2\beta y_1)$ is the Whittaker function that vanishes at the origin and diverges exponentially as $y_1\to\infty$. From Eq. (15), we know that

$$\lim_{y_1\to\infty}\mathscr{Y}^C(y_1)\to C_t^C N_W W_{-\kappa,1/2}(2\beta y_1)Y_{00}(\hat{y}_1) \quad (26)$$

and, therefore, the integral relation for C_t^C becomes

$$C_t^C = -\frac{4M}{3}\frac{\sqrt{4\pi}}{N_W}\frac{\Gamma(1+\kappa)}{2\beta}$$
$$\times \int_0^\infty y_1 dy_1 M_{-\kappa,1/2}(2\beta y_1)$$
$$\times \left[\langle \Phi_d \vec{y}_1 \,|\, V_{23} \,|\, \chi_2 + \eta \rangle + \left\langle \Phi_d \vec{y}_1 \left| \frac{\alpha}{x_3} - \frac{\alpha}{y_1} \right| \chi_1 \right\rangle \right] . \quad (27)$$

We note that as $\alpha \to 0$ (and therefore $\kappa \to 0$), Eq. (27) reduces to the non-Coulomb result of Ref. 17 [Eq. (24)] since $M_{-\kappa,1/2}(2\beta y_1) \to 2\sinh(\beta y_1)$ in this limit.

Two points should be made about Eq. (27) before closing this subsection. Firstly, we emphasize again the importance of N_W in Eq. (27) as defined[21] by Eq. (17). Secondly, the polarization term (the second term within the brackets) is not unique and depends on the particular decomposition of Schrödinger's equation into equivalent coupled Faddeev equations for the three components of $\Psi(\overline{12},3)$. In fact, in Appendix A, we show that there are infinitely many decompositions between that given above and that of Sasakawa and Sawada,[22] where, in the latter case, the formula for C_t^C does not contain an explicit polarization term. We prove that all these representations for C_t^C are equivalent. This point is important, because in the next subsection we use the Sasakawa-Sawada decomposition.[23]

B. *S*- and *D*-wave five-channel derivation

After the rather detailed derivation given in the previous subsection for the spin independent, *s*-wave case, the five-channel, configuration-space derivation for the *S*- and *D*-wave asymptotic-normalization integral relations for ^{3}H and ^{3}He will be more abbreviated. Only essential aspects will be presented. For notational details beyond the immediate needs of this paper, we refer the reader to the Appendix of Payne *et al.*[20]

We begin by defining the *S*- and *D*-wave asymptotic normalization constants for ^{3}H and ^{3}He. The ^{3}H and ^{3}He bound states have $J^\pi = \frac{1}{2}^+$ and the deuteron has $J^\pi = 1^+$. Therefore, the ^{3}H asymptotic normalization constants are defined as

$$\lim_{y_1\to\infty} \Psi^{[1/2]}_{^3\mathrm{H}}(\vec{y}_1,\vec{x}_1) \to C_S N_{ZR} \frac{e^{-\beta y_1}}{y_1} [[Y_0(\hat{y}_1)\times \chi^{1/2}(1)]^{[1/2]} \times \Phi^{[1]}(\vec{x}_1)]^{[1/2]} \frac{\eta'}{\sqrt{2}}$$
$$+ C_D N_{ZR} \frac{e^{-\beta y_1}}{y_1} \left[1 + \frac{3}{\beta y_1} + \frac{3}{\beta^2 {y_1}^2} \right] [[Y_2(\hat{y}_1)\times \chi^{1/2}(1)]^{[3/2]} \times \Phi^{[1]}(\vec{x}_1)]^{[1/2]} \frac{\eta'}{\sqrt{2}} , \quad (28)$$

where C_S and C_D are the triton *S*- and *D*-wave asymptotic normalization constants, respectively, $Y_l(\hat{y})$ is a spherical harmonic (*m* suppressed due to coupling), $\chi^{1/2}$ is a spin-$\frac{1}{2}$ state, $\Phi^{(1)}$ is the deuteron wave function, and η' is the isospin-$\frac{1}{2}$ function for three particles, where particles 2 and 3 (the deuteron) are coupled to isospin 0. Similarly, the ^{3}He *S*- and *D*-wave asymptotic normalization constants C_S^C and C_D^C, respectively, are defined as

$$\lim_{y_1\to\infty} \Psi^{[1/2]}_{^3\mathrm{He}}(\vec{y}_1,\vec{x}_1) \to C_S^C N_W \frac{W_{-\kappa,1/2}(2\beta y_1)}{y_1} [[Y_0(\hat{y})\times \chi^{1/2}(1)]^{1/2} \times \Phi^{[1]}(\vec{x}_1)]^{[1/2]} \frac{\eta'}{\sqrt{2}}$$
$$+ C_D^C N_W \frac{W_{-\kappa,5/2}(2\beta y_1)}{y_1} [[Y_2(\hat{y}_1)\times \chi^{1/2}(1)]^{[1/2]} \times \Phi^{[1]}(\vec{x}_1)]^{[1/2]} \frac{\eta'}{\sqrt{2}} . \quad (29)$$

We can now proceed with the derivation of the C_S^C and C_D^C integral relations and recover the tritium ones by letting $\kappa \to 0$.

