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Analytic treatment of multistep processes in hadron-nucleus scattering

R. D. Amado, J. A. McNeil, and D. A. Sparrow
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
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Multistep contributions to elastic and inelastic scattering of intermediate energy hadrons from collective nuclei are treated analytically to all orders. Exploiting the eikonal formalism and nuclear geometry, we obtain closed form expressions for the relevant amplitudes without the need for coupled channels. Agreement with experiment, where available, is excellent, and interesting predictions are made for large momentum transfer experiments and "poor" energy resolution.

NUCLEAR REACTIONS Analytical expressions for hadron-nucleus coupled channels calculations. ^{166}Er, ^{182}W, ^{154}Sm (p,p'), calculated $\sigma(\theta)$ exclusive to 0^+, 2^+, 4^+, and inclusive.

I. INTRODUCTION

Inelastic scattering among strongly coupled nuclear levels cannot be treated in distorted wave Born approximation, but rather must be calculated to all orders in the inelastic coupling. The standard procedure for doing this uses the method of coupled channels. Recently, we have shown that intermediate energy elastic scattering and inelastic scattering among weakly coupled states can be treated analytically, in closed form, by exploiting the nuclear geometry expressed in the context of the eikonal approximation.[1–4] In this paper we extend these methods to strongly coupled collective states obtaining closed form, analytic expressions for the scattering amplitudes for elastic and inelastic scattering for all orders in the coupling. This not only avoids the algebraic and numerical complexities of the coupled channel method but also provides analytic insight into the features that dominate the physical processes. Our results agree with available data and make interesting predictions for future experiments.

Strongly coupled collective states have the property that the couplings among states are not independent but rather are the manifestations of some simple collective feature, for example, the deformation. Furthermore, the energy separation of the collective states is very small compared with that of the incident intermediate energy projectile. Equivalently, one can say that the projectile-nucleus interaction times are very short compared with the times characteristic of the collective motion. Exploiting this fact at intermediate energy, one can study the scattering of the projectile for *fixed* collective coordinates, e.g., deformation, rather than for fixed nuclear eigenstates, and then project that scattering amplitude onto states of definite collective motion. This divides the problem in two parts: first, the calculation of scattering from a nucleus with fixed collective coordinates and second, the projection of that scattering amplitude onto nuclear eigenstates. In our previous work we have seen that hadron-nucleus scattering at intermediate energy can be calculated analytically for large momentum transfer starting from the eikonal expression for the amplitude and exploiting the asymptotic properties of the integral and the nuclear geometry via the method of stationary phase. These methods still apply for fixed collective coordinates. Since the scattering amplitude obtained in this way is analytic, the subsequent projection into nuclear eigenstates is straightforward. For asymptotic momentum transfers we obtain explicit expressions for elastic and inelastic scattering of intermediate energy hadronic projectiles from a nucleus to all orders in the collective coordinate (deformation). It is easy to check that our results reduce to our previous one step and two step calculation when expanded, but we also show that dispersive effects (the virtual excitation of many collective states during the scattering process) are correctly incorporated and are of considerable importance.

In Sec. II our analytic methods are applied to the eikonal scattering amplitude for hadron scattering from a nucleus with fixed collective coordinates. We specialize to the case of an elliptically deformed (quadrupole) nucleus but the result is easily generalized to higher multipole moments. We then show how the analytic amplitude we obtain can be projected onto discrete nuclear states so as to calculate elastic and inelastic scattering in the rotational band to all orders in the deformation. In Sec. III we apply these methods to ^{154}Sm, ^{182}W, and ^{166}Er, nuclei that represent a wide choice of quadrupole and hexadecapole deformations. Agreement with experiment where available [800 MeV ^{154}Sm(p,p')] is excellent, and the importance of dispersive effects (virtual multistep processes) is demonstrated by a comparison calculation without such effects. In Sec. IV we discuss our results, present some conclusions, and point out some new experimental avenues that exploit the difficult energy resolution problems intrinsic to scattering from large strongly deformed nuclei. In the Appendix we show how our method can be applied to strongly coupled vibrational states.

II. THEORY

Consider the scattering of a fast particle from a nucleus with a static deformation. In the high energy limit we may not only use the eikonal approximation to describe the scattering, but we may also neglect the energy difference between the collective states of the nucleus compared with the bombarding energy. This is equivalent to neglecting the dynamics of the collective degrees of freedom or equivalently to keeping them fixed during the scattering. We can, therefore, calculate the scattering for fixed collective coordinates, and then project that scattering amplitude onto states of proper collective motion (e.g., states of good angular momentum.) In a different language this is an adiabatic or Born-Oppenheimer approximation that exploits the fact that the collective motion is slow compared with the projectile motion. For protons at a few hundred MeV, on a typical collective nucleus, this must be an excellent approximation. The resulting amplitude will have summed the scattering to all orders in the collective motion (within the adiabatic approximation) and will, therefore, be equivalent to a full coupled channels calculation coupling together all the collective states.

