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Use of distorted waves in the theory of inelastic scattering

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A distorted wave description of inelastic scattering of nucleons from nuclei is formulated so that the microscopic content of the various ingredients can be made explicit. Special care is taken to ensure that physical processes are not overcounted as a consequence of the use of distorted waves in both the initial and final channels. Two attitudes to applications of the theory are taken. In the first, it is assumed that phenomenological distorted waves are employed and attention is focused upon the microscopic transition potential and the final distorted wave. Theoretically based recommendations for practical calculations of both these quantities are given. Secondly, we present a completely microscopic treatment wherein the truncations of the microscopic distorting potentials and the transition potential, at the single scattering level, are consistent with the underlying theoretical framework which links them. Our approach is designed to embody the distorted wave impulse approximation as a suitable lowest order result. Again, recommendations for practical calculations are given.

[NUCLEAR REACTIONS Inelastic scattering, distorted wave Born approximation, distorted wave impulse approximation, multiple scattering.]

I. INTRODUCTION

This paper follows an extended formal theoretical discussion of inelastic scattering, the preceding paper.[1] The main thrust of that paper was the development of a concise and general theoretical framework for the distorted wave theory within which it would be possible to develop practical, well-defined truncations which were both physically and mathematically acceptable for different kinds of scattering situations. Thus, in that paper one of the items to which much attention is given is the scattering of a composite projectile from a composite target in which some of the constituent particles of the projectile may be fermions *identical* to some of the fermions constituting the target. This treatment of the Pauli principle is complete and yet is tractable enough to lead to prescriptions for the inclusion of the Pauli exchange effects in practical calculations. However, since that work concentrates so heavily on formal aspects, it may not be as easy as the authors would like for readers to extract specific prescriptions for incorporating these ideas into practical calculations. Similarly, a large portion of that paper is devoted to the elimination of the dependence of the theory on the existence of underlying potential interactions. This degree of generality, which is expected to be of value in the investigation of mesonic scattering, for example, may be less than helpful in the analysis of proton scattering data.

Such considerations have prompted the authors to offer the present paper, whose main purpose will be to attempt to treat inelastic scattering in as elementary a fashion as is possible without sacrifice of the integrity of the presentation. Another desideratum is to establish close contact with the current ways of thinking and the established language in the area. To this end, the starting point in this paper is the two-potential formula, since that has been the way that the pioneering works[2] and the most familiar monographs[3] have begun their considerations. In this way we hope not only to investigate critically the current conventions in the field, but more im-

portantly to build upon the undoubted strengths and insights that have been developed in the course of an enormous amount of study.[4] The aim of this paper is to provide a basis for improved calculations which may be used to abstract new and/or more reliable information from the more precise data now available.

In order to keep the language and notation very much uncluttered so as to make maximum contact between the formalism and the physical basis thereof, we have chosen to think primarily about inelastic proton-nucleus scattering, although we shall avoid any unique particularizations to that special case. The generalizations to other inelastic processes (and to inclusion of particle identity) require no further discussion here.

In Sec. II, the transition matrix element for inelastic scattering is studied through the intermediary of a distorting potential W_*, which is taken to be diagonal in the target states. This leads to the standard result

$$T_{fi}^{\text{inel}} = \langle \psi_*^{(-)} | P_*(V - W_*) | \Psi^{(+)} \rangle , \tag{1.1}$$

which along with the precise definitions of the quantities therein is the result given in Eq. (2.19). This result is reexpressed in Eq. (2.21) as

$$T_{fi}^{\text{inel}} = \langle \psi_*^{(-)} | P_* VP | \chi^{(+)} \rangle + \langle \psi_*^{(-)} | P_*(V - W_*)Q | \Psi^{(+)} \rangle . \tag{1.2}$$

The first term on the right side of Eq. (1.2) represents the distorted wave Born approximation (DWBA). We comment upon the fact that whereas the distorted wave state vector, $|\chi^{(+)}\rangle = P|\Psi^{(+)}\rangle$, is unambiguously defined, the state $\langle \psi_*^{(-)}| = \langle \psi_*^{(-)}| P_*$, which is an incoming wave eigenstate of $H_0 + P_* W_* P_*$, depends upon the choice of W_*. In practice W_* is chosen in a heuristic, physically appealing manner. However, to the best of our knowledge there exists no theoretical discussion of the DWBA based on minimization[5] of the second term in Eq. (1.2). Without some specific criterion for the choice of W_* which permits the second term of Eq. (1.2) to serve as a control on the DWBA, this approximation stands on a very weak theoretical foundation. Much of the thrust of the work of the present authors on this problem may be characterized as an attempt to remedy this lack.