We begin from

$$\Psi_{^3\mathrm{He}}(\vec{y}_1,\vec{x}_1) = (1 + P^- + P^+)\psi^C(\vec{y}_1,\vec{x}_1) , \quad (30)$$

where

$$\psi^C(\vec{y}_1,\vec{x}_1) = \sum_{\nu=1}^{5} \psi_\nu^C(y_1,x_1)\,|\,\nu\rangle , \quad (31)$$

and the orbital-spin angular momentum-isospin states $|\nu\rangle$ are defined in Table III. The Schrödinger equation is decomposed such that the Faddeev components, $\psi^C(\vec{y}_1,\vec{x}_1)$, satisfy Saskawa-Sawada-type equations[22]

$$[H_0+V(\vec{x}_1)+V_2(y_1)+V_3(y_1)-E]\psi^C(\vec{y}_1,\vec{x}_1) = -[V(\vec{x}_1)-V_3(y_2)]\psi^C(\vec{y}_2,\vec{x}_2) - [V(\vec{x}_1)-V_2(y_3)]\psi^C(\vec{y}_3,\vec{x}_3)\ , \quad (32)$$

where

$$\psi^C(\vec{y}_2,\vec{x}_2)=P^-\psi^C(\vec{y}_1,\vec{x}_1)\ , \quad (33)$$

$$\psi^C(\vec{y}_3,\vec{x}_3)=P^+\psi^C(\vec{y}_1,\vec{x}_1)\ , \quad (34)$$

$$V_i(y_j)=P_C^{i,j}\frac{\alpha}{y_j}\ , \quad (35)$$

and

$$V(\vec{x}_1)=\sum_{\nu=1}^{5}|\nu\rangle V_\nu(\vec{x}_1)\langle\nu| + P_C^{2,3}\frac{\alpha}{x_1}\ . \quad (36)$$

The projection operator $P_C^{i,j}$ is the product of the proton projection operators for the particles i and j. In Eq. (35) the subscript i reminds us that the actual Coulomb force is between particles i and j, and that asymptotically

$$V_i(y_j)\to P_C^{i,j}\alpha/x_k \quad (i,j,k \text{ cyclic})\ .$$

Therefore, asymptotically,

$$P_C^{2,3}\alpha/x_1 - P_C^{2,3}\alpha/y_3 \to O(x_1^{-2})\ ; \quad (37)$$

i.e., it is "short" ranged. The $V_\nu(\vec{x}_1)$ represents the nuclear interaction appropriate to the quantum numbers ν. In conjunction with this three-body formalism, we complete our starting point with the deuteron wave function defined as

$$\Phi_d(\vec{x}_1)=\frac{u_S(x_1)}{x_1}|S\rangle+\frac{u_D(x_1)}{x_1}|D\rangle\ , \quad (38)$$

where $|S\rangle$ and $|D\rangle$ represent the standard S- and D-state orbital-spin angular momentum-isospin functions.

The asymptotic-normalization-constant integral relations are now derived by coupling the third nucleon to the deuteron to form the two states

$$|S,d\rangle=\frac{u_S(x_1)}{x_1}|2\rangle+\frac{u_D(x_1)}{x_1}|3\rangle \quad (39)$$

and

$$|D,d\rangle=\frac{u_S(x_1)}{x_1}|4\rangle+\frac{u_D(x_1)}{x_1}|5\rangle\ , \quad (40)$$

respectively (see Table III). The states $|\nu\rangle$ contain the isospin, spin, and orbital angular momentum wave functions for the spectator ($\vec{y}_1$) and the interacting pair ($\vec{x}_1$). Therefore, following the same procedure as in subsection A above, we project from the left with $\langle d,L|$ ($L=S$ or D) on Eq. (32) and obtain

$$\left[\frac{d^2}{dy_1^2}-\frac{L(L+1)}{y_1^2}-\beta^2-\frac{2\kappa\beta}{y_1}\right]\mathscr{Y}_L^C(y_1) = \frac{4M}{3}y_1\langle d,L;\vec{y}_1|V(\vec{x}_1) \times(P^-+P^+)|\psi^C(1,23)\rangle\ , \quad (41)$$

TABLE III. The 5 states in $J_\nu-j_\nu$ coupling[a] which compose the three-nucleon wave function when the N-N interaction is limited to $^1S_0$ and $^3S_1-{}^3D_1$. $J=J_\nu\otimes j_\nu=\frac{1}{2}$, the total angular momentum of the triton. $T=T_\nu\otimes t_\nu=\frac{1}{2}$, the total isospin of the three-nucleon system, where t_ν is the total isospin of particles 2 and 3; the small isospin-$\frac{3}{2}$ component is neglected. In the MT models inclusion of this component has been found to have a negligible effect on C_S.

ν	$(l_\nu,S_\nu)j_\nu$	$(L_\nu,S_\nu)J_\nu$	$(t_\nu,T_\nu)T$
1	(0,0)0	$(0,\frac{1}{2})\frac{1}{2}$	$(1,\frac{1}{2})\frac{1}{2}$
2	(0,1)1	$(0,\frac{1}{2})\frac{1}{2}$	$(0,\frac{1}{2})\frac{1}{2}$
3	(2,1)1	$(0,\frac{1}{2})\frac{1}{2}$	$(0,\frac{1}{2})\frac{1}{2}$
4	(0,1)1	$(2,\frac{1}{2})\frac{3}{2}$	$(0,\frac{1}{2})\frac{1}{2}$
5	(2,1)1	$(2,\frac{1}{2})\frac{3}{2}$	$(0,\frac{1}{2})\frac{1}{2}$

[a]Note that the coupling order is reversed compared to Ref. 20.

where

$$\mathscr{Y}_L^C(y_1) \equiv y_1 \langle d,L;\vec{y}_1 | \psi^C(1,23)\rangle \quad (42)$$

and

$$\psi^C(\vec{y}_1,\vec{x}_1) = \langle \vec{y}_1,\vec{x}_1 | \psi^C(1,23)\rangle . \quad (43)$$