In the eikonal approximation the scattering of a spinless particle from momentum $\vec{p}-\vec{q}/2$ to $\vec{p}+\vec{q}/2$, from a nucleus with *fixed* collective coordinate Ω is given by

$$\langle \vec{p}+\frac{\vec{q}}{2},\Omega \,|\, T \,|\, \vec{p}-\frac{\vec{q}}{2},\Omega \rangle = \frac{ip}{2\pi}\int b\,db\,d\phi\, e^{iqb\cos\phi}(1-e^{-\chi(b,\phi,\Omega)}) \,, \quad (1)$$

where the particle position is described by cylindrical coordinates z, b, and ϕ, and $\vec{p}$ is along z. χ is proportional to the integral along z (at fixed b and ϕ) of the projectile nucleus potential for fixed collective coordinate Ω. Hence, that integral depends not only on b and ϕ but also on the collective coordinates, in this case the orientation of the nucleus. In the first order multiple scattering or "$t\rho$" approximation, χ is given by

$$\chi(b,\phi,\Omega)=\gamma t(b,\phi,\Omega) \,, \quad (2a)$$

with

$$t(b,\phi,\Omega)=\int_{-\infty}^{\infty}\rho(\vec{r},\Omega)dz \,, \qquad \gamma=\tfrac{1}{2}\sigma_{\text{tot}}(1-ir) \,, \quad (2b)$$

where ρ is the nuclear density for fixed Ω, normalized to A nucleons, as a function of $\vec{r}$, the projectile coordinate. In γ, σ_{tot} and r are the isospin averaged elementary projectile-nucleon total cross section and the ratio of real to imaginary part, respectively. The above discussion identifies terms in the limit of short ranged nucleon-nucleon forces. Incorporation of the finite ranges does not change the structure of the equations in any way; it only obscures the identification of the various terms in the expression with physical quantities. For simplicity, we will continue to talk in terms of the interaction strength γ and density ρ, but it must be recognized that the following discussion is equally valid for the folded $t\rho$.

Let us now specialize to the case of the greatest physical interest—an axially symmetric nucleus with quadrupole deformation. The collective coordinate Ω characterizes the orientation of the nucleus with respect to a laboratory fixed axis, and may be specified in terms of polar angles (θ,Φ) or a unit vector $\hat{R}$. In this case the nuclear density ρ in the laboratory system will depend on the collective coordinate Ω entirely through the second order

Legendre polynomial $P_2(\hat{r}\cdot\hat{R})$.

In an exclusive scattering we must project the T matrix of Eq. (1) onto nuclear states of definite LM. Since the target starts in the ground state $(L,M=0,0)$, we can only go to a state in the ground state rotational band (L even). To do this projection we write (taking the same z axis for the M projections)

$$\left\langle \vec{p}+\frac{\vec{q}}{2},LM \mid T \mid \vec{p}-\frac{\vec{q}}{2},00\right\rangle \equiv T_{LM}=\int d\Omega\, Y^*_{LM}(\Omega)\left\langle \vec{p}+\frac{\vec{q}}{2},\Omega \mid T \mid \vec{p}-\frac{\vec{q}}{2},\Omega\right\rangle Y_{00}(\Omega)\ . \quad (3)$$

The problem of solving the scattering to all orders in the collective motion is now reduced to evaluating the integrals in (1) and (3). The ϕ integral can be done directly by noting that $P_2(\hat{r}\cdot\hat{R})$ will depend only on $\phi-\Phi$, the difference between the projectile and nuclear azimuthal coordinate, by virtue of our choice of z axis. We can also write $Y_{LM}(\Omega)=P_{LM}(\theta)e^{iM\Phi}$. If we change variables to $\Phi'=\Phi-\phi$ and $\Omega'=(\theta,\Phi')$, we have

$$\begin{aligned} T_{LM}&=\frac{ip}{2\pi}\int d\Omega' Y^*_{LM}(\Omega')\int b\, db\,(1-e^{-\gamma t(b,\Omega')})\int d\phi\, e^{iqb\cos\phi-iM\phi}Y_{00}(\Omega') \\ &=ipi^M\int d\Omega' Y^*_{LM}(\Omega')Y_{00}(\Omega')\int b\, db\, J_M(qb)(1-e^{-\gamma t(b,\Omega')})\ , \end{aligned} \quad (4)$$

after making use of the integral representation of the Bessel function. To make further progress, we concentrate on large q and use the asymptotic methods that we have employed for elastic scattering and one and two step excitations.[1-3]

In keeping with our earlier work we assume the nuclear density has a smooth central region and a well developed surface. In r space this can be described by a density function with a singularity near the real axis. For large q the corresponding singularity of $t(b)$ dominates the b integral of Eq. (4), which can then be evaluated by the method of stationary phase. The added feature in our treatment here is that the singularity position now depends on the collective coordinate Ω. A specific realization of such a density is the familiar Fermi form, but now with angle dependent radius (c) and skin thickness β

$$\rho=\rho_0\left\{1+\exp\left[\frac{r-c(1+\delta)}{\beta(1+\delta)}\right]\right\}^{-1}\ , \quad (5)$$

where $\delta=\Lambda P_2(\hat{r}\cdot\hat{R})$. ρ_0 is a normalization constant and Λ is a dimensionless measure of the deformation. For a spherical Fermi distribution the nearest singularity is a pole at $b_0=c+i\pi\beta$ and thus in the context of our analytic approach it is natural to deform both the radius and skin thickness $[(b_0\rightarrow b_0(1+\delta)]$ as we have in Eq. (5). Nevertheless, similar results would be obtained if only the radius were deformed. Below we will present phenomenological evidence as well for this particular choice, but we wish to stress that there is no fundamental constraint preventing distortion of c alone in this formalism should the need arise.

