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Time-dependent scattering theory for identical particles

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A time-dependent multichannel nonrelativistic quantum scattering theory is established for systems which contain identical particles. The symmetry properties of the theory are rooted in the abstract algebraic relationships between permutations. The formalism of second quantization is thus avoided. Appropriate wave and scattering operators are defined, including those for systems with long range (e.g., Coulomb) interactions. A correctly symmetrized dynamical equation for the symmetric form of the transition operator is derived and used to develop an optical potential formalism.

[NUCLEAR REACTIONS Nonrelativistic multichannel scattering theory for identical particles, wave and scattering operators, dynamical equations, optical potentials.]

I. INTRODUCTION

During the past few years considerable progress has been made in the nonrelativistic multichannel quantum scattering theory for systems of distinguishable particles. A time-dependent theory has emerged that is physically transparent and mathematically precise.[1–9] By now there are also several formulations of the exactly equivalent time-independent theory available.[10–26]

On the other hand, the scattering theory for systems containing indistinguishable particles is in a less satisfactory state, in our opinion. It is generally believed that the treatment of particle identity in an N-particle system poses only technical problems. While this is certainly true for bound state problems, the inclusion of exchange symmetry in scattering problems is far from trivial. The reason is that the asymptotic form of the scattering wave function (boundary condition) does not accommodate the permutation symmetry imposed by the symmetry group of the Hamiltonian of the system. As a consequence there is no longer a single channel Hamiltonian which describes the asymptotic waves and the well known problems connected with rearrangement arise. The conventional procedure described in textbooks is to construct properly symmetrized scattering amplitudes from those of distinguishable particle theory.[3,4,26] It is clear, however, that such a procedure does not provide a solution to the physical problem and is therefore unsatisfactory from a fundamental point of view, as well as highly impractical except for Born-type approximations.

Our goal in this paper is to remedy this situation and to bring the scattering theory for indistinguishable particles to a state equally as satisfactory as that of distinguishable particle theory. This is done by combining ideas of Ekstein[1] and Coester and Schlessinger[27] on the time-dependent formalism with ideas embodied in the time-independent formalism of Bencze and Redish.[28] The result is a time-dependent theory that involves only objects with the correct permutation symmetry and that is clearly defined both physically and mathematically. The abstract structure, and hence many desirable concrete structural features, of distinguishable particle theory are preserved. In particular, the transition to time-independent theory can be made in the standard way[29] and the previous time-independent results[28] can be recovered. Our results and their physical interpretation, however straightforward

they may seem, could not have been derived without the modern techniques developed by Bencze[30] and Bencze and Redish.[28,31]

We have organized this paper as follows. In Sec. II the two-Hilbert-space formalism for the scattering of distinguishable particles is recalled. Notation is thereby established. The relevant portions of the theory of permutation symmetry are reviewed in Sec. III. Section IV is devoted to the scattering theory for identical particles. Correctly symmetrized forms of the wave and scattering operators are developed, as is a dynamical equation for the symmetric form of the transition operator. In Sec. V an optical potential formalism is developed along the lines of that of Kowalski and co-workers.[32,33] Concluding remarks are in Sec. VI.

II. TWO-HILBERT-SPACE FORMALISM

In recent years the two-Hilbert-space formulation of nonrelativistic multichannel quantum scattering theory has been extensively developed by several authors. The features of this formalism that are essential for this paper are briefly summarized in this section. For details the reader should consult Refs. 1, 22, 27, 29, and 34.

The formalism of this section refers to systems of distinguishable particles. The dynamics of a system of N distinguishable particles is governed by a Hamiltonian operator H_N. The operator H_N is self-adjoint and is bounded from below on a separable Hilbert space $\mathscr{H}_N$ of N-particle wave functions.

A space $\mathscr{H}$ of asymptotic wave functions is constructed as a direct sum,

$$\mathscr{H}=\bigoplus_A \mathscr{H}_A, \quad (\mathscr{H}_A=P_A\mathscr{H}_N) . \tag{2.1}$$

The $\mathscr{H}_A$ are the arrangement channel (cluster) subspaces of $\mathscr{H}_N$ and the operators P_A are orthogonal projections. The asymptotic dynamics on $\mathscr{H}$ is governed by a unitary operator $e^{iHt}U(t)$, where

$$e^{iHt}U(t)=U(t)e^{iHt} . \tag{2.2}$$

The form of operator $e^{iHt}U(t)$ is specified by the equation

$$e^{iHt}U(t)\Psi=\bigoplus_A e^{iH_At}U_A(t)\psi_A , \tag{2.3}$$

where $\Psi=\oplus_A\psi_A$ is any vector in $\mathscr{H}$. The operators H_A are the arrangement channel (cluster) Hamiltonians. The unitary operators $U_A(t)$ represent the corrections to e^{iH_At} that are necessitated if the interactions have long range.[6–9,29,34] Such corrections are, in particular, necessary for the Coulomb interactions. If there are no long-range interactions, then $U(t)=I$, where I denotes the identity on $\mathscr{H}$. Finally, the commutation relations

$$[H_A,P_A]=0=[U_A(t),P_A] \tag{2.4}$$

hold.