The equivalent of Eq. (24) then follows directly from the Green's function

$$G_L(y_1,y_1') = \left[\prod_{n=1}^{L}(1+\kappa/n)\right]\Gamma(1+\kappa)\frac{M_{-\kappa,L+1/2}(2\beta y_{1<})W_{-\kappa,L+1/2}(2\beta y_{1>})}{2^{L+1}\beta(2L+1)!!} . \quad (44)$$

Specifically,

$$\mathscr{Y}_L^C(y_1) = -\frac{4M}{3}\int_0^\infty y_1' dy_1' G_L(y_1,y_1')\langle d,L;\vec{y}_1 | V(\vec{x}_1)(P^- + P^+) | \psi^C(1,23)\rangle . \quad (45)$$

From Eq. (42), in combination with Eq. (29), we know that

$$\lim_{y_1\to\infty} \mathscr{Y}_L^C(y_1) \to C_L^C N_W W_{-\kappa,L+1/2}(2\beta y_1) . \quad (46)$$

Therefore,

$$C_L^C = -\frac{4M}{3}\frac{\left[\prod_{n=1}^{L}(1+\kappa/n)\right]\Gamma(1+\kappa)}{N_W 2^{L+1}\beta(2L+1)!!}\int_0^\infty y_1 dy_1 M_{-\kappa,L+1/2}(2\beta y_1)\langle d,L;\vec{y}_1 | V(\vec{x}_1)(P^- + P^+) | \psi^C(1,23)\rangle . \quad (47)$$

The non-Coulomb integral relations follow immediately by letting $\kappa \to 0$. Since

$$M_{0,L+1/2}(2\beta y_1) = \frac{(2L+1)!}{L!}\sqrt{2\pi\beta y_1} I_{L+1/2}(\beta y_1) , \quad (48)$$

where $I_\nu(z)$ is a modified Bessel function, we obtain

$$C_L = -\frac{2M}{3}\frac{\sqrt{\pi}}{\beta}\int_0^\infty y_1^{3/2} dy_1 I_{L+1/2}(\beta y_1)\langle d,L;\vec{y}_1 | V(\vec{x}_1)(P^- + P^+) | \psi(1,23)\rangle . \quad (49)$$

Equations (47) and (49) are used to calculate the results given in the next section. This is a straightforward numerical task, because the terms represented by $\langle d,L;\vec{y}_1 | V(\vec{x}_1)(P^- + P^+) | \psi(1,23)\rangle$ (excluding the deuteron spatial function) with or without the Coulomb interaction are the same terms that appear in the coupled equations that are solved to obtain the components ψ_ν^C or ψ_ν.

III. NUMERICAL RESULTS

The results of our numerical calculations are given in Tables IV – VI. They are broken apart according to increasing complexity. The results for the s-wave models of Malfliet and Tjon[24] (MT II-IV and MT I-III) are given in Table IV, the RSC three channel (RSC3) in Table V, and the RSC five channel (RSC5) in Table VI. In what follows NC, PC, and FC denote no Coulomb, point Coulomb, and finite-size (proton dipole charge distribution[23]) Coulomb interaction in the proton-proton potential.

One of the main results of this work is clear already from the simple s-wave MT model results in Table IV. Coulomb effects *increase* the S-wave trinucleon asymptotic normalization constant by *only* approximately one percent. Thus, to a very good approximation, the ^3H and ^3He S-wave asymptotic normalization constants are the same. Introduction of the nucleon finite size through use of a dipole form factor in constructing the Coulomb potential leads to only a slight change in the Coulomb predictions ($\lesssim 0.2\%$). The more significant effect on the values of the S-wave asymptotic norms evident in the table is due to the presence of repulsion in the NN interactions, which pushes probability into the exterior of the wave function. The MT II-IV model has no repulsion, whereas MT I-III does possess short-range repulsion; Comparison of asymptotic norms for these two models indicates that repulsion leads to an increase of

TABLE IV. S-wave asymptotic normalization constant values for the triplet-singlet models of Malfliet and Tjon.

Case[a]	B_3 (MeV)	κ	C_S or C_S^C	$(C_S$ or $C_S^C)^2$
		[MT II-IV (B_d=2.2110 MeV)]		
NC	11.880	0	1.873	3.508
PC	10.849	0.043 925	1.890	3.572
FC	11.030	0.043 473	1.894	3.587
		[MT I-III (B_d=2.2306 MeV)]		
NC	8.535	0	1.959	3.838
PC	7.870	0.054 364	1.974	3.897
FC	7.903	0.054 207	1.976	3.904

[a]NC≡no Coulomb; PC≡point Coulomb; FC≡finite Coulomb (dipole proton form factor).

$\lesssim 5\%$ in the S-wave asymptotic norm value. Finally, our MT I-III (NC) value of 1.959 is in good agreement with the value 1.97 quoted in Ref. 16, indicating consistency with previous work.

Further aspects of the asymptotic normalization not apparent in Table IV but substantiated by numerical calculations are worthwhile pointing out. Firstly, an increase in the binding energy due to an increase in the strength of either the triplet or the singlet central force interaction leads to an increase in C_S. Increasing the binding draws both the wave function and the zero range comparison function in toward the origin; the zero range function is drawn in more because it is singular at the origin. The probability of the wave function adjacent to the asymptotic region is enhanced with respect to the comparison function, leading to the increase in C_S. Alternatively this is equivalent to an increase in the $^3H \rightarrow n+d$ coupling constant, since the residue at the ^3H pole in the nd scattering amplitude is directly proportional to C_S^2 (see Ref. 4). Secondly, the repulsive Coulomb interaction modifies both the zero-range comparison function and the asymptotic behavior of the physical wave function. At the same time it pushes probability away from the origin into the asymptotic region in the case of the physical wave function; this increases C_S^C compared to C_S *when* the binding energies are identical.