The result quoted in Eq. (1.2) has been obtained from the "prior" form of T_{fi}^{inel}. The analogous result in the "post" form is obtained from Eq. (2.22) to be

$$T_{fi}^{\text{inel}} = \langle \Psi_*^{(-)} | P_* VP | \chi^{(+)} \rangle + \langle \Psi_*^{(-)} | Q_*(V - PWP)P | \chi^{(+)} \rangle , \tag{1.3}$$

where again the definitions of the quantities which appear in Eq. (1.3) are given in Sec. II. The first term on the right-hand side of Eq. (1.3) also represents the DWBA, as did the corresponding term in Eq. (1.2). The "post" and "prior" forms of the DWBA will be identical, one to the other, if we require that

$$|\chi^{(+)}\rangle \equiv P|\Psi^{(+)}\rangle \tag{1.4}$$

and

$$\langle \psi_*^{(-)}| \equiv \langle \Psi_*^{(-)}| P_* . \tag{1.5}$$

These identifications permanently fix the meanings of $P_* W_* P_*$ and PWP.

The apparently innocuous requirement that the "post" and "prior" forms of the DWBA be identical makes the distorted wave discussion completely microscopic and hence leaves no latitude for the adjustment of W or W_*. Thus acceptance of this restriction upon the form of the theory means that we are no longer at liberty to choose the auxiliary diagonal interaction W so as to minimize the effect of the second term on the right-hand side of Eq. (1.3). This in turn implies that we necessarily take what we get for the DWBA result and that the correction to the DWBA represented by the second term has become an integral part of the theory, which must (and now can) be studied and included in the theory. The remainder of Sec. II of this paper is devoted to that task. The result of that exercise is given in Eq. (2.43), as

$$T_{fi}^{\text{inel}} = \langle \hat{\chi}_*^{(-)} | P_* \hat{U} P | \chi^{(+)} \rangle , \tag{1.6}$$

where $|\chi^{(+)}\rangle$ is the elastically scattered distorted wave presented in Eq. (2.29), and $\hat{U}$ is an effective interaction given in Eq. (2.44), whose diagonal matrix element, $P_* \hat{U} P_*$, generates the final distortion in Eq. (1.6) and whose off-diagonal matrix element, $P_* \hat{U} P$, acts as a transition operator in Eq. (1.6). We emphasize that Eq. (1.6) is *exact*. Evaluation of this matrix element depends upon the three quantities $P_* \hat{U} P_*$, $P_* \hat{U} P$, and PUP. Since $\hat{U}$ and U are many-body operators of great complexity, the viability of this argument depends critically upon our ability to present useful approximations to the above three quantities which are either directly calculable or accessible through related measurements. This, in one form or another, is the burden of the later sections of this paper.

In the initial phase of this work we take for

granted the accessibility of the quantity *PUP*, the optical potential for elastic scattering in the initial configuration. The elastic optical potentials are circumscribed by an enormous amount of high quality data as well as by extensive basic theoretical and computational work.[6] For this reason, we have limited our treatments in Secs. II–IV to those in which the initial state distorted wave is that corresponding to the standard optical potential for elastic scattering, thus leaving the operator $\hat{U}$ as the principal object of our interest. Further, because *PUP* may be thought of as well known, we have sought to relate $P_* \hat{U} P_*$ to *PUP*, so that we may take maximum advantage of the elastic information available to us. In Sec. IV we study the excited state distorted wave and, among other things, discuss how it differs from the right-hand distorted wave as well as how it may be related thereto. The discussion in Sec. IV indicates a number of reasonably satisfactory options open to us to obtain an acceptable approximation to the final state distorted wave; this is based principally on a multiple scattering approach together with a knowledge of the relation between the structure of the ground and excited states of the target.