Communication between $\mathscr{H}$ and $\mathscr{H}_N$ is provided by the linear operator $J:\mathscr{H}\to\mathscr{H}_N$,

$$J\Psi=J\bigoplus_A\psi_A\equiv\sum_A\psi_A , \tag{2.5}$$

and its adjoint $J^*:\mathscr{H}_N\to\mathscr{H}$,

$$J^*\psi=\bigoplus_A P_A\psi . \tag{2.6}$$

The multichannel wave operators $\Omega^{\pm}:\mathscr{H}\to\mathscr{H}_N$ are defined by

$$\Omega^{\pm}\equiv \mathrm{s}\text{–}\lim_{t\to\pm\infty} e^{iH_Nt}Je^{-iHt}U(-t) . \tag{2.7}$$

These operators are partially isometric ($\Omega^{+*}\Omega^{+}=I=\Omega^{-*}\Omega^{-}$) and have the intertwining property ($H_N\Omega^{\pm}=\Omega^{\pm}H$). An alternative representation is given by

$$\Omega^{\pm}\Psi=\sum_A\Omega_A^{\pm}\psi , \tag{2.8}$$

where $\Psi=\oplus_A\psi_A$ is any vector in $\mathscr{H}$. The arrangement channel (cluster) wave operators $\Omega_A^{\pm}:\mathscr{H}_A\to\mathscr{H}_N$ are given by

$$\Omega_A^{\pm}=\mathrm{s}\text{–}\lim_{t\to\pm\infty} e^{iH_Nt}P_Ae^{-iH_At}U_A(t) . \tag{2.9}$$

The partial isometry of $\Omega^{\pm}$ is equivalent to the property

$$\Omega_B^{+*}\Omega_A^{+}=\Omega_B^{-*}\Omega_A^{-}=\delta_{BA}P_A , \tag{2.10}$$

where δ_{BA} is the Kronecker delta. The intertwining property of $\Omega^{\pm}$ is equivalent to the property

$$H_N\Omega_A^{\pm}=\Omega_A^{\pm}H_A . \tag{2.11}$$

The multichannel scattering operator $S:\mathscr{H}\to\mathscr{H}$ is defined by

$$S=\Omega^{+*}\Omega^{-} . \tag{2.12}$$

An alternative representation is

$$S\Psi=\bigoplus_B\left[\sum_A S_{BA}\psi_A\right] , \tag{2.13}$$

where $\Psi=\oplus_A\psi_A$ is any vector in $\mathscr{H}$ and $S_{BA}=\Omega_B^{+*}\Omega_A^{-}$.

Consider now systems in which all interactions have short range, so that $U(t)=I$. The scattering

operator S then has a time-independent representation in terms of the usual transition operators.[22,29] From the various forms of the transition operators we choose the symmetric form

$$T(z)\equiv(z-H)J^*(z-H_N)^{-1}J(z-H)-(z-H)\ , \tag{2.14}$$

where $\mathrm{Im}z\neq 0$ has been assumed. The operator $T(z)$ satisfies the dynamical equation

$$T(z)=(J^*J-1)(z-H)+W+WR(z)T(z)\ , \tag{2.15}$$

where $R(z)\equiv(z-H)^{-1}$, and where W satisfies

$$WJ^*=J^*H_N-HJ^*\ . \tag{2.16}$$

Equation (2.14) can also be written in terms of the arrangement channel (cluster) matrix elements $T_{BA}(z)$,

$$T_{BA}(z)=(1-\delta_{BA})P_AP_B(z-H_A)+W_{BA}+\sum_C W_{BC}(z-H_C)^{-1}T_{CA}(z)\ , \tag{2.17}$$

where δ_{BA} is the Kronecker delta.

Different specific versions of this abstract formalism are generated by different choices of H and W. In the Chandler-Gibson (CG) theory[22,29] the spaces $\mathscr{H}_A$ are true channel subspaces in which the bound state wave functions of the bound clusters are incorporated. The operator W in this case is defined by

$$(W^{\mathrm{CG}})_{BA}=P_B(H_N-H_B)\left[\sum_C P_C\right]^{-1}P_A\ . \tag{2.18}$$

In the Bencze-Redish-Sloan (BRS) theory[16,17] the spaces $\mathscr{H}_A$ are each identical to $\mathscr{H}_N$, so that $P_A=I_N$, the identity on $\mathscr{H}_N$. The operator W is defined by

$$(W^{\mathrm{BRS}})_{BA}=(-1)^{m_A}(m_A-1)!V_A^B\ , \tag{2.19}$$

where m_A is the number of clusters in arrangement A and V_A^B is the sum over all interactions that are in H_A but not in H_B. The Kouri-Levin-Tobocman (KLT) theory[19,20] is obtained by setting $\mathscr{H}_A=\mathscr{H}_N$ and restricting the index A to those arrangements with only two clusters. The operator W in this theory has the form

$$(W^{\mathrm{KLT}})_{BA}=(H_N-H_B)\omega_{BA}\ , \tag{2.20}$$

where ω_{BA} is a numerical valued matrix with the property

$$\sum_A \omega_{BA}=1\ . \tag{2.21}$$

Because our abstract formalism embraces all of these specific theories, our conclusions in this paper will apply to all of them.

III. PERMUTATION SYMMETRY IN N-PARTICLE SYSTEMS

In quantum systems identical particles are indistinguishable so that any permutation of them cannot affect any physical property of the system. Consequently, the permutation of identical particles is a symmetry operation and the set of all such permutations forms a symmetry group of the system's Hamiltonian, i.e., the permutation group of the system. Therefore, if in an N-particle system some or all the particles are identical, certain symmetry requirements have to be fulfilled by the treatment of the dynamics. The algebraic properties of permutation symmetry have been discussed extensively in previous works[28,30] so that here we only summarize the most important results to be used.