The values of C_S and C_S^C for the RSC3 model given in Table V show the same behavior with respect to Coulomb effects as the MT models: a $\lesssim 1\%$ enhancement. (There is no C_D in this model since there is no $L_v=2$, as is clear from Table III.) The most striking difference compared with the MT models is the *lower* values of C_S and C_S^C, about 11 or 12% smaller than the MT I-III model results attributable to the introduction of the tensor force in the triplet interaction. (The S-state probability is only 92%, which accounts for ≅8% of this decrease.) Compared to the work of Sasakawa *et al.*[15] (see Table II), we predict only a 0.5% increase in going from C_S to C_S^C (PC), whereas they predict a 3.4% increase.[25]

Completing the picture of our calculations are the RSC5 results given in Table VI which lead to both the S- and D-wave asymptotic normalization constants. The RSC5 results for the S-wave values do not differ greatly from the RSC3 case, being only slightly larger ($\lesssim 2\%$). This increase is con-

TABLE V. S-wave asymptotic normalization constant values for the RSC ($^1S_0$,$^3S_1$-$^3D_1$) potential model in the truncated three-channel approximation.

Case	B_3 (MeV)	κ	C_S or C_S^C	$(C_S$ or $C_S^C)^2$
NC	6.384	0	1.736	3.014
PC	5.775	0.068 514	1.744	3.042
FC	5.797	0.068 300	1.746	3.048
B_d=2.2245 MeV.				

TABLE VI. S- and D-wave asymptotic normalization constant values for the RSC ($^1S_0, ^3S_1$-$^3D_1$) potential model in the full five-channel calculation.

Case	B_3 (MeV)	κ	C_S or C_S^C	(C_S or $C_S^C)^2$	C_D or C_D^C	D_2 or D_2^C (fm^2)	$f(\kappa)$	β^2 (fm^{-2})
NC	7.022	0	1.758	3.090	0.0658	−0.243	1	0.1541
PC	6.390	0.063 259	1.771	3.136	0.0609	−0.2404	0.9354	0.1338
FC	6.414	0.063 077	1.773	3.144	0.0614	−0.2407	0.9356	0.1346
PC/NC[a]	6.390	0	1.713	2.934	0.0561	−0.245	1	0.1338
NC/PC	7.022	0.058 938	1.823	3.323	0.0715	−0.239	0.9396	0.1541
B_d=2.2245 MeV								

[a]PC/NC≡point Coulomb wave function treated as non-Coulomb in extracting asymptotic normalization constants; NC/PC≡non-Coulomb wave function treated as having Coulomb present in extracting asymptotic normalization constants.

sistent with the increase in binding. Moreover, the Coulomb effects are similar, $\lesssim 1\%$ increase, and our non-Coulomb results are in agreement with those of Refs. 13 and 16 (See Table II). The most striking prediction of the RSC5 model is that the D-wave asymptotic norm is *lowered* by $\sim$7–8 % due to Coulomb effects. It should be emphasized that our calculations for the RSC5 model with Coulomb constitute the first predictions of C_S^C and C_D^C for this model. When the Coulomb interaction is absent, our RSC5 value for C_D again agrees with the results of Refs. 13 and 16 (see Table II). We predict that $C_D(C_D^C)$ is positive relative to $C_S(C_S^C)$. Also, as a check on various possible approximations, we have calculated the asymptotic norms with the integral relation for no Coulomb, Eq. (49), but the RSC5 wave function generated with Coulomb, and vice versa. Clearly, neither represents a good approximation for computing Coulomb effects (see PC/NC and NC/PC lines in Table VI).

From the physics viewpoint, several aspects stand out in the above: (1) $C_S > 1$, (2) $C_S^C \cong C_S$, (3) $C_D << 1$, and (4) $C_D^C < C_D$. Qualitatively, these results can be understood as follows: (1) The S-wave asymptotic norm is greater than unity because the effective nucleon-deuteron wave function is not singular at the origin like the zero-range comparison function and the excess probability appears in the tail of the wave function. Therefore, even under their respective normalization constraints,[26] the nucleon-deuteron wave function turns out to be larger (by the factor C_s) in the asymptotic region. (2) The fact that $C_S^C \cong C_S$ is a reflection of the nearly complete cancellation of the increase in C_S^C due to Coulomb repulsion (at fixed binding energy) and the decrease in C_S^C due to the reduced binding energy in the presence of the Coulomb interaction. (3) The fact that $C_D << 1$ simply reflects the relative sizes of the S- and D-wave nucleon-deuteron wave functions. Recall that the D-wave asymptotic norm is defined relative to the S-wave zero-range function [see Eq. (28)]. (4) The fact that $C_D^C < C_D$ follows from the same explanation as given for $C_S^C \cong C_S$, but in the present D-wave case the centrifugal barrier lessens the effectiveness of the Coulomb repulsion in "pushing-out" the maximum of the proton-deuteron wave function (relative to the neutron-deuteron wave function). Thus C_D^C must be less than C_D due to the dominant binding energy effect.[26]