Following our earlier work we write for (4)

$$T_{LM}=-ipi^M\int d\Omega' Y^*_{LM}(\Omega')Y_{00}(\Omega') \times[G_M(q,\gamma,\Omega')+G^*_M(q,\gamma^*,\Omega')]\ , \quad (6)$$

where

$$G_M(q,\gamma,\Omega')=\tfrac{1}{2}\int b\, db\, H^{(1)}_M(qb)e^{-\gamma t(b,\Omega')}\ . \quad (7)$$

We have expressed the Bessel function in (4) in terms of the Hankel functions and used $H^{(2)}_M=H^{(1)*}_M$ and the fact that γ is the only complex quantity under the b integral in (4). We have also dropped the one in (4) since that contributes only in the forward direction in elastic scattering. To evaluate (7) for large q we use the method of stationary phase with the asymptotic expansion of the Hankel function. The stationary phase point comes near the closest singular point of $t(b,\Omega')$ in the upper half b plane. That singular point arises from the corresponding singularity of ρ, which for simplicity we evaluate using expression (5), which has a simple pole at $r/1+\delta=c+i\pi\beta\equiv b_0$ with residue $-\beta$. These results in pole position and residue are easily generalized to other functional forms for the density.[5] For any density representing a medium to large nucleus, the residue will be approximately minus the skin thickness. For the

Fermi case, the singular part of t, t_0, can be written

$$t_0 \cong -\beta\rho_0 \int_{-\infty}^{\infty} \frac{dz}{\left[\dfrac{r}{1+\delta} - b_0\right]}, \quad r^2 = z^2 + b^2 . \tag{8}$$

The difficulty with evaluating this expression by our usual methods is that δ depends on z through P_2. Using the addition theorem for P_2 and expressing the $Y_{2M}(\hat{r})$ in terms of z and b we can write

$$\delta = \frac{\Lambda}{r^2}(\Delta_0 b^2 + \Delta_1 bz + \Delta_2 z^2) , \tag{9}$$

where the Δ_i depend on θ and Φ'. The singularity of (8) in b comes when the denominator and its first derivative with respect to z vanish. The two roots of the denominator then coincide, pinching the integration contour. In the absence of the Δ_1 term in (9), the denominator depends only on z^2, and hence, its stationary point is $z=0$, just as it is in the $\Lambda=0$ case treated in the work of Amado, Dedondor, and Lenz (ADL). Since the singular point is a stationary point, the addition of the linear $\Lambda\Delta_1$ term can only change the location of the singular point of t_0 in b to second order in $\Lambda\Delta_1$. To first order in Λ, then, the singularity comes from $z=0$ and we may replace δ in (8) by $\delta_0 = \Lambda\Delta_0$, its value at $z=0$.[6] In that case t_0 is precisely the t_0 of ADL, that is, of the undistorted case, except that $b_0 \to b_0(1+\delta_0)$ and $\beta \to \beta(1+\delta_0)$. Using the fact that $H_M^{(1)}$ differs asymptotically from $H_0^{(1)}$ only by $e^{-iM\pi/2}$ we have for the G_M of (7), for large qb,

$$G_M(q,\gamma,\Omega') = e^{-iM(\pi/2)} G_0[q,\gamma,b_0(1+\delta_0)] , \tag{10}$$

where G_0 is the G_0 of ADL [Eq. (40)] but with b_0 replaced everywhere (including in β) by $b_0(1+\delta_0)$. In obtaining this very simple result for the eikonal amplitude at fixed Ω, the only assumption we have made over the usual asymptotic assumption familiar from ADL is that a shift in location of the singular point of the profile function that is second order in the deformation may be neglected.

It now remains to do the state projection of Eq. (6), using the form for G of (10). To do this projection we need the explicit angular dependence of (10) carried in Δ_0. It is a straightforward exercise to show that

$$\Delta_0 = \tfrac{1}{2}(3\sin^2\Theta\cos^2\Phi' - 1) \tag{11}$$

in terms of the nuclear collective coordinates. Since $\sin^2\Theta\cos^2\Phi' = x^2/r^2$, Δ_0 is just the second der Legendre polynomial P_2, but with respect to the x axis. [$P_2 = \frac{1}{2}(3\hat{x}^2 - 1)$.] The projections needed in Eq. (4) are then of the form

$$I_{LM} = e^{-iM(\pi/2)} \int d\Omega' Y_{LM}^*(\Omega') Y_{00}(\Omega') \times G_0\{q,\gamma,b_0[1+\Lambda P_2(\hat{x})]\} . \tag{12}$$

To carry out the integration we express the $Y_{LM}(\Omega')$ in terms of the $Y_{LM}(\Omega_x)$ referring to the x axis. We have

$$Y_{LM}(\Omega') = \sum_{M'} Y_{LM'}(\Omega_x) \mathscr{D}_{M'M}^{L}(\alpha\beta\gamma) \tag{13}$$

in terms of the standard rotation matrices $\mathscr{D}$. The arguments of $\mathscr{D}$ are the Euler angles that effect the rotation. In our case we need only rotate the z axis into the x axis and that requires a rotation of $\pi/2$ about the y axis, which means $\alpha=\gamma=0$, $\beta=\pi/2$. With $\alpha=\gamma=0$, we may express $\mathscr{D}$ in terms of the reduced d. We then have for (12)

$$I_{LM} = e^{-iM\pi/2} \sum_{M'} d_{M'M}^{L*}(\pi/2) \int d\Omega_x Y_{LM'}^*(\Omega_x) Y_{00}(\Omega_x) G_0\{q,\gamma,b_0[1+\Lambda P_2(\hat{x})]\} . \tag{14}$$