Let $\mathscr{S}$ denote the permutation group of the system. If all the particles are identical, then $\mathscr{S}\equiv S_N$ is the full symmetric group of N objects. In case there are different sorts of identical particles present, clearly $\mathscr{S}$ will be (isomorphic to) the direct product of several symmetric groups. The permutation symmetry of the system implies that

$$[H_N,p]=0,\ \forall p\in\mathscr{S}\ , \tag{3.1}$$

where p denotes the linear unitary operator associated with the permutation. Since linear unitary operator representations of $\mathscr{S}$ can always be constructed, in the following we shall denote both the permutation and its unitary operator representative by the same symbol.

It is convenient to introduce the operators

$$\hat{p}=f_pp\ , \tag{3.2}$$

where the phase factor f_p is -1 if p involves an odd number of fermion permutations and $+1$ otherwise. In this way bosons and fermions can be treated in a unified manner.

The symmetrization postulate for physically admissible wave functions[30,35] can then be formulated as the requirement that

$$\hat{p}\psi=\psi,\ \forall p\in\mathscr{S}\ . \tag{3.3}$$

As is well known, the Young symmetrizer of the group S,

$$R_N^{\mathscr{S}}=|\mathscr{S}|^{-1}\sum_{p\in\mathscr{S}}\hat{p}\ , \tag{3.4}$$

where $|\mathscr{S}|$ denotes the order of the group, is an idempotent element of the group algebra,

$$(R_N^{\mathscr{S}})^2=R_N^{\mathscr{S}}=R_N^{\mathscr{S}*}\ , \tag{3.5}$$

with the important property $\hat{p}R_N^{\mathscr{S}}=R_N^{\mathscr{S}}$. Using (3.3) and (3.5) the symmetrization postulate can be reformulated as the requirement that each physically admissible wave function of the system should transform according to a one-dimensional irreducible representation of the permutation group $\mathscr{S}$. Wave functions of physical interest are, therefore, those which belong to the set $R_N^{\mathscr{S}}\mathscr{H}_N$.

The asymptotic states of a multiparticle system can be characterized in terms of partitions (clusterings) of the particles. Consequently, in the various formulations of multiparticle scattering theory the relevant quantities are usually labeled by partitions or chains of partitions. Let $\mathscr{A}$ denote the set of these labels α,

$$\mathscr{A}=\{\alpha\}\ . \tag{3.6}$$

Then it is clear that the permutation group $\mathscr{S}$ of the system maps $\mathscr{A}$ into itself. Depending on the way of labeling, some permutations leave a label α unchanged. The set of such permutations,

$$\mathscr{S}_\alpha=\{p\in\mathscr{S}\,|\,p\alpha=\alpha\}\ , \tag{3.7}$$

form a subgroup $\mathscr{S}_\alpha\subseteq\mathscr{S}$ of the permutation group. The corresponding Young operator

$$R^{\mathscr{S}_\alpha}=|\mathscr{S}_\alpha|^{-1}\sum_{p\in\mathscr{S}_\alpha}\hat{p} \tag{3.8}$$

is an idempotent element of the group algebra so that its operator representation is a projection.

If α is a partition label A, then clearly $\mathscr{S}_A$ is a symmetry group of the channel Hamiltonian H_A, as well as (in case of long-range interactions) the operator family $U_A(t)$,

$$[H_A,p]=[U_A(t),p]=0,\ \forall p\in\mathscr{S}_A\ . \tag{3.9}$$

For an N-particle system that contains identical particles, the symmetry is carried by the permutation group $\mathscr{S}$ of the system. Since the permutation of identical particles, even if it changes labels, does not change any physical property of the system, quantities labeled by α and α' with $\alpha'=p\alpha$, $p\in\mathscr{S}$, are physically equivalent. It is easy to see[30] that the binary relation

$$\alpha'\mathscr{R}\alpha,\ \text{iff}\ \exists\ p\in\mathscr{S},\ p\alpha=\alpha' \tag{3.10}$$

is an equivalence relation that splits up the label set $\mathscr{A}$ into equivalence classes. From the above considerations it follows that the physically relevant information is carried by quantities which depend only on the equivalence classes. Let $[\alpha]$ denote the equivalence class to which α belongs. By Lagrange's theorem[30] the number of elements of the equivalence class $[\alpha]$ is given by

$$N_{[\alpha]}=\frac{|\mathscr{S}|}{|\mathscr{S}_\alpha|}\ , \tag{3.11}$$

where $\mathscr{S}_\alpha$ is the subgroup defined by (3.7). The label α in $[\alpha]$ can be arbitrarily chosen since if $\alpha'\mathscr{R}\alpha$ then necessarily $\mathscr{S}_\alpha$ and $\mathscr{S}_{\alpha'}$ are conjugate subgroups ($\mathscr{S}_{\alpha'}=p\mathscr{S}_\alpha p^{-1}$ for some $p\in\mathscr{S}$). From the above results follow the important factorization formulas

$$R_N^{\mathscr{S}}=\frac{1}{N_{[\alpha]}}\sum_{\alpha'\in[\alpha]}\hat{p}_{\alpha'\alpha}R^{\mathscr{S}_\alpha}\ , \tag{3.12}$$

$$R_N^{\mathscr{S}}=\frac{1}{N_{[\alpha]}}\sum_{\alpha'\in[\alpha]}R^{\mathscr{S}_\alpha}\hat{p}_{\alpha\alpha'}\ , \tag{3.13}$$

where $p_{\alpha'\alpha}$ denotes the permutation which maps α into α' and $\{p_{\alpha\alpha'}\}$ is a transversal of the subgroup $\mathscr{S}_\alpha$. Since the choice of transversals is not unique, it is often convenient to introduce canonical labels α^0 to fix the representation.