IV. COMPARISON WITH EXPERIMENT

First, we compare the S-wave asymptotic normalization values for the triplet-singlet models of Malfliet and Tjon (Table IV) with the experimental values (Table I). We see immediately that the MT values all lie outside the experimental limits, even though the MT I-III model is as realistic an S-wave model as one might want to construct. This can be attributed to the absence of the tensor force in the NN interaction. When the tensor force is present, as in the RSC3 model (Table V), the values of C_S or C_S^C are considerably reduced relative to those of the MT models. Both the S-state wave function probability and the three-body binding energy are smaller. Moreover, within present experimental precision, the RSC3 predictions are consistent with the measurements. The improved treatment of the tensor force in the RSC5 model (Table VI) results in only slightly larger S-wave asymptotic norm values compared to the RSC3 model; therefore, they are also in agreement with experiment. Overall, the RSC models are consistent with present experimental data, and they predict that Coulomb effects are $\lesssim 1\%$ for the S-

wave asymptotic normalization constants. This latter aspect emphasizes the importance of increasing the precision in determining asymptotic norms *and* the accuracy of the triton value.

So far, the trinucleon D-wave asymptotic normalization constants have not been directly extracted from experiment, only their ratio with the respective S-wave asymptotic norms. This is achieved by choosing the parameter D_2 (which is *approximately* related to the C_D/C_S ratio) to give optimal fits to tensor analyzing powers for $(\vec{d},{}^3\mathrm{H})$ and $(\vec{d},{}^3\mathrm{He})$ reactions. The parameter D_2 is defined as[27]

$$D_2=-\frac{\int_0^\infty dy_1 y_1{}^4 u_2(y_1)}{15\int_0^\infty dy_1 y_1{}^2 u_0(y_1)}\ , \tag{50}$$

where the effective nucleon-deuteron wave functions are defined through the overlap (isospin projections suppressed)

$$\langle \vec{y}_1\chi^{1/2}_{m_N}(1)\Phi^{[1]}_{m_d}\mid\Psi^{[1/2]}_m(i)\rangle=u_0(y_1)\langle \tfrac{1}{2}m_n 1m_D\mid \tfrac{1}{2}m\rangle Y_{00}(\hat{y}_1)$$
$$+u_2(y_1)\sum_M\langle \tfrac{1}{2}m_N 1m_D\mid \tfrac{3}{2}M\rangle\sum_{m_l}\langle 2m_l\tfrac{3}{2}M\mid\tfrac{1}{2}m\rangle Y_{2m_l}(\hat{y}_1)\ . \tag{51}$$

This definition applies to both ^{3}H and ^{3}He, i.e., $i={}^3\mathrm{H}$ or ${}^3\mathrm{He}$.[10,28] The ^{3}H and ^{3}He cases are distinguished through the asymptotic behaviors of the $u_l(y_1)$:

$(i={}^3\mathrm{H})$,

$$u_0(y_1)\underset{y_1\to\infty}{\longrightarrow} C_S N_{ZR}\frac{e^{-\beta y_1}}{y_1}\ , \tag{52}$$

$$u_2(y_1)\underset{y_1\to\infty}{\longrightarrow} C_D N_{ZR}\frac{e^{-\beta y_1}}{y_1}\left[1+\frac{3}{\beta y_1}+\frac{3}{\beta^2 y_1{}^2}\right]\ , \tag{53}$$

and

$(i={}^3\mathrm{He})$,

$$u_0(y_1)\underset{y_1\to\infty}{\longrightarrow} C_S^C N_W\frac{W_{-\kappa,1/2}(2\beta y_1)}{y_1}\ , \tag{54}$$

$$u_2(y_1)\underset{y_1\to\infty}{\longrightarrow} C_D^C N_W\frac{W_{-\kappa,5/2}(2\beta y_1)}{y_1}\ . \tag{55}$$

Then, to the extent that the $y_1{}^4$ in the integrand of the numerator and the $y_1{}^2$ in the integrand of the denominator justify replacing the $u_l(y_1)$ by their asymptotic forms, we can derive[29]

$$D_2\cong-\frac{C_D}{\beta^2 C_S} \tag{56}$$

and

$$D_2^C\cong-\frac{C_D^C}{\beta^2 C_S^C}f(\kappa)\ , \tag{57}$$

where

$$f(\kappa)=6\frac{{}_2F_1(2,\kappa-2;5+\kappa;-1)}{(4+\kappa)(3+\kappa){}_2F_1(2,\kappa;3+\kappa;-1)}\underset{\kappa\to 0}{\longrightarrow}1\ . \tag{58}$$

Clearly, within the limits of the approximations leading to Eq. (56) and the approximations in extracting D_2 experimentally, the RSC5 value of $D_2=-0.243$ fm^2 is in agreement with the experimentally extracted value of -0.279 ± 0.012 fm^2. Especially significant is the overall sign. Unfortunately, the experimental values of D_2^C are not as well determined as D_2 (see Table I), so all that we can say is that our theoretical values are not inconsistent with experiment at this stage. Perhaps more important is the theoretical prediction that

$$D_2^C\cong D_2 \tag{59}$$

to better than 1.1%. Furthermore, it is interesting to see how compensating Coulomb factors lead to this result: The reduction in binding energy increases $|D_2^C|$ over $|D_2|$ by 15%, but $f(\kappa)$ reduces the ratio by $\sim$7% *and* the ratio is reduced another $\sim$8% due to $C_D^C<C_D$. Thus, the approximate equality in Eq. (59) results.