The only dependence on Φ_x in (14) is in the Y_{LM}, we may therefore evaluate the Φ integral, which will require $M'=0$, to give

$$I_{LM} = e^{-iM(\pi/2)} \frac{\sqrt{2L+1}}{2} d_{0M}^{L*}(\pi/2) \int d\hat{x}\, P_L(\hat{x}) G_0\{q,\gamma,b_0[1+\Lambda P_2(\hat{x})]\} . \tag{15}$$

This can be further simplified by noting that

$$d_{0M}^{L}(\pi/2) = \left[\frac{4\pi}{2L+1}\right]^{1/2} Y_{LM}(\theta=\pi/2,\ \varphi=0) \tag{16}$$

and that $Y_{LM}(\theta=\pi/2, \phi=0)=0$ for $L+M$ odd. We therefore get the well known selection rule that $L+M$ must be even. L must also be even by parity in (15). $Y_{LM}(\theta=\pi/2, \phi=0)$ is the quantity we called a_0 in the work of Amado, Lenz, McNeil, and Sparrow (ALMS) and Amado, McNeil, and Sparrow (AMS-I).

To take full account of the x dependence in (15) via all the places that b_0 and β appear in G_0 [see ADL, Eq. (40)] is complicated, but the integral in (15) can easily be done numerically since G_0 is known analytically. Alternatively, and far more simply, one can exploit the fact that the principal dependence of G_0 on q and b_0 is via the exponential factor e^{iqb_0} to write for (15)

$$I_{LM} \simeq e^{-iM(\pi/2)} \frac{(2L+1)^{1/2}}{2} d^{L*}_{OM}\left(\frac{\pi}{2}\right) \hat{G}_0(q,\gamma,b_0) J_L \,, \tag{17}$$

where J_L is given by

$$J_L = e^{iqb_0} \int_{-1}^{1} dx\, P_L(x) e^{iqb_0 \Lambda P_2(x)} \,, \tag{18}$$

and where

$$\hat{G}_0(q,\gamma,b_0) = e^{-iqb_0} G_0(q,\gamma) \,. \tag{19}$$

If Λ is very small, we can expand the exponential under the integral in (18) and recover our one and two step results. In particular, from the orthogonality of the Legendre polynomials and since L must be even, we see that if we expand the exponential the first nonvanishing power of Λ will be $\Lambda^{L/2}$. That is, for $L=2$ we have one step results, for $L=4$ we have two step results, etc. For the one step $L=2$ case the expansion of the exponent yields a factor of iqb_0^{Λ}, while for the two-step $L=4$ we get $-q^2 b_0^2 \Lambda^2/2$. In ALMS and AMS-I we discuss the importance of these q factors in distinguishing the slopes of the one- and two-step processes and the role of the i in the one-step process in reversing the relative phase of G and G^* so as to give the reversal of maxima and minima from elastic scattering normally referred to as the Blair phase rule. We also comment there on the importance of the additional phase shift between G and G^* that arises from the complex factor of b_0 (or $b_0{}^2$ in the two step process). This additional phase, associated with diffusivity, is required in many cases to fit the data. We would not recover it here had we deformed the radius parameter c only in the density (5). This is the empirical justification for deforming both c and β together. In the spirit of our treatment, this is also natural by analyticity. The precise connection with earlier work is that

$$\Lambda = -\frac{\lambda}{\gamma} \frac{5}{(4\pi)^{1/2}} \tag{20}$$

in terms of γ and λ of ALMS.

Evaluation of J_L requires integrals of the form

$$K_n = \int_0^1 dx\, x^{2n} e^{\mu x^2}, \quad \mu \equiv i\tfrac{3}{2} qb_0 \Lambda \,. \tag{21}$$

These may be expressed in terms of the error function and its derivatives with respect to μ or directly evaluated numerically. Alternatively, one could expand the exponential yielding a power series in the coupling, and recover our previous one and two step results. Although this power series converges for all μ, it converges slowly, requiring some care in its evaluation.

This slow oscillating convergence signals that for asymptotic μ the integral may profitably be evaluated by the method of stationary phase. To do this we use the explicit form of P_2 to write

$$J_L = e^{iqb_0(1-\Lambda/2)} \int_{-1}^{1} dx\, P_L(x) e^{\mu x^2} \,. \tag{22}$$

For $|\mu| >> 1$ and $\mathrm{Re}\mu < 0$, the stationary phase point is $x=0$, and the integral is carried out by evaluating P_L at $x=0$ and extending the limits to $\pm\infty$. Since $\mathrm{Im}(b_0)>0$, $\mu<0$ is equivalent to $\Lambda>0$, or prolate deformation, which is the more common case in nature. We find

$$J_L \simeq e^{iqb_0(1-\Lambda/2)} P_L(0) \left[\frac{-2\pi}{3iqb_0\Lambda}\right]^{1/2} . \tag{23}$$