It is important to study how the quantities labeled by partitions or chains of partitions transform under permutations of the particles. The operators and other relevant quantities typical to N-particle scattering theories are uniquely determined by giving the partitions or the labels of the system. If a permutation of the particles changes the labels, the corresponding quantities change as well. The simplest, and most natural, transformation properties are

$$\begin{aligned}&pM^\alpha p^{-1}=M^{p\alpha}\ ,\\ &pM^{\alpha\beta}p^{-1}=M^{p\alpha,p\beta}\ ,\end{aligned} \tag{3.14}$$

with $p\in\mathscr{S}$ and $\alpha\in\mathscr{A}$. The symbols M^α and $M^{\alpha\beta}$ denote arbitrary operators characterized by one or two labels. The class of operators satisfying Eq. (3.14) is called "label transforming" and plays a very important role in the multiparticle scattering theory of identical particles. In subsequent sections the labels α will always be taken to be the partition label A of the arrangement channels (clusterings).

IV. SCATTERING THEORY FOR IDENTICAL PARTICLES

A. The operator $Q^{\mathscr{S}}$

The following label transforming properties are essential to the structure of the theory:

$$p_{BA}R^{\mathscr{S}_A}=R^{\mathscr{S}_B}p_{BA}\ , \tag{4.1}$$

$$p_{BA}H_A=H_Bp_{BA}\ , \tag{4.2}$$

$$p_{BA}U_A(t)=U_B(t)p_{BA}\ , \tag{4.3}$$

$$p_{BA}P_A=P_Bp_{BA}\ , \tag{4.4}$$

$$R^{\mathscr{S}_A}\mathscr{H}_A=\mathscr{H}_A\ . \tag{4.5}$$

Equation (4.1) is a consequence of the definition[30] of $R^{\mathscr{S}_A}$. Equations (4.2) and (4.3) are consequences of the commonly assumed forms of H_A and $U_A(t)$.[6–9,29,36]. Equation (4.4) is a straightforward relation between spaces and does not imply any symmetry of particluar functions under the action of $\mathscr{S}$. Equation (4.5) is automatic for the BRS and KLT theories, but it implies in the CG theory that the bound state wave functions of the clusters are invariant under the actions of $\mathscr{S}_A$. We adopt Eqs. (4.1)–(4.5) as basic assumptions.

Consider now the operators $Q^{\mathscr{S}}_{BA}$ defined by

$$Q^{\mathscr{S}}_{BA}\equiv N^{-1}_{[A]}\hat{p}_{BA}R^{\mathscr{S}_A}\ ,\ \text{if } B\in[A]\ , \tag{4.6}$$

$$\equiv 0\ ,\ \text{if } B\notin[A]\ . \tag{4.7}$$

It is clear from Eqs. (4.4) and (4.5) that $Q^{\mathscr{S}}_{BA}$ maps $\mathscr{H}_A$ into $\mathscr{H}_B$. The operator $Q^{\mathscr{S}}$, defined by

$$Q^{\mathscr{S}}\Psi=Q^{\mathscr{S}}\bigoplus_A\psi_A\equiv\bigoplus_B\left[\sum_A Q^{\mathscr{S}}_{BA}\psi_A\right], \tag{4.8}$$

is therefore a mapping from $\mathscr{H}$ into $\mathscr{H}$. Equations (3.8), (3.9), (4.2), and (4.3) imply

$$Q^{\mathscr{S}}_{BA}H_A=H_BQ^{\mathscr{S}}_{BA} \tag{4.9}$$

and

$$Q^{\mathscr{S}}_{BA}U_A(t)=U_B(t)Q^{\mathscr{S}}_{BA}\ . \tag{4.10}$$

Consequently,

$$[Q^{\mathscr{S}},H]=0=[Q^{\mathscr{S}},U(t)]\ . \tag{4.11}$$

Since the operators $\hat{p}$ form a unitary representation of the group $\mathscr{S}$, it follows that $\hat{p}^*_{BA}=\hat{p}_{AB}$ and, hence, that $Q^{\mathscr{S}*}_{BA}=Q^{\mathscr{S}}_{AB}$. This implies that $Q^{\mathscr{S}}$ is self-adjoint. We now consider

$$(Q^{\mathscr{S}}Q^{\mathscr{S}})_{BA}=\sum_C Q^{\mathscr{S}}_{BC}Q^{\mathscr{S}}_{CA}\ . \tag{4.12}$$

The right side of Eq. (4.12) is zero unless $B\in[A]$. If $B\in[A]$, then

$$\sum_C Q^{\mathscr{S}}_{BC}Q^{\mathscr{S}}_{CA}=N_{[A]}{}^{-2}\sum_{C\in[A]}\hat{p}_{BC}R^{\mathscr{S}_C}\hat{p}_{CA}R^{\mathscr{S}_A}\ , \tag{4.13}$$

$$=N_{[A]}{}^{-2}\sum_{C\in[A]}\hat{p}_{BA}R^{\mathscr{S}_A}\ . \tag{4.14}$$

In going from Eq. (4.13) to (4.14) we have made use of Eq. (4.1) and the group property of $\mathscr{S}$. Since $N_{[A]}$ is the number of members of the equivalence class $[A]$, it follows immediately that the right side of Eq. (4.14) is just $Q^{\mathscr{S}}_{BA}$. Consequently, $(Q^{\mathscr{S}})^2=Q^{\mathscr{S}}$ and $Q^{\mathscr{S}}$ is an orthogonal projection on $\mathscr{H}$.