The preceding remarks indicate the possibility that the left-hand distorted wave state of Eq. (1.6) may be obtained somewhat independently of a complete knowledge of $\hat{U}$. This is especially the case when the diagonal projection of $\hat{U}$, $P_* \hat{U} P_*$, is much larger than the off-diagonal projection $P_* \hat{U} P$. Sec. III concerns itself with the operator $\hat{U}$ with particular, but not exclusive, emphasis on $P_* \hat{U} P$. A multiple scattering treatment of $\hat{U}$ is presented which is closely related to the Watson series. Truncations and approximations to $\hat{U}$ which make calculation of the transition potential $P_* \hat{U} P$ and the distorting potential $P_* \hat{U} P_*$ practicable are suggested.

Finally, we present a *consistent* single scattering treatment of the inelastic distorted wave matrix element in Sec. V. Here we show how a consistent truncation leads to a distorted wave matrix element of the form given in Eq. (5.19), viz.,

$$T_{fi}^{\text{inel}} = \langle \xi_*^{(-)} \mid A P_* t P \mid \xi^{(+)} \rangle \ , \tag{1.7}$$

where t is the free nucleon-nucleon t matrix, and the distorted waves are given by

$$\mid \xi^{(+)} \rangle = \mid \vec{k}, \phi \rangle + G_0 (A-1) P t P \mid \xi^{(+)} \rangle \ , \tag{1.8}$$

which is the scaled first order Kerman, McManus, and Thaler[7] (KMT) elastic scattering distorted wave, and

$$\langle \hat{\xi}_*^{(-)} \mid = \langle \vec{k}', \phi_* \mid + \langle \hat{\xi}_*^{(-)} \mid (A-1) P_* \tau P_* G_0 \ , \tag{1.9}$$

which is a scaled first order KMT-type distorted wave for the excited state, but is not of the standard type since τ takes the place of the free t. This result, most especially in the case where $P_* \tau P_* \simeq P_* t P_*$ is a good approximation, indicates that a calculationally straightforward approximation, containing a good deal of the physics, is possible through strongly physically motivated manipulation of the elements of the formal theory.

II. TWO POTENTIAL FORMULATION

The fundamental quantity of interest for inelastic scattering of protons from nuclei is the transition matrix element T_{fi}^{inel} which can be expressed as

$$T_{fi}^{\text{inel}} = \langle \vec{k}', \phi_* \mid V \mid \Psi^{(+)} \rangle \ . \tag{2.1}$$

Here $\mid \Psi^{(+)} \rangle$ is the full state vector for the projectile plus nucleus system and has the incident boundary condition of a plane wave for the projectile with momentum $\vec{k}$ impinging upon the target nucleus in its ground state ϕ. Contained in $\mid \Psi^{(+)} \rangle$ are outgoing spherical waves in all possible open channels in one of which the target is in the bound excited state ϕ_* with an outgoing nucleon of final momentum $\vec{k}'$. The external channel interaction potential V is the sum of the interaction potentials between the projectile and each of the A target nucleons,

$$V = \sum_{i=1}^{A} v_{oi} \ , \tag{2.2}$$

under the "two-body forces only" assumption. The total Hamiltonian for the system is given by

$$H = h_0 + H_A + V \tag{2.3}$$

or

$$H = H_0 + V \ , \tag{2.4}$$

where h_0 is the operator for the kinetic energy of the projectile relative to the target's center of mass, and H_A is the intrinsic Hamiltonian of the target.

The state vector $\mid \Psi^{(+)} \rangle$ in Eq. (2.1) satisfies the Lippmann-Schwinger equation

$$\mid \Psi^{(+)} \rangle = \mid \vec{k}, \phi \rangle + G_0 V \mid \Psi^{(+)} \rangle \ , \tag{2.5}$$

where

$$G_0 = \frac{1}{E + i\eta - H_0} = \frac{1}{E + i\eta - h_0 - H_A} \ . \tag{2.6}$$

In a completely standard fashion, we introduce the projector onto the target ground state ϕ,

$$P = |\phi\rangle\langle\phi| \ , \tag{2.7}$$

along with its conjugate projector Q,

$$Q = 1 - P \ . \tag{2.8}$$

These are projectors in the space of the A target nucleons. We also require the projectors P_* and Q_* defined as

$$P_* = |\phi_*\rangle\langle\phi_*| \tag{2.9}$$

and

$$Q_* = 1 - P_* \ . \tag{2.10}$$

Our main aim in this section is to convert Eq. (2.1) into a distorted wave expression for T_{fi}^{inel}. One method[8] for so doing involves separation of the external potential V into two parts, as

$$V = W_* + (V - W_*) = V_1 + V_2 \ , \tag{2.11}$$

where W_* is a two-body distorting potential for projectile-target relative motion. The *only* requirement we impose on the choice of W_* at this stage is that it be diagonal in the basis of target eigenstates (so that it cannot contribute directly to the inelastic transition); otherwise, W_* remains free.