Note the absence of any L dependence in the phase of J_L. Since all the other factors in the amplitude are common, all excitations have the same phase and, therefore, the cross sections will have the same shape. Their diffractive patterns will be in step or phase locked asymptotically. Furthermore, the magnitudes of the cross sections when summed over final m states are only weakly L dependent. The relevant factor is

$$(2L+1)P_L(0)^2 = \frac{2L+1}{4^L} \begin{bmatrix} L \\ L/2 \end{bmatrix}^2 , \tag{24}$$

where $\binom{L}{L/2}$ is the binomial coefficient. For $L=0$, this factor is 1 and it increases monotonically to $4/\pi$ for very large L. Asymptotically not only will the cross section be phase locked, but they will have roughly the same magnitudes. The multiple virtual excitations have only the impact of altering the effective geometry, that is, $b_0 \to b_0\,(1-\Lambda/2)$. For prolate ($\Lambda > 0$) deformations in the asymptotic regime, both the radius and diffusivity are reduced. This runs counter to the conventional wisdom which holds that deformed nuclei appear as an ordinary nucleus but with a larger diffusivity due to the "smearing out" of the deformation on averaging. For the intermediate momentum transfer thus far sampled experimentally, the falloff of the cross sections are, in fact, larger than average, indicating a larger effective diffusivity, but this is due primarily to the influence of multiple excitations, not the averaging over orientations. In the time of passage of the projectile, the nucleus is fixed in orientation and lacks the time necessary to smear the surface. In averaging the amplitude over orientations for large Λ, qc cancellations occur over all orientations except for the stationary point $x=0$. This corresponds to scattering from the "waist" of the ellipsoid, where the effective geometry is reduced by the $(1-\Lambda/2)$ factor. In Sec. III the example of 800 MeV ^{166}Er(p,p') scattering illustrates the onset of asymptopia where the slope of the elastic distribution is reduced beyond $q=2.6$ fm^{-1}. The 800 MeV ^{182}W(p,p') example is even more dramatic but complicated somewhat by the additional large hexadecapole contribution.

In the actual examples we study in Sec. III, μ is seldom large enough to permit direct use of the asymptotic method. Therefore, we evaluate the J_L integral in (18) numerically. To account for the remaining x dependence in G_0 [see Eq. (15)] we use the asymptotic form $b_0 \to b_0(1-\Lambda/2)$ in $\hat{G}_0$ of (19). In numerical tests with (15) we find this gives remarkably accurate results for a wide range of q and Λ. In these terms the full amplitude can be written

$$T_{LM} = -ip\frac{\sqrt{2L+1}}{2}d^{L*}_{OM}(\pi/2) \times \{\hat{G}_0[q,\gamma,b_0(1-\Lambda/2)]J_L + \hat{G}_0^*[q,\gamma^*,b_0(1-\Lambda/2)]J_L^*\}\ . \tag{25}$$

The discussion thus far for quadrupole deformations easily generalizes to higher moments. For an example of empirical interest we consider adding the hexadecapole deformation, thus in G_0 we use

$$\delta = \Lambda_2 P_2 + \Lambda_4\left[\frac{b_0}{c}\right]^2 P_4\ , \tag{26}$$

where the factor of $(b_0/c)^2$ arises from a Tassie description of the transition and is well motivated empirically. The arguments concerning the fixed orientation of the target and the application of the ADL analytic methods go through as before. The scattering amplitude is given by (25) but now J_L is given by

$$J_L = \int_{-1}^{1} dx\, P_L(x)\exp\left\{iqb_0\left[1+\Lambda_2 P_2(x)+\Lambda_4\left[\frac{b_0}{c}\right]^2 P_4(x)\right]\right\} \tag{27}$$

and is evaluated numerically, and we use $\hat{G}_0\{q,\gamma,b_0[1+\Lambda_2 P_2(0) + \Lambda_4 b_0{}^2/c^2 P_4(0)]\}$. In the realistic examples presented below, this generalized form is used. Complete generalization to arbitrary even L poles with or without the Tassie factors is straightforward.

III. RESULTS

In this section we present sample calculations of elastic scattering and inelastic excitation of the first two excited states by 800 MeV protons for three rotational nuclei, ^{166}Er, ^{182}W, and ^{154}Sm. These three are chosen to represent a distribution of deformation parameters. Of these nuclei, data for comparison exists only in the case of ^{154}Sm. (Our calculations are also successful at describing the ^{176}Yb data, but those results are not included here.)

The first example is a calculation of 800 MeV protons on ^{166}Er to the ground and first two excited states. The geometric and deformation parameters are $c=6.1$ fm, $\beta=0.60$ fm, $\Lambda_2=0.21$, and $\Lambda_4=-0.003$.[7] The nucleon-nucleon strength $\gamma=2.09+i0.38$ fm^2 is fixed throughout. The nearly vanishing value for the hexadecupole moment insures that only the distortion and multiple quadrupole excitations contribute. In Fig. 1 we plot the 800 MeV ^{166}Er(p,p) elastic cross section along with the 2^+ and 4^+ inelastic cross sections as well as the incoherent sum. For small momentum transfer q, we see the characteristic power law difference of the slopes between the elastic scattering, the one step 2^+, and two step 4^+.[2] However, since Λ_2 is fairly large these perturbation theory features soon