Suppose now that $\Psi=\bigoplus_A\psi_A$ belongs to $\mathscr{H}$ and consider

$$JQ^{\mathscr{S}}\Psi=\sum_B\left[\sum_A Q^{\mathscr{S}}_{BA}\psi_A\right], \tag{4.15}$$

$$=\sum_A\left[\sum_B Q^{\mathscr{S}}_{BA}\right]\psi_A\ . \tag{4.16}$$

Equation (3.12), with $\alpha=A$, and Eqs. (4.6) and (4.7) imply that

$$\sum_B Q^{\mathscr{S}}_{BA}=R^{\mathscr{S}}_N\ . \tag{4.17}$$

Combining Eqs. (4.16) and (4.17) yields the fundamental result

$$R^{\mathscr{S}}_NJ=JQ\ . \tag{4.18}$$

The operator $Q^{\mathscr{S}}$ has another important representation. For each equivalence class $[A]$ let a canonical label A^0 be chosen. Define the direct sum Hilbert space

$$\mathscr{F}\equiv\bigoplus_{A^0}\mathscr{H}_{A^0}\ . \tag{4.19}$$

Next define the linear operator $\rho:\mathscr{F}\to\mathscr{H}$ through the following equations.

$$\rho\Phi=\rho\bigoplus_{A^0}\phi_{A^0}\equiv\oplus_B\left[\sum_{A^0\in[B]}\rho_{BA^0}\phi_{A^0}\right]. \tag{4.20}$$

$$\rho_{BA^0}\equiv N^{-1/2}_{[B]}\hat{p}_{BA^0}R^{\mathscr{S}^0_A}\ ,\ \text{if } A^0\in[B]\ , \tag{4.21}$$

$$\equiv 0\ ,\ \text{if } A^0\notin[B]\ . \tag{4.22}$$

It is straightforward to prove that

$$\rho\rho^*=Q^{\mathscr{S}}\ \text{and}\ \rho^*\rho=R^{\mathscr{S}}\ , \tag{4.23}$$

where

$$R^{\mathscr{S}}\Phi=R^{\mathscr{S}}\bigoplus_{A^0}\phi_{A^0}\equiv\bigoplus_{A^0}R^{\mathscr{S}^0_A}\phi_{A^0}\ . \tag{4.24}$$

A further consequence of Eq. (4.23) is that

$$Q^{\mathscr{S}}\rho=\rho\ \text{and}\ \rho^*Q^{\mathscr{S}}=\rho^*\ . \tag{4.25}$$

The operators $Q^{\mathscr{S}}$ and ρ play a central role in subsequent developments in this paper.

B. Wave and scattering operators

Where some or all of the particles of the system are identical, the wave operators appropriate to the system are $R_N^{\mathscr{S}}\Omega^{\pm}$.[4,27] A straightforward consequence of Eqs. (2.7), (3.1), (4.11), and (4.18) is that

$$R_N^{\mathscr{S}}\Omega^{\pm}=\Omega^{\pm}Q^{\mathscr{S}} . \tag{4.26}$$

The asymptotic wave functions of the system must therefore lie in the space $Q^{\mathscr{S}}\mathscr{H}$. It is convenient to represent these states in terms of wave functions in the space $\mathscr{F}$. Wave operators $\Omega^{\mathscr{S}\pm}:\mathscr{F}\to R_N^{\mathscr{S}}\mathscr{H}_N$ are therefore defined by

$$\Omega^{\mathscr{S}\pm}\equiv\Omega^{\pm}\rho . \tag{4.27}$$

By the partial isometry of $\Omega^{\pm}$ and Eq. (4.23),

$$\Omega^{\mathscr{S}+*}\Omega^{\mathscr{S}+}=\Omega^{\mathscr{S}-*}\Omega^{\mathscr{S}-}=R^{\mathscr{S}} , \tag{4.28}$$

so that the wave operators $\Omega^{\mathscr{S}\pm}$ are partial isometries with initial space $R^{\mathscr{S}}\mathscr{F}$. The scattering operator $S^{\mathscr{S}}:\mathscr{F}\to\mathscr{F}$ is defined by

$$S^{\mathscr{S}}\equiv\Omega^{\mathscr{S}+*}\Omega^{\mathscr{S}-} . \tag{4.29}$$

Equation (4.27) can be written in terms of arrangement channel (cluster) matrix elements. Thus,

$$\Omega^{\mathscr{S}\pm}\Phi=\Omega^{\mathscr{S}\pm}\bigoplus_{A^0}\phi_{A^0}=\sum_{A^0}\Omega_{A^0}^{\mathscr{S}\pm}\phi_{A^0} , \tag{4.30}$$

where

$$\Omega_{A^0}^{\mathscr{S}\pm}=\sum_B\Omega_B^{\pm}\rho_{BA^0} . \tag{4.31}$$

An elementary consequence of Eqs. (2.9), (3.1), (3.9), (4.2), and (4.4) is that $\Omega_B^{\pm}$ satisfies

$$\Omega_B^{\pm}\rho_{BA^0}=\rho_{BA^0}\Omega_{A^0}^{\pm} . \tag{4.32}$$

By Eq. (4.21), therefore,

$$\Omega_{A^0}^{\mathscr{S}\pm}=N_{[A^0]}^{-1/2}\sum_{B\in[A^0]}\Omega_B^{\pm}\hat{p}_{BA^0}R^{\mathscr{S}}{}_{A^0}$$

$$=N_{[A^0]}^{1/2}R_N^{\mathscr{S}}\Omega_{A^0}^{\pm} . \tag{4.33}$$

The content of Eq. (4.33) is familiar.[28]