The method of two potentials is to substitute the two components of V from Eq. (2.11) into Eq. (2.1) for T_{fi}^{inel} and also into Eq. (2.5) for $|\Psi^{(+)}\rangle$. The standard manipulations for scattering from two potentials[9] then give us

$$T_{fi}^{\text{inel}} = \langle \vec{k}', \phi_* | W_* | \psi^{(+)} \rangle + \langle \psi_*^{(-)} | (V - W_*) | \Psi^{(+)} \rangle \ , \tag{2.12}$$

where

$$|\psi^{(+)}\rangle = |\vec{k}, \phi\rangle + G_0 W_* |\psi^{(+)}\rangle \tag{2.13}$$

and

$$\langle \psi_*^{(-)}| = \langle \vec{k}', \phi_*| + \langle \psi_*^{(-)}| W_* G_0 \ . \tag{2.14}$$

In Eq. (2.12) the first term represents the inelastic scattering due to W_* only.

Let us now consider the first term of Eq. (2.12). From Eq. (2.13) and the diagonal property of W_*, $(P_* W_* P = 0)$, we have

$$P|\psi^{(+)}\rangle = |\psi^{(+)}\rangle; \ P_*|\psi^{(+)}\rangle = 0 \ ; \tag{2.15}$$

and hence

$$\langle \vec{k}', \phi_* | W_* | \psi^{(+)} \rangle = \langle \vec{k}', \phi_* | P_* W_* P | \psi^{(+)} \rangle = 0 \ . \tag{2.16}$$

Thus W_* does indeed play the role of a distorting potential which makes no direct contribution to inelastic scattering and we have the exact relation

$$T_{fi}^{\text{inel}} = \langle \psi_*^{(-)} | V - W_* | \Psi^{(+)} \rangle \ . \tag{2.17}$$

Again from the diagonal property of W_* and Eq. (2.14), we have

$$\langle \psi_*^{(-)}| P_* = \langle \psi_*^{(-)}|; \ \langle \psi_*^{(-)}| P = 0 \ . \tag{2.18}$$

Thus Eq. (2.17) can be written

$$T_{fi}^{\text{inel}} = \langle \psi_*^{(-)} | P_* (V - W_*) | \Psi^{(+)} \rangle \ , \tag{2.19}$$

and it should be clear that only the $P_* W_* \equiv P_* W_* P_*$ part of W_* matters in Eqs. (2.17) or (2.19). Eventually, the exact many-body state vector $|\Psi^{(+)}\rangle$ must be approximated, and the choice of W_* should be coordinated with that approximation so that the description of T_{fi}^{inel} is as physically sensible and as numerically accurate as is practically possible. These are difficult considerations and the standard reaction models give insufficient attention to these points. The standard distorted wave Born approximation (DWBA) can be obtained from Eq. (2.19) by the truncation of the fully distorted state vector $|\Psi^{(+)}\rangle \to P|\Psi^{(+)}\rangle \equiv |\chi^{(+)}\rangle$, so that Eq. (2.19) becomes

$$T_{fi}^{\text{inel}} \simeq \langle \psi_*^{(-)} | P_* V P | \chi^{(+)} \rangle \ , \tag{2.20}$$

where the contribution from W_* has been eliminated since $P_* W_* P = 0$. Here $|\chi^{(+)}\rangle$ is a product of the target ground state and a projectile-target relative motion wave function which, by definition, is the elastic scattering wave function.

Apparently, in the DWBA the final distorting potential W_* can be chosen in a manner which is *not at all* influenced by the content of the two-body wave function $|\chi^{(+)}\rangle$. Heuristic physical arguments are used at this point in standard treatments to choose $P_* W_* P_*$ as an optical potential. This optical potential is usually chosen for practical reasons to be the optical potential for elastic scattering from the target ground state. There is, however, very little in the way of a theoretical basis for this procedure.