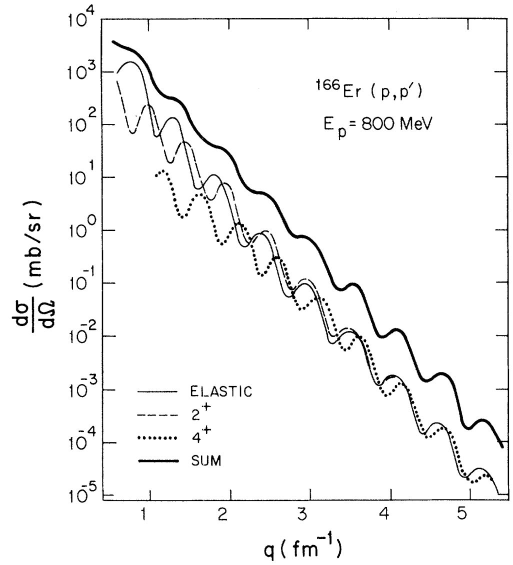

FIG. 1. Calculations using Eqs. (25) and (27) of 800 MeV protons on ^{166}Er to the 0^+ ground and nearest 2^+ and 4^+ excited states as well as the incoherent sum are shown. The nuclear parameters were taken from Ref. 7. The cross sections have the same shape for large momentum transfers, which returns the diffractive structure to the incoherent sum.

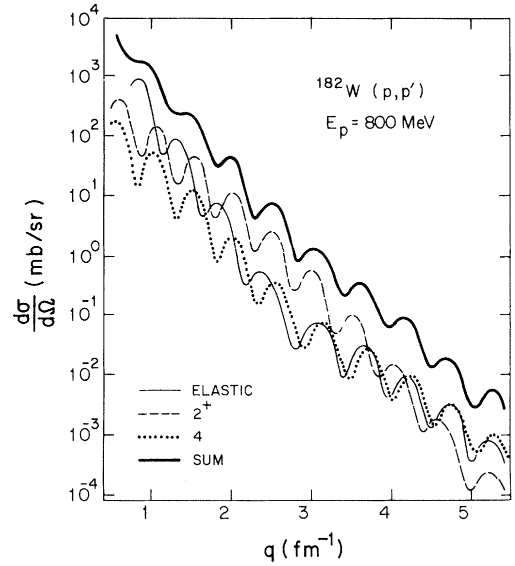

FIG. 2. Calculations using Eqs. (25) and (27) of the ground and 2^+ and 4^+ excited state cross sections as well as their incoherent sum for 800 MeV protons on ^{182}W are shown. The nuclear parameters were taken from Ref. 8. The large quadrupole and hexadecupole moments couple to the elastic scattering in such a way to produce the dramatic change in the slope at momentum transfer $q=2.8$ fm^{-1}. This nucleus was chosen as an example because of the large negative hexadecapole moment.

disappear with increasing q and beyond $q=3.5$ fm^{-1} the phases and shapes of the diffraction patterns are locked together and correspond to a decrease in effective radius and diffusivity as our asymptotic formula requires. Thus, even a poorly resolved experiment which measures the incoherent sum of elastic and 2^+ and 4^+ excitations would show some diffractive structure for large momentum transfer. We know of no data currently available for ^{166}Er.

To illustrate some of the variety of effects possible we present in Fig. 2 the elastic and low lying 2^+ and 4^+ excitations of ^{182}W by 800 MeV protons. The geometric and deformation parameters are $c=6.4$ fm, $\beta=0.48$ fm, $\Lambda_2=0.17$, and $\Lambda_4=-0.16$.[8] With $|\Lambda_2|\simeq|\Lambda_4|$ the situation is more complicated. At small momentum transfer we see the power law difference between elastic and inelastic scattering, but now since Λ_4 is large, both 2^+ and 4^+ can go by the one step process. In the intermediate momentum transfer region $1.5<q<4$ fm^{-1} the 2^+ excitation dominates the incoherent sum, while the elastic and 4^+ cross sections quickly phase lock (by $q=5.0$ fm^{-1} the phase of the 2^+ is also locked with the others). Note the discontinuity in the slope of the elastic cross section around $q=2.8$ fm^{-1}; this is due to the virtual multiple excitations which dominate the elastic cross sections beyond $q=2.8$ fm^{-1} and is characteristic of the onset of asymptopia. In this example a poorly resolved experiment would show some diffractive structure for $q \gtrsim 2.0$ fm^{-1}. For 2.0 fm^{-1} $< q < 4.2$ fm^{-1} this is due to the dominance of the incoherent sum by the 2^+ excitation, and for momentum transfers larger than $q=4.2$ fm^{-1} the structure results from total phase locking, indicative of asymptopia. Again no data is available.