Equation (4.33) can be used to derive similar formulas for the matrix elements $S_{B^0A^0}^{\mathscr{S}}$ of $S^{\mathscr{S}}$. The results are well known relations,[28] an example being

$$S_{B^0A^0}^{\mathscr{S}}=N_{[B^0]}^{-1/2}N_{[A^0]}^{1/2}\sum_{B\in[B^0]}R^{\mathscr{S}}{}_{B^0}\hat{p}_{B^0B}S_{BA^0} . \tag{4.34}$$

Another interesting property of $S^{\mathscr{S}}$ follows in an elementary way from Eq. (4.25). The property is that

$$S^{\mathscr{S}*}S^{\mathscr{S}}=\rho^*S^*S\rho \text{ and } S^{\mathscr{S}}S^{\mathscr{S}*}=\rho^*SS^*\rho . \tag{4.35}$$

It follows that if S is unitary on $\mathscr{H}$, then $S^{\mathscr{S}}$ is unitary on $R^{\mathscr{S}}\mathscr{F}$. Thus, the symmetrization required by indistinguishability of particles, a nondynamical symmetry, does not destroy the unitarity of the scattering operator, an important dynamical symmetry.

The symmetrization also preserves the asymptotic completeness property.[34]

We now define unitary operators e^{iFt} and $V(t)$ on $\mathscr{F}$:

$$e^{iFt}\Phi=e^{iFt}\bigoplus_{A^0}\phi_{A^0}$$

$$\equiv\bigoplus_{A^0}e^{iH_{A^0}t}\phi_{A^0} , \tag{4.36}$$

and

$$V(t)\Phi=V(t)\bigoplus_{A^0}\phi_{A^0}$$

$$\equiv\bigoplus_{A^0}U_{A^0}(t)\phi_{A^0} . \tag{4.37}$$

It follows that e^{iFt} and $V(t)$ have the properties

$$[e^{iFt},V(t)]=0 , \tag{4.38}$$

$$\rho e^{iFt}=e^{iHt}\rho , \tag{4.39}$$

$$\rho V(t)=U(t)\rho . \tag{4.40}$$

Equations (2.7), (4.27), and (4.38)—(4.40) can be combined to yield the important equation

$$\Omega^{\mathscr{S}\pm}=\operatorname*{s-lim}_{t\to\pm\infty}e^{iH_Nt}J\rho e^{-iFt}V(t) . \tag{4.41}$$

Furthermore, by Eq. (4.39) and the intertwining property for $\Omega^{\pm}$, relation $H_N\Omega^{\mathscr{S}\pm}=\Omega^{\mathscr{S}\pm}F$ holds. When the interactions have short range, the cluster properties of $\Omega^{\mathscr{S}\pm}$ and $S^{\mathscr{S}}$ have also been verified.[36]

The form of Eq. (4.41) is the same as that of Eq. (2.7). The difference lies in the fact that in Eq. (4.41) entire equivalence classes (represented by canonical members A^0) are treated as single physical channels. This is in complete agreement with the notion that these equivalence classes are the

fundamental physical entities.

The foregoing results imply that the scattering theory for systems with identical particles has the same abstract structure as that for systems of distinguishable particles. To pass from the distinguishable particle theory to the identical particle theory one replaces $\mathscr{H}$ by $\mathscr{F}$, H by F, $U(t)$ by $V(t)$, and J by

$$J^{\mathscr{S}} \equiv J\rho \ . \tag{4.42}$$

C. Transition operators

According to Secs. II and IV B, the symmetric transition operator that corresponds to $S^{\mathscr{S}}$ when the interactions have short range is

$$T^{\mathscr{S}}(z) \equiv (z-F)J^{\mathscr{S}*}(z-H_N)^{-1}J^{\mathscr{S}}(z-F)-(z-F) \ ,$$
$$=\rho^* T(z)\rho \ . \tag{4.43}$$

Here we have considered $T^{\mathscr{S}}$ as an operator on $R^{\mathscr{S}}\mathscr{F}$, on which $\rho^*\rho$ is the identity. This symmetrized transition operator satisfies the dynamical equation

$$T^{\mathscr{S}} = \rho^*(J^*J-I)\rho(z-F) + W^{\mathscr{S}} + W^{\mathscr{S}}(z-F)^{-1}T^{\mathscr{S}} \ , \tag{4.44}$$

where

$$W^{\mathscr{S}}J^{\mathscr{S}*} = J^{\mathscr{S}*}H_N - FJ^{\mathscr{S}*} \ . \tag{4.45}$$

Thus,

$$W^{\mathscr{S}}J^{\mathscr{S}*} = \rho^* WJ^* R_N^{\mathscr{S}} = \rho^* WQ^{\mathscr{S}}J^* \ , \tag{4.46}$$

from which it follows that

$$W^{\mathscr{S}}J^{\mathscr{S}*} = \rho^* W\rho J^{\mathscr{S}*} \ . \tag{4.47}$$

The simplest choice of W is

$$W^{\mathscr{S}} = \rho^* W\rho \ . \tag{4.48}$$

With this choice the scattering theory is completely specified in terms of correctly symmetrized objects.