It should be clear from the nature of the derivation of Eqs. (2.19) and (2.20) that significant effects can enter through those parts of $|\Psi^{(+)}\rangle$ which are described by $Q|\Psi^{(+)}\rangle$. We can write Eq. (2.19) as

$$T_{fi}^{\text{inel}} = \langle \psi_*^{(-)} | P_* V P | \chi^{(+)} \rangle + \langle \psi_*^{(-)} | P_* (V - W_*) Q | \Psi^{(+)} \rangle \ . \tag{2.21}$$

The role of the second term on the right-hand side

of Eq. (2.21) can be determined if we know something more about the microscopic basis of the development. For example, the distorted wave impulse approximation[7] (DWIA) which we shall eventually study arises from consideration of the second term of Eq. (2.21). The formulation that we arrive at in this paper can be derived from Eq. (2.21) through an explicit choice for the microscopic content of W_*. However, because the final nuclear state is an excited state, and so is not ordinarily accessible as a target state in a scattering experiment, the choice of the final state distorting potential cannot be guided so directly by physical motivation as can the choice of the initial state distorting potential. Accordingly, we may find it useful to interchange the treatment given to the initial and final channels in the development so far. This is particularly easy to accomplish in the present case since the "post" and "prior" forms of the transition operator are identical for inelastic scattering. A presentation of the "post" form of the argument of Eqs. (2.1)–(2.21) is given in Appendix A. Every step in the "prior" development above has its analog in the "post" development.

The "post" result analogous to Eq. (2.21) is given in Eq. (A7) as

$$T_{fi}^{\text{inel}} = \langle \Psi_*^{(-)} | P_* V P | \eta^{(+)} \rangle + \langle \Psi_*^{(-)} | Q_* (V - W) P | \eta^{(+)} \rangle , \quad (2.22)$$

the first term of which represents the "post" form of the DWBA, viz.,

$$T_{fi}^{\text{inel}} \simeq \langle \Psi_*^{(-)} | P_* V P | \eta^{(+)} \rangle . \quad (2.23)$$

Here, as was the case for Eq. (2.20), one of the distorted waves (in this case $\langle \Psi_*^{(-)} | P_*$) is fixed, but the other (in this case $P | \eta^{(+)} \rangle$) is not, since the initial distorting potential W can be chosen in a manner which is *not at all* influenced by the content of the two-body wave function $\langle \Psi_*^{(-)} | P_*$.

In practice the DWBA is used in the form

$$T_{fi}^{\text{inel}} \simeq \langle \psi_*^{(-)} | V | \eta^{(+)} \rangle , \quad (2.24)$$

where the distorted waves $\langle \psi_*^{(-)} | \equiv \langle \psi_*^{(-)} | P_*$ and $| \eta^{(+)} \rangle \equiv P | \eta^{(+)} \rangle$ are calculated from optical potentials $P_* W_* P_*$ and PWP, respectively. These potentials are chosen in a heuristic, physically appealing manner. However, to the best of our knowledge there exists no theoretical discussion of the DWBA based either on the idea that $| \eta^{(+)} \rangle \equiv P | \Psi^{(+)} \rangle \equiv | \chi^{(+)} \rangle$ and that $P_* W_* P_*$ can be chosen so as to minimize the effect of the second term in Eq. (2.21), or else based on $\langle \psi_*^{(-)} | \equiv \langle \Psi_*^{(-)} | P_*$ and a choice of PWP which minimizes the second term in Eq. (2.22).

One attitude that we could take to relieve ourselves of the ambiguity in the "post" and "prior" forms of the DWBA would be to insist on the identity of these two forms. That is we might enforce the equality

$$\langle \psi_*^{(-)} | VP | \Psi^{(+)} \rangle = \langle \Psi_*^{(-)} | P_* V | \eta^{(+)} \rangle , \quad (2.25)$$

through the even more stringent requirement that

$$\langle \psi_*^{(-)} | = \langle \Psi_*^{(-)} | P_* \quad (2.26)$$

and

$$| \eta^{(+)} \rangle = P | \Psi^{(+)} \rangle \equiv | \chi^{(+)} \rangle . \quad (2.27)$$

This identification of either $\langle \psi_*^{(-)} |$ as $\langle \Psi_*^{(-)} | P_*$ in the prior form, or $| \eta^{(+)} \rangle = | \chi^{(+)} \rangle$ as $P | \Psi^{(+)} \rangle$ in the post form, removes any ambiguity from the theory and makes it completely microscopic. We no longer have the option of the free choice of $P_* W_* P_*$ (or PWP), with the concomitant freedom to adjust the distorting potential so as to change the relative importance of the second term in Eq. (2.21) [Eq. (2.22)]. This also implies that the second term can no longer be ignored in the hope that we can make a choice of distorting potential which will justify this neglect. Thus while the choice of the distorting potential implied by Eq. (2.26) [Eq. (2.27)] does yield an unambiguous DWBA, and immediately makes the theory microscopic, it requires that we *must* include the second term in our considerations.