As our final example we consider elastic scattering and 2^+ and 4^+ excitations of ^{154}Sm bombarded by 800 MeV protons for which data is available.[9] The geometric parameters fit to the data are $c=6.0$ fm, $\beta=0.64$ fm, $\Lambda_2=0.21$, $\Lambda_4=0.06$, and so represents a case intermediate to the first two ex-

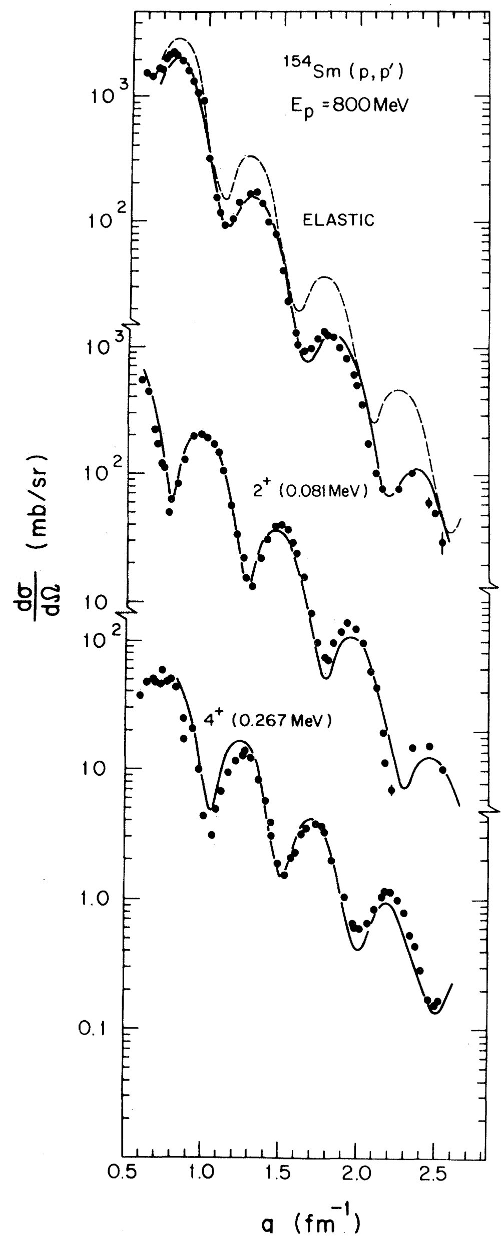

FIG. 3. We compare data from Ref. 9 with calculations using Eqs. (25) and (27) for 800 MeV p-^{154}Sm elastic and 2^+ (0.081) and 4^+ (0.267) inelastic scattering. We also present a calculation of the elastic scattering without dispersive effects (dashed curve). The nuclear parameters fit to the data were $c=6.0$ fm, $\beta=0.64$ fm, $\Lambda_2=0.21$, and $\Lambda_4=0.06$.

amples. (Our parameters are to be compared with $\Lambda_2=0.19$ and $\Lambda_4=0.09$ of Ref. 10.) In Fig. 3 we present the data along with the theoretical calculations using Eqs. (25) and (27). The agreement is excellent. In Fig. 3 we also present calculations of the elastic scattering without the multiple virtual excitations. The empirical necessity of including dispersive processes is clear. The dispersive processes have the effect in this intermediate momentum transfer region of increasing the effective diffusivity and decreasing the effective radius. In asymptopia we expect both the effective radius and diffusivity to be decreased.

IV. DISCUSSION

Exploiting the fast passage of an intermediate energy hadron by a collective nucleus, we have been able to give a closed form (integrals) expression in the eikonal formalism for elastic scattering and inelastic excitation of the nucleus to all orders in the collective coordinate deformation. Using our analytic methods and the nuclear geometry, these integrals can be evaluated. Comparison with data, where it is available, yields excellent agreement. We find, in agreement with others, that dispersive effects (the virtual effects of intermediate excitations) are very important.[9,11] In addition, we have shown how these effects grow with momentum transfer, and furthermore, that at very large momentum transfer the elastic and inelastic cross sections become essentially identical. This "phase locking" suggests that a "moderate" energy resolution experiment (which cannot resolve the elastic and low-lying inelastic scattering) will show diffractive structures at these large momentum transfers.

This asymptotic result may surprise some who believe the large momentum transfers correspond to short distance behavior, and that one should see details of the excitation mechanism rather than the average properties associated with the deformation. This might be so if the large momentum transfer collision occurred in one hard step, but our result, and earlier work on multiple scattering both within and outside the eikonal formalism, shows that it is far more probable to achieve the large momentum transfer by a coherent sum of small steps than in one large step. It is precisely these many steps our formalism sums up.

Our treatment is primarily given for deformed nuclei and for rotational states. In the Appendix we treat vibrations. It is of interest to combine these, perhaps using the interacting boson model.

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APPENDIX: VIBRATIONAL STATES

In this appendix we develop the theory for treating the excitation of collective vibrations by an intermediate energy hadronic probe to all orders in the coupling. Our concern here is primarily with the formalism rather than with comparison with experiment.

Consider a nucleus with a set of quadrupole vibrations separated by energy ϵ. The creation (destruction) operators for these vibrations are $a_\mu^+ (a_\mu)$, $\mu=0$, ± 1, and ± 2. They obey

$$[a_\mu, a_\gamma^\dagger] = \delta_{\mu\gamma} . \tag{A1}$$

The Hamiltonian describing the interaction of an hadronic projectile of mass m with this nucleus is

$$H = \frac{P^2}{2m} + V(r) + \lambda w(r) \sum_\mu [Y_{2\mu}^*(\theta,\varphi) a_\mu + Y_{2\mu}(\theta,\varphi) a_\mu^\dagger] + \sum_\mu \epsilon a_\mu^\dagger a_\mu \ , \tag{A2}$$

where r, θ, and ϕ are the projectile coordinates and p is its momentum. $V(r)$ is an average distorting potential and $\lambda w(r)$ is the coupling potential between the projectile and the oscillator. We wish to solve the scattering problem to all orders in λ.