Equation (4.43) can be written in terms of arrangement channel (cluster) matrix elements. When this is done and the label transforming properties [Eqs. (4.1), (4.2), and (4.3)] are taken into account, the well known results[28,36] expressing $T^{\mathscr{S}}_{B^0A^0}$ as coherent sums of $T^{\mathscr{S}}_{BA}$ are recovered. For example,

$$T^{\mathscr{S}}_{B^0A^0}(z) = N_{[B^0]}^{-1/2} N_{[A^0]}^{1/2} \times \sum_{B\in[B^0]} R^{\mathscr{S}_{B^0}} \hat{p}_{B^0B} T_{BA^0}(z) \ . \tag{4.49}$$

Equations (4.44) and (4.48) can also be written in terms of matrix elements. The result is

$$T^{\mathscr{S}}_{B^0A^0} = (N_{[B^0]}N_{[A^0]})^{1/2} P_{B^0} R^{\mathscr{S}_{B^0}} R_N^{\mathscr{S}} R^{\mathscr{S}_{A^0}} P_{A^0} - \delta_{B^0A^0} R^{\mathscr{S}_{A^0}} P_{A^0} + W^{\mathscr{S}}_{B^0A^0} + \sum_{C^0} W^{\mathscr{S}}_{B^0C^0}(z-H_{C^0})^{-1} T^{\mathscr{S}}_{C^0A^0} \tag{4.50}$$

and

$$W^{\mathscr{S}}_{B^0A^0} = (N_{[B^0]}N_{[A^0]})^{-1/2} \times \sum_{B,A} R^{\mathscr{S}}_{B^0} \hat{p}_{B^0B} W_{BA} \hat{p}_{AA^0} R^{\mathscr{S}}_{A^0} \ . \tag{4.51}$$

An important question is whether Eq. (4.48) preserves the channel coupling structure of the distinguishable particle equation Eq. (2.15). There are indications that it is preserved for the BRS theory[28] and is not for the KLT theory.[37] The definitive answer is a matter for future research.

V. OPTICAL POTENTIAL FORMALISM

Though the optical model of elastic scattering dates back to the 1940's, the formal theoretical justification of it was elaborated only much later by Feshbach.[38] The projection operator formalism developed by Feshbach made it possible to reduce the elastic scattering of two nuclear particles to an effective two-body problem. The corresponding effective two-body interaction, the (generalized) optical potential, however, could only be given by a formal expression. Its construction, in fact, would have required the solution of a multiparticle collision problem.

Owing to the rapid development in the field of multiparticle scattering theory there are now a number of exact N-particle formalisms available. These exact N-particle equations make it possible to recast the optical potential formalism in the form of an explicit theory. Thus, recently there has been an increased interest in developing dynamical theories for the optical potential.[32,33,39] While these considerations are straightforward for systems of distinguishable particles, particle identity introduces some complications into the formalism, as it has been recently demonstrated by Kowalski.[33]

As an application of the results described in previous chapters, here we develop a simple and explicit formalism for the optical potential. The two-

Hilbert-space formalism makes the physical ideas as well as mathematical technique especially transparent. The case of identical particles presents no extra difficulties.

Once the corresponding Hilbert spaces have been constructed which account for the permutation symmetry of the N-particle system, the abstract structure of the theory is the same as in the case of distinguishable particles. For simplicity of notation, therefore, we first develop the theory in terms of the distinguishable particle theory.

Let us consider the symmetric form $T(z)$ of the transition operator [cf. Eq. (2.14)]. The dynamical equation satisfied by $T(z)$ is Eq. (2.15). Since $T(z)$ is a multicomponent quantity, Eq. (2.15) represents, in fact, a set of coupled equations. Now, if one is interested in the description of elastic scattering, only a single component is effectively needed. Of course, if Eq. (2.15) is solved, one also has the elastic scattering T operator. In the optical potential formalism, however, one first eliminates the unwanted quantities and obtains a single equation with effective interaction for the elastic T operator. In Feshbach's theory, the elimination procedure is facilitated by the projection operator technique, though it results only in formal expressions. In the frame of exact N-particle scattering theory, the projection operator technique will become a powerful tool and can lead to practical results.[40]

Let us introduce the projection operator P that when applied to T selects the required component. By definition

$$[P,R]=0 . \tag{5.1}$$

Let us define the operator Γ by the equation

$$\Gamma=W+W(I-P)R\Gamma . \tag{5.2}$$

The formal solution of Eq. (5.2) can be written as

$$\Gamma=[I-W(I-P)R]^{-1}W . \tag{5.3}$$

By making use of Eq. (5.3) one can then obtain the equation

$$T=A+\Gamma+\Gamma RPT , \tag{5.4}$$

where

$$A\equiv[I+\Gamma(1-P)R](J^*J-I)(z-H) . \tag{5.5}$$

The derivation of Eq. (5.4) is very simple.[33] Starting with Eq. (2.15) one writes

$$T=(J^*J-I)R^{-1}+W + WRPT+WR(1-P)T . \tag{5.6}$$

Inverting part of the kernel we obtain

$$T=[1-W(1-P)R]^{-1} \times[(J^*J-I)R^{-1}+W+WRPT] . \tag{5.7}$$

Equation (5.4) now follows directly from Eqs. (5.3) and (5.5). Multiplying Eq. (5.4) from the left and right by P and rearranging terms yields

$$PTP=P(A+\Gamma)P +[P(A+\Gamma)P-PAP]RPTP . \tag{5.8}$$

Another partial inversion of the kernel yields

$$PTP=[I+PAPR]^{-1}P(A+\Gamma) \times P[I+RPTP] , \tag{5.9}$$

$$=U+URPTP . \tag{5.10}$$

The optical potential operator $U=[I+PAPR]^{-1}P(A+\Gamma)P$ satisfies the equation

$$U=P(A+\Gamma)P-PAPRU . \tag{5.11}$$

Equations (5.2), (5.10), and (5.11) are the fundamental equations of Kowalski,[33] written in two-Hilbert-space notation.