The above discussion emphasizes the importance of the need for further experimental work on the various asymptotic norm parameters. At the same time, we cannot expect to find significant Coulomb effects in C_S^C or D_2^C measurements; only C_D is predicted to exhibit any measurable ($\sim$8%) Coulomb effects. It would be especially valuable to have *direct* extractions of C_D and C_D^C from experiment. We also note that realistic force models underbind the triton and, in accordance with our previous discussion, correcting this deficiency may lead to an increase in the calculated values of C_S and C_S^C.

V. DISCUSSION AND CONCLUSIONS

In this paper, we have presented the theory of the trinucleon asymptotic normalization constants, both *S*- and *D*-wave, with and without Coulomb effects. Special emphasis is placed on the importance of defining these constants relative to *normalized* zero-range comparison functions, especially in the Coulomb case where up to now this has not been done. This is particularly important since, otherwise, incorrect conclusions about the relative importance of Coulomb effects can be reached. In addition, through a complete comparison of our five-channel Reid-soft-core (RSC) model results with the "best" available experimental results, it is clear that higher precision measurements of the *S*-wave asymptotic norms are needed along with an accurate measurement of the ^{3}H $\rightarrow n+d$ *S*-wave value to resolve the present conflict between values. Moreover, it would be extremely valuable to have a direct extraction of the *D*-wave asymptotic normalization constants for both ^{3}H and ^{3}He, since Coulomb effects are predicted to be largest in this wave. Besides the above general comments, the main conclusions of this work can be summarized as follows:

(1) *S*-wave asymptotic norms: Coulomb effects lead to less than a 1% *increase* of the ^{3}He asymptotic norm relative to that of ^{3}H. Both the three- and five-channel RSC results are consistent with existing experimental values; in particular, excellent agreement occurs for ^{3}He where the data are most reliable.

(2) *D*-wave asymptotic norms: Coulomb effects lead to an approximately 8% *reduction* of the ^{3}He asymptotic norm relative to that of ^{3}H. No data exist.

(3) DWBA *D*-state parameter D_2: D_2 is predicted to be negative for the five-channel RSC model in agreement with experiment. It is approximately equal to the negative ratio of the *D*-wave to *S*-wave asymptotic norms divided by the mass times the nucleon-deuteron separation energy in fm^{-2}. Therefore, the *D*-wave asymptotic norm is positive relative to the *S* wave. The predicted values of D_2 are consistent with experiment; especially for ^{3}H, where the experimental value is best known, the agreement is excellent within the approximations involved. Due to compensating factors, we predict that D_2 for ^{3}He is *reduced* in magnitude by only approximately 1% compared to that of ^{3}H.

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APPENDIX A: EQUIVALENCE OF COULOMB FORMALISMS IN THE DERIVATION OF INTEGRAL RELATIONS FOR ^{3}He ASYMPTOTIC NORMALIZATION CONSTANTS

As mentioned near the end of Sec. I A, there are infinitely many decompositions of the Schrödinger equation leading to equivalent coupled equations for the three Faddeev components of $\psi(\overline{12},3)$ when the Coulomb interaction is present. They can be viewed as lying between the extremes of Eqs. (9) and the Sasakawa-Sawada form[22] as given in Eq. (32) with Eqs. (9) being the Coulomb formulation without distortion, the Sasakawa-Sawada form being full distortion, and the intermediate cases being partial distortion. Specifically, we introduce a parameter p, $0 \leq p \leq 1$, and write

$$\Psi(\overline{12},3)=\chi_1^p+\chi_2^p+\eta^p \,, \tag{A1}$$

where

$$\left[E-H_0-V_{23}-pV_{12}^C-(1-p)\frac{\alpha}{y_1}\right]\chi_1^p=V_{23}(\chi_2^p+\eta^p) \,, \tag{A2a}$$

$$\left[E-H_0-V_{31}-pV_{12}^C-(1-p)\frac{\alpha}{y_2}\right]\chi_2^p=V_{31}(\chi_1^p+\eta^p) \,, \tag{A2b}$$

$$[E-H_0-V_{12}-V_{12}^C]\eta^p=V_{12}(\chi_1^p+\chi_2^p)+(1-p)\left[V_{12}^C-\frac{\alpha}{y_1}\right]\chi_1^p+(1-p)\left[V_{12}^C-\frac{\alpha}{y_2}\right]\chi_2^p . \tag{A2c}$$

Clearly, Eqs. (9) are recovered when $p=1$ and the Sasakawa-Sawada equations correspond to $p=0$. Furthermore, it is clear from Eq. (A2a) that we would expect

$$C_t^C=-\frac{4M}{3}\frac{\sqrt{4\pi}}{N_W}\frac{\Gamma(1+\kappa)}{2\beta}\int_0^\infty y_1dy_1M_{-\kappa,1/2}(2\beta y_1)\times\left\{\langle\Phi_d\vec{y}_1\,|\,V_{23}\,|\,\chi_2^p+\eta^p\rangle+p\left\langle\Phi_d\vec{y}_1\left|\frac{\alpha}{x_3}-\frac{\alpha}{y_1}\right|\chi_1^p\right\rangle\right\} . \tag{A3}$$

Thus, in the Sasakawa-Sawada formulation there is no polarization term in the integral relation for C_t^C. This last statement hinges on the assumption that it is sufficient to consider only the χ_1^p equation ($p=0$) in deriving C_t^C. This is not obvious for $p<1$ due to the presence of the polarization terms on the right-hand side of the η^p equation. The purpose of this appendix is to prove that Eq. (A3) is the correct expression for C_t^C when $0\le p\le 1$. The proof is equivalent to showing that the two polarization terms on the right-hand side of the η^p equation go to zero faster than y_1^{-1} in the limit $y_1\to\infty$; i.e., the χ_1^p and χ_2^p functions are critical to the discussion.