The basic assumption we have been making is that the last term in H, the nuclear excitation energy, may be neglected compared with the projectile energy, in which case the scattering amplitude is most easily found not in a representation in which $a^+ a$ is diagonal, but rather as an operator in the nuclear space[12]. In the eikonal approximation that amplitude is

$$T = -\frac{ik}{2\pi} \int d^2 b \, e^{i\vec{q}\cdot\vec{b}} \left\{ 1 - \exp\left[-\chi(b) - \sum_\mu \left[\Lambda_\mu^\dagger a_\mu + \Lambda_\mu a_\mu^\dagger \right] \right] \right\} , \tag{A3}$$

where k is the projectile average momentum and is in the z direction, q is the momentum transfer, and

$$\chi(b) = \frac{mi}{k} \int_{-\infty}^{\infty} dz \, V(b,z) ,$$

$$\Lambda_\mu(\vec{b}) = \frac{\lambda mi}{k} \int_{-\infty}^{\infty} dz \, w(b,z) Y_{2\mu}(\theta,\varphi) . \tag{A4}$$

Once one neglects the $a^+ a$ term in H, expression (A3) follows by the standard eikonal methods.

To evaluate the integral for Λ_μ one must take account of the z dependence of $Y_{2\mu}$. If w depends only on $r = (b^2+z^2)^{1/2}$, the $Y_{2\pm 1}$ terms will integrate to zero since they are odd in z. If we further assume, as in the rotational case, that the leading contribution will come for near $z=0$ we obtain (putting $z=0$ in the explicit forms of the $Y_{2\mu}$)

$$\Lambda_0 = \frac{\lambda mi}{k} \left[-\tfrac{1}{2}\sqrt{5/4\pi} \right] \int_{-\infty}^{\infty} dz \, w(b,z) \equiv -\chi_t(b) ,$$

$$\Lambda_{\pm 2} = \frac{\lambda mi}{k} \left[\tfrac{1}{2}\sqrt{15/8\pi} e^{\pm 2i\varphi} \right] \int_{-\infty}^{\infty} dz \, w(b,z) \equiv \sqrt{3/2} e^{\pm 2i\varphi} \chi_t(b) , \tag{A5}$$

$$\Lambda_{\pm 1} = 0 .$$

If we define

$$A = -a_0 + \sqrt{3/2}(a_2 e^{-2i\varphi} + a_{-2} e^{2i\varphi}) \tag{A6}$$

we can write for the transition operator

$$T = \frac{ik}{2\pi} \int d^2 b \, e^{i\vec{q}\cdot\vec{b}} e^{-\chi(b)} e^{-\chi_t(b)(A+A^\dagger)} , \tag{A7}$$

where we have dropped the 1 that contributes only to forward elastic scattering and assumed $\chi_t = \chi_t^*$.

To evaluate the transition amplitude to particular final states of the target, we need matrix elements of (A7) between the ground state $|0\rangle$, which contains no oscillator quanta, and some excited state $|N\rangle$ containing N quanta, $\langle N \mid T \mid 0\rangle$. To find this it is convenient to use the Baker, Campbell, Hausdorff formula[13]

$$e^{-\chi_t(b)(A^\dagger+A)}=e^{-\chi_t(b)A^+}e^{-\chi_t(b)A}e^{-\chi_t{}^2(b)[A^\dagger,A]/2} . \tag{A8}$$

Since $[A^+,A]=-4$ and since $A\mid 0\rangle=0$, we get

$$\begin{aligned}\langle N \mid T \mid 0\rangle &= \frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}e^{-\chi(b)}e^{2\chi_t{}^2(b)}\langle N \mid e^{-\chi_t(b)A^\dagger}\mid 0\rangle ,\\ &=\frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}e^{-\chi(b)}e^{2\chi_t{}^2(b)}\frac{1}{N!}[-\chi_t(b)]^N\langle N \mid (A^+)^N\mid 0\rangle .\end{aligned} \tag{A9}$$

To specify the state N we must also give its total angular momentum ($J=0$, 2, 4, 6 . . .) and the z component M. We thus have

$$\langle N,J,M \mid (A^+)^N \mid 0\rangle = e^{iM\varphi}f_{N,M}^{(J)} , \tag{A10}$$

where f is a combinatoric factor and M must be even by the structure of A (A6). The f's are easily calculated for particular cases. We get, finally,

$$\langle N,J,M \mid T \mid 0\rangle = ik(-i)^M\frac{f_{N,M}^{(J)}}{N!}\int_0^\infty db\, bJ_M(qb)(-\chi_t)^N e^{-\chi(b)+2\chi_t{}^2(b)} , \tag{A11}$$

where we used the integral representation of the Bessel function to do the ϕ integral. Given a dynamical theory for $\chi(b)$ and $\chi_t(b)$, the integral in (A11) can be evaluated by the stationary phase methods we have used before. It is particularly simple in a derivative coupling model in which $\chi_t(b)=(d/db)\chi(b)$ and $X(b)$ is the usual density thickness function.

It is of considerable interest to combine the algebraic results presented in this appendix with the rotational al methods of the paper to develop a "unified" theory. We are working to do this in the context of the interacting boson model.

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