The fundamental equations for systems with identical particles follow from those for distinguishable particles. The operators W, J, and R are to be replaced by $W^{\mathscr{S}}$, $J^{\mathscr{S}}$, and $(z-F)^{-1}$, respectively. The projection operator P is to be interpreted as selecting the desired channel A^0 in $\mathscr{F}$. The resulting equations for the matrix elements of the operators are

$$T^{\mathscr{S}}_{A^0A^0}=U^{\mathscr{S}}_{A^0}+U^{\mathscr{S}}_{A^0}(z-H_{A^0})^{-1}T^{\mathscr{S}}_{A^0A^0} , \tag{5.12}$$

$$U^{\mathscr{S}}_{A^0}=A^{\mathscr{S}}_{A^0}+\Gamma^{\mathscr{S}}_{A^0A^0} -A^{\mathscr{S}}_{A^0}(z-H_{A^0})^{-1}U^{\mathscr{S}}_{A^0} , \tag{5.13}$$

$$\Gamma^{\mathscr{S}}_{B^0A^0}=W^{\mathscr{S}}_{B^0A^0}+\sum_{C^0}(1-\delta_{C^0A^0})W^{\mathscr{S}}_{B^0C^0} \times(z-H_{C^0})^{-1}\Gamma^{\mathscr{S}}_{C^0A^0} , \tag{5.14}$$

$$A^{\mathscr{S}}_{A^0}=\sum_{C^0}[\delta_{A^0C^0}+(1-\delta_{A^0C^0})\Gamma^{\mathscr{S}}_{A^0C^0}(z-H_{C^0})^{-1}] \times[(N_{[C^0]}N_{[A^0]})^{1/2}P_{C^0}R^{\mathscr{S}_{C^0}}R_N^{\mathscr{S}}R^{\mathscr{S}_{A^0}}P_{A^0} -\delta_{C^0A^0}R^{\mathscr{S}_{A^0}}P_{A^0}](z-H_{A^0}) . \tag{5.15}$$

The optical potential $U^{\mathscr{S}}_{A^0}$ is defined by Eqs. (5.13)–(5.15), which contains only correctly sym-

metrized quantities. Once this potential is known $T_{A^0A^0}$ can be calculated via the Lippmann-Schwinger equation Eq. (5.12).

It is important to note that the above simple considerations are completely general and identical particle effects are easily incorporated. The formal tools used to obtain dynamical equations for the optical potential are essentially the generalization of those of Feshbach in the frame of an exact multiparticle scattering theory.

Two final delicate points should be mentioned. While in the various N-particle formalisms it turns out that Eqs. (5.13) and (5.14) have well-behaved kernels, the existence and uniqueness of the solution needs some more careful study. Second, the operators $U^{\mathscr{S}}_{A^0}$, $\Gamma^{\mathscr{S}}_{B^0A^0}$, and $A^{\mathscr{S}}_{A^0}$ seem to have no cut corresponding to the channel A^0. This was a major point of Kowalski.[33]

VI. CONCLUSIONS

In a multiparticle system the permutations of identical particles form a symmetry group of the Hamiltonian. In addition to the symmetrization postulate for the total wave function, the permutation symmetry of the system has other important consequences. The permutation symmetry induces an equivalence relation on the set of asymptotic arrangement channels of the system. Consequently, only the equivalence classes can be considered as physical entities. In particular, the physical properties of the system have to be described by operators which depend only on the equivalence classes. This necessitates a reformulation of the asymptotic condition and leads naturally to the properly symmetrized wave and scattering operators for physical processes. These operators can be expressed as suitable linear combinations of the corresponding operators of distinguishable particle theory, but are much fewer in number.[41] These "physical" operators are, in the usual terminology, coherent sums of operators for the "direct" and "exchange" processes.

In the present work we have built the permutation symmetry into the definition of the scattering states. This has made it possible to elaborate the previous work of Ekstein[1] and Coester and Schlessinger[26] into a detailed scattering theory for identical particles.

The main results of the paper are as follows. Using a two-Hilbert-space technique, in which the abstract structure of the theory is especially clear, we have shown that the incorporation of the permutation symmetry of the system into the formalism does not destroy the unitarity of the scattering operator. In addition, we have demonstrated that the symmetrization of the theory can be performed even if there are long-range Coulomb interactions in the system. This last result is not obvious even if it is usually taken for granted. We have also written down the correctly symmetrized equation for the transition operator. As a byproduct of these considerations, we have also shown that the two-Hilbert-space technique and a suitable generalization of Feshbach's projection operator formalism lead in a natural way to an optical potential formalism. Owing to the abstract structure of the theory identical particle effects are easily included in the optical potential formalism if the relevant quantities are label transforming. Our results can be regarded as an abstract generalization of Kowalski's optical potential formalism.[33]

The most important result of the present work is, however, the combination of the abstract algebraic consequences of the permutation symmetry of the system with a clear time-dependent physical picture of the scattering process. As a result, a time-dependent scattering theory of systems with identical particles emerges that is clearly defined, both physically and mathematically, in terms of objects with the correct symmetries. That the construction of this theory seems almost trivial is testimony to the power of the two-Hilbert-space formalism and the abstract algebraic theory of the permutation group.[30] In this way the elaborate formalism of second quantization is not needed. Moreover, the transition to time-independent theory can be made in a rigorous way and the previous time-independent results[28] are recovered.

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