The correctness of Eq. (A3) will be demonstrated by starting from the $p=1$ expression and showing that it can be transformed into the $p=0$ case. For the $p=1$ case, we use $\chi_1{}^1=\chi_1$, etc., as in Sec. I A, and for the $p=0$ case $\chi_1^0=\tilde{\chi}_1$, etc. Then, by comparing Eqs. (A2) with Eqs. (9), we get

$$\chi_1=\tilde{\chi}_1+\frac{1}{E-H_0-\alpha/x_3}\left[\frac{\alpha}{x_3}-\frac{\alpha}{y_1}\right]\tilde{\chi}_1 , \tag{A4a}$$

$$\chi_2=\tilde{\chi}_2+\frac{1}{E-H_0-\alpha/x_3}\left[\frac{\alpha}{x_3}-\frac{\alpha}{y_2}\right]\tilde{\chi}_2 , \tag{A4b}$$

$$\eta=\tilde{\eta}-\frac{1}{E-H_0-\alpha/x_3}\left[\frac{\alpha}{x_3}-\frac{\alpha}{y_1}\right]\tilde{\chi}_1-\frac{1}{E-H_0-\alpha/x_3}\left[\frac{\alpha}{x_3}-\frac{\alpha}{y_2}\right]\tilde{\chi}_2 . \tag{A4c}$$

Equations (A4) permit us to derive

$$\langle\Phi_d\vec{y}_1\,|\,V_{23}\,|\,\chi_2+\eta\rangle+\left\langle\Phi_d\vec{y}_1\left|\frac{\alpha}{x_3}-\frac{\alpha}{y_1}\right|\chi_1\right\rangle=\langle\Phi_d\vec{y}_1\,|\,V_{23}\,|\,\tilde{\chi}_2+\tilde{\eta}\rangle-\left[H_{0_{y_1}}+B_{pd}+\frac{\alpha}{y_1}\right]g(y_1) , \tag{A5}$$

where

$$g(y_1)=\langle\Phi_d\vec{y}_1\,|\,\chi_1-\tilde{\chi}_1\rangle \tag{A6}$$

$$=\left\langle\Phi_d\vec{y}_1\left|\frac{1}{E-H_0-\alpha/x_3}\left[\frac{\alpha}{x_3}-\frac{\alpha}{y_1}\right]\right|\tilde{\chi}_1\right\rangle \tag{A7}$$

$$=\left\langle\Phi_d\vec{y}_1\left|\frac{1}{E-H_0-\alpha/y_1}\left[\frac{\alpha}{x_3}-\frac{\alpha}{y_1}\right]\right|\chi_1\right\rangle . \tag{A8}$$

The equivalence of the $p=1$ and $p=0$ equations is proved by showing that the integral

$$I=\int_0^\infty y_1dy_1M_{-\kappa,1/2}(2\beta y_1)\times\left[H_{0_{y_1}}+B_{pd}+\frac{\alpha}{y_1}\right]g(y_1) \tag{A9}$$

vanishes.

By transferring the Hamiltonian operator onto the Whittaker function, we obtain

$$I=-\frac{3}{4M}\int_0^\infty y_1dy_1\left[\left[\frac{d^2}{dy_1{}^2}-\beta^2-\frac{2\kappa\beta}{y_1}\right]M_{-\kappa,1/2}(2\beta y_1)\right]g(y_1)+y_1M_{-\kappa,1/2}(2\beta y_1)\frac{dg(y_1)}{dy_1}\bigg|_0^\infty-y_1{}^2\frac{d}{dy_1}\left[\frac{M_{-\kappa,1/2}(2\beta y_1)}{y_1}\right]g(y_1)\bigg|_0^\infty . \tag{A10}$$

The integral term on the right-hand side of Eq. (A10) vanishes identically, because the expression in square brackets is the Whittaker-function differential equation and thus $[\cdots]\equiv 0$. The two boundary terms are the key to the proof. Now it is known that

$$M_{-\kappa,1/2}(2\beta y_1) \underset{y_1\to 0}{\longrightarrow} 2\beta y_1 , \tag{A11}$$

$$M_{-\kappa,1/2}(2\beta y_1) \underset{y_1\to\infty}{\longrightarrow} \frac{(2\beta y_1)^{\kappa}}{\Gamma(1+\kappa)} e^{\beta y_1} , \tag{A12}$$

$$\frac{d[M_{-\kappa,1/2}(2\beta y_1)/y_1]}{dy_1} \underset{y_1\to 0}{\longrightarrow} 2\beta^2\kappa , \tag{A13}$$

and

$$\frac{d[M_{-\kappa,1/2}(2\beta y_1)/y_1]}{dy_1} \underset{y_1\to\infty}{\longrightarrow} \frac{\beta(2\beta)^{\kappa} e^{\beta y_1}}{\Gamma(1+\kappa) y_1^{1-\kappa}} . \tag{A14}$$

Furthermore, it is possible to show by means of Eq. (A8) and the explicit form[30] of the Coulomb Green's function for the operator $(E-H_0-\alpha/y_1)^{-1}$ that

$$g(0) < \infty , \tag{A15}$$

$$\left.\frac{dg(y_1)}{dy_1}\right|_{y_1=0} < \infty , \tag{A16}$$

$$g(y_1) \underset{y_1\to\infty}{\longrightarrow} \text{constant} \times \frac{e^{-\bar{\beta} y_1}}{y_1^{5/2}} , \tag{A17}$$

and

$$\frac{dg(y_1)}{dy_1} \underset{y_1\to\infty}{\longrightarrow} -\bar{\beta} \times \text{constant} \times \frac{e^{-\bar{\beta} y_1}}{y_1^{5/2}} , \tag{A18}$$

where $\bar{\beta} > \beta$ and explicitly $\bar{\beta}^2 = 2\mu_y(B_d + B_{pd})$. With the facts given in the last eight equations, clearly the boundary terms vanish and

$$I \equiv 0 . \tag{A19}$$

Therefore, the $p=1$ and $p=0$ expressions for C_t^C in Eq. (A3) are equivalent. This obviously generalizes for any $p \in [0,1]$ and our proof is complete.