We now proceed with our discussion in the post form. This is convenient because the potential PWP can be identified as the usual microscopically based optical potential for elastic scattering from the nuclear ground state. Hence in this form, we can take maximum advantage of the wealth of empirical and theoretical information and insights concerning the elastic optical potential which have been developed over a long time.

In Appendix B we show that the choice

$$| \eta^{(+)} \rangle = P | \Psi^{(+)} \rangle \equiv | \chi^{(+)} \rangle$$

implies that

$$PWP = PUP , \quad (2.28)$$

where PUP is the elastic optical potential from which one may calculate $| \chi^{(+)} \rangle$ as

$$| \chi^{(+)} \rangle = | \vec{k}, \phi \rangle + G_0 PUP | \chi^{(+)} \rangle \quad (2.29)$$

with

$$U = V\frac{1}{1-G_0QV} = \frac{1}{1-VG_0Q}V$$
$$= V + V\frac{1}{G_0^{-1}-QV}QV \tag{2.30}$$

or

$$U = V + VG_0QU \ . \tag{2.31}$$

Equation (2.30) is the usual formal definition[10,11] of the microscopic optical potential operator.

If we now insert the microscopic choice $W = PUP$, where U is given by Eq. (2.30), in the matrix element for T_{fi}^{inel} as given in Eq. (2.22), we obtain

$$T_{fi}^{\text{inel}} = \langle \Psi_*^{(-)} | [V - PU]P \, | \chi^{(+)} \rangle$$
$$= \left\langle \Psi_*^{(-)} \middle| \left[QV - PV\frac{1}{G_0^{-1}-QV}QV \right] P \middle| \chi^{(+)} \right\rangle$$
$$= \left\langle \Psi_*^{(-)} \middle| \left[1 - PV\frac{1}{G_0^{-1}-QV} \right] QV \middle| \chi^{(+)} \right\rangle$$
$$= \left\langle \Psi_*^{(-)} \middle| (G_0^{-1} - QV - PV)\frac{1}{G_0^{-1}-QV}QV \middle| \chi^{(+)} \right\rangle$$
$$= \left\langle \Psi_*^{(-)} \middle| G^{-1}\frac{1}{G_0^{-1}-QV}QV \middle| \chi^{(+)} \right\rangle$$
$$= \left\langle \Psi_*^{(-)} \middle| G^{-1}G_0\frac{1}{1-QVG_0}QV \middle| \chi^{(+)} \right\rangle . \tag{2.32}$$

We then observe that the Lippman-Schwinger equation for $\langle \Psi_*^{(-)} |$, Eq. (A2), implies that

$$\langle \Psi_*^{(-)} | \, G^{-1}G_0 = \langle \Psi_*^{(-)} | \, (1 - VG_0) = \langle \vec{k}', \phi_* | \ , \tag{2.33}$$

so that Eq. (2.32) becomes

$$T_{fi}^{\text{inel}} = \left\langle \vec{k}', \phi_* \middle| \frac{1}{1-QVG_0}QV \middle| \chi^{(+)} \right\rangle$$
$$= \left\langle \vec{k}', \phi_* \middle| Q\frac{1}{1-VQG_0}V \middle| \chi^{(+)} \right\rangle$$
$$= \left\langle \vec{k}', \phi_* \middle| \frac{1}{1-VQG_0}V \middle| \chi^{(+)} \right\rangle$$
$$= \langle \hat{\Psi}_*^{(-)} | \, V \, | \chi^{(+)} \rangle \ . \tag{2.34}$$

The new state vector

$$\langle \hat{\Psi}_*^{(-)} | \equiv \langle \vec{k}', \phi_* | \, \frac{1}{1-VQG_0}$$

has been introduced into Eq. (2.34). This many-body wave function thus obeys a Lippmann-Schwinger equation of the form

$$\langle \hat{\Psi}_*^{(-)} | = \langle \vec{k}', \phi_* | + \langle \hat{\Psi}_*^{(-)} | \, VG_0Q \ . \tag{2.35}$$

That is, $\langle \hat{\Psi}_*^{(-)} |$ is formally very similar to $\langle \Psi_*^{(-)} |$, which is defined in Eq. (A2), except that $\langle \hat{\Psi}_*^{(-)} |$ evolves out of $\langle \vec{k}', \phi_* |$ only via those components of the residual channel interaction V which connect with target states in the Q space. Propagations which would allow the target to reach the ground state are specifically forbidden. The reason for this is that all such processes already have been accounted for by use of the elastic scattering wave function $|\chi^{(+)}\rangle$ as the initial distorted wave.[1] We note that Eq. (2.34) is an exact expression and that it has the same form as a DWBA expression for inelastic scattering. Indeed, if the replacement

$$\langle \hat{\Psi}_*^{(-)} | \to \langle \hat{\Psi}_*^{(-)} | \, P_* \equiv \langle \hat{\chi}_*^{(-)} |$$

in Eq. (2.32) were an excellent approximation, then the DWBA would then be an excellent approximation when the two-body state vector $\langle \hat{\chi}_*^{(-)} |$ is used for the final distortion. Of course, $\langle \hat{\Psi}_*^{(-)} |$ is obviously not a two-body state vector, as can be seen from inspection of Eq. (2.35) since the projector Q not only permits propagation of the target in the state ϕ_*, but also allows propagation to all other target states except the ground state.

The task of bringing Eq. (2.34) into the form of a useful distorted wave expression now entails the projection of $\langle \hat{\Psi}_*^{(-)} |$ onto the target state ϕ_* to extract the two-body portion of this state vector. We begin with the decomposition

$$\langle \hat{\Psi}_*^{(-)} | = \langle \hat{\Psi}_*^{(-)} | \, P_* + \langle \hat{\Psi}_*^{(-)} | \, Q_* \ , \tag{2.36}$$

and use Eq. (2.35) to interpret the second term as

$$\langle \hat{\Psi}_*^{(-)} | \, Q_* = \langle \hat{\Psi}_*^{(-)} | \, P_* VG_0QQ_*$$
$$+ \langle \hat{\Psi}_*^{(-)} | \, Q_* VG_0QQ_* \ . \tag{2.37}$$

The formal solution of Eq. (2.37) yields

$$\langle \hat{\Psi}_*^{(-)} | \, Q_* = \langle \hat{\Psi}_*^{(-)} | \, P_* VG_0QQ_* \frac{1}{1-VG_0QQ_*} \ , \tag{2.38}$$

and when this expression is substituted back into Eq. (2.36) the result is

$$\langle \hat{\Psi}_*^{(-)} | = \langle \hat{\Psi}_*^{(-)} | \, P_* \{ (1 - VG_0QQ_*) + VG_0QQ_* \}$$
$$\times \frac{1}{1-VG_0QQ_*} \tag{2.39}$$

or

$$\langle\hat{\Psi}_*^{(-)}| = \langle\hat{\Psi}_*^{(-)}| P_* \frac{1}{1-VG_0QQ_*} , \tag{2.40}$$

so that we have finally

$$\langle\hat{\Psi}_*^{(-)} = \langle\hat{\chi}_*^{(-)}| \frac{1}{1-VG_0QQ_*} . \tag{2.41}$$

Here we have introduced the definition

$$\langle\hat{\chi}_*^{(-)}| = \langle\hat{\Psi}_*^{(-)}| P_* , \tag{2.42}$$

for the two-body projection of $\langle\hat{\Psi}_*^{(-)}|$ onto the final excited state of the target. We will examine this wave function shortly. When Eq. (2.41) is substituted into Eq. (2.34) the *exact* inelastic transition amplitude becomes[1]

$$T_{fi}^{\text{inel}} = \langle\hat{\chi}_*^{(-)}| P_* \hat{U} P |\chi^{(+)}\rangle , \tag{2.43}$$

where we have introduced the operator $\hat{U}$ defined by

$$\hat{U} = \frac{1}{1-VG_0QQ_*} V = V \frac{1}{1-QQ_*G_0V} , \tag{2.44}$$

or

$$\hat{U} = V + VG_0QQ_*\hat{U} . \tag{2.45}$$

Comparison of Eq. (2.44) with Eq. (2.30) shows that there is a very close relationship between the distorted transition operator $\hat{U}$ for inelastic scattering and the elastic scattering optical potential operator U. We will exploit this relationship later when considering the microscopic content of this formalism. The final distorted wave $\langle\hat{\chi}_*^{(-)}|$ needs to be examined before we can consider the usefulness of the distorted wave expression of Eq. (2.43). Using Eqs. (2.42) and (2.35) we have