Physically, the equivalence of the undistorted decomposition and distorted decomposition means that more and more of the polarization term is imbedded in the first term of Eq. (A3) through the functions χ_2^p and η^p until $p=0$; then *all* of the polarization contribution is "hidden" in the first term.

APPENDIX B: COMPUTATION OF N_W

The following series approximation is useful in computing N_W/N_{ZR}:

$$R_C \equiv N_W/N_{ZR} = \left[\frac{\Gamma(3+\kappa)\Gamma(2+\kappa)}{2\,{}_3F_2(\kappa,2,1+\kappa;3+\kappa,2+\kappa;1)} \right]^{1/2} \tag{B1}$$

$$\cong 1 + (1-C)\kappa + \frac{\kappa^2}{12}[6(1-C)^2 - \pi^2 + 12] + \frac{\kappa^3}{12}\left\{ (1-C)[2(1-C)^2 - \pi^2 + 12] - 4\pi^2 + 20\zeta(3) + 16 \right\} + \cdots , \tag{B2}$$

where C is Euler's constant $(=0.577\,215\,664\,9\ldots)$ and ζ is the zeta function $[\zeta(3) = 1.202\,056\,903\,1\ldots]$. The exact value of R_C compared to the 2-term (through κ^2) and 3-term (through κ^3) series approximations are given for a representative value of $\kappa(=\frac{1}{16})$ by

$$R_C = 1.027\,499\,66 \quad \text{exact} , \tag{B3a}$$

$$\cong 1.027\,466\,62 \quad \text{2-term} , \tag{B3b}$$

$$\cong 1.027\,499\,47 \quad \text{3-term} . \tag{B3c}$$

Clearly (B2) is an adequate approximation under virtually all circumstances, while the 2-term approximation is adequate for most applications.

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$$\left[\frac{\Gamma(3+\kappa)\Gamma(2+\kappa)}{2\,{}_3F_2(\kappa,2,1+\kappa;3+\kappa,2+\kappa;1)}\right]^{1/2}$$

[see Eq. (17) of the present paper] into their definition of the asymptotic norms. For ^{3}He, where $\kappa=0.0551$, this factor has the value 1.0241. We caution the reader that Locher and Mizutani, Ref. 4, have adopted the Plattner *et al.* convention. However, we *strongly recommend* that all asymptotic normalization constants be defined relative to zero-range comparison functions normalized to unity whether the Coulomb interaction is present or not, to insure overall consistency.

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[21]Specifically, replacement of $W_{-\kappa,1/2}(2\beta y_1)$ by $(2\beta y_1)^{-\kappa}e^{-\beta y_1}$ in Eq. (15) requires that N_W be replaced by N'_W, where

$$(N'_W)^{-2}=\int_0^\infty dy_1(2\beta y_1)^{-2\kappa}e^{-2\beta y_1}$$

or

$$N'_W=N_{ZR}/[\Gamma(1-2\kappa)]^{1/2}\,.$$

To understand the error made by not using the proper normalization, suppose κ has its experimental value, i.e., $\kappa=0.0551$. Then, $N'_W/N_W=0.941$, which is a large discrepancy since we find actual Coulomb effects to be small. The treatment of this particular aspect is not clear in the paper of Sasakawa *et al.*, Ref. 15 above.

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[23]G. L. Payne, J. L. Friar, and B. F. Gibson, Phys. Rev. C 22, 832 (1980). We emphasize that the configuration-space equations solved for the ψ_ν^C (i.e., with the Coulomb interaction present) are those derived from various $\langle\nu|$ projections of Eq. (32) in the present text, *not* those given in the Appendix of the above paper. The equations projected from Eq. (32) are of the Sasakawa-Sawada form, Ref. 22. [Our RSC3 result for the point Coulomb ^{3}He binding energy given in Table V agrees precisely with the result of T. Sasakawa, H. Okuno, and T. Sawada, Phys. Rev. C 23, 905 (1981).] Because this form of the equation builds in Coulomb distortion between the proton and deuteron in a more complete way than those in Payne *et al.*, we have adopted this approach in calculating our Coulomb wave functions. The difference between Coulomb energies calculated from perturbation theory and from Faddeev eigenvalues is virtually eliminated using this approach.

[24]The parameters for these potentials are listed in Ref. 20; the reader should note that these are not the same as one might infer from Ref. 2.

[25]It is possible that Sasakawa *et al.* have adopted the convention of Refs. 5 and 7; see Ref. 12 above. If so, their value of 1.765 should be divided by 1.0302 to get 1.713. Then, their predicted Coulomb effects are consistent with ours; namely, an increase of $\lesssim 0.5\%$.

[26]Though the zero-range comparison function is normal-

ized to unity, the effective S-wave nucleon-deuteron wave function is normalized to the fraction of S-wave nucleon-deuteron component in the three-body wave function. This fraction is less than unity; in simple models it could be of the order of 0.5. Likewise for the D-wave component.

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