$$\langle\hat{\chi}_*^{(-)}| = \langle\vec{k}',\phi_*| + \langle\hat{\Psi}_*^{(-)}| VG_0QP_* \tag{2.46}$$

and substituting Eq. (2.41) into the second term we get

$$\langle\hat{\chi}_*^{(-)}| = \langle\vec{k}',\phi_*| + \langle\hat{\chi}_*^{(-)}| P_*\hat{U}P_*G_0 , \tag{2.47}$$

where we have made use of Eq. (2.44) and the identity $QP_* \equiv P_*$. Thus the distorting potential which generates the final distorted wave can be obtained from the diagonal matrix element (with respect to the excited state ϕ_* of the target) of the same operator $\hat{U}$ whose off-diagonal matrix element describes the inelastic transition.

The result given in Eqs. (2.43)–(2.47) is derived in Appendix C, without specific reference to the two-potential formulation. By so doing we obtain a result equally applicable to elastic and inelastic scattering. This permits us to make certain comparisons of the two situations with greater facility.

We emphasize that the distorted wave matrix element in Eq. (2.43) is an *exact* formal expression and that it has the standard *form* used in present calculations. Specifically, Eq. (2.43) can be written in position space as

$$T_{fi}^{\text{inel}} = \int d^3r'd^3r\, g^{(-)}(\vec{k}',\vec{r}') \times \hat{u}(\vec{r}',\vec{r}) f^{(+)}(\vec{k},\vec{r}) , \tag{2.48}$$

where

$$\hat{u}(\vec{r}',\vec{r}) = \langle\vec{r}',\phi_*|\hat{U}|\phi,\vec{r}\rangle , \tag{2.49}$$

$$f^{(+)}(\vec{k},\vec{r}) = \langle\vec{r},\phi|\chi^{(+)}\rangle , \tag{2.50}$$

and

$$g^{(-)}(\vec{k}',\vec{r}) = \langle\hat{\chi}_*^{(-)}|\phi_*,\vec{r}'\rangle . \tag{2.51}$$

Here $\vec{r}$ and $\vec{r}'$ are the nucleon-nucleus relative coordinates in the initial and final channels, respectively. The utility of this form for the transition amplitude depends upon the ease with which the transition potential $\hat{u}(\vec{r}',\vec{r})$ and the final distorted wave $g^{(-)}(\vec{k}',\vec{r}')$ can be calculated. The full complexity of the many-body problem is contained in $\hat{u}(\vec{r}',\vec{r})$ and in the next section we develop an expansion of this quantity which follows closely the microscopic theories for the elastic scattering optical potential.

III. MICROSCOPIC CONSIDERATIONS

The *exact* distorted wave transition matrix element for inelastic scattering given in Eq. (2.43), for T_{fi}^{inel},

$$T_{fi}^{\text{inel}} = \langle\hat{\chi}_*^{(-)}| P_*\hat{U}P|\chi^{(+)}\rangle , \tag{3.1}$$

where

$$|\chi^{(+)}\rangle = |\vec{k},\phi\rangle + G_0PUP|\chi^{(+)}\rangle , \tag{3.2}$$

with

$$U = V + VG_0QU \tag{3.3}$$

and

$$\langle\hat{\chi}_*^{(-)}| = \langle\vec{k}',\phi_*| + \langle\hat{\chi}_*^{(-)}| P_*\hat{U}P_*G_0 , \tag{3.4}$$

with

$$\hat{U} = V + VG_0QQ_*\hat{U} , \tag{3.5}$$

is the result of Sec. II under study here. Clearly the

matrix element T_{fi}^{inel} can be calculated once the three effective interactions $P_* \hat{U} P_*$, $P_* \hat{U} P$, and PUP are known. These effective two-body interactions are the final distorting potential, the distorted inelastic transition potential, and the initial distorting potential, respectively. The initial distorting potential is, by definition, the elastic optical potential and we assume here that this potential is adequately described through an analysis of elastic data, and/or a standard microscopic treatment of elastic scattering. Let us now consider the microscopic content of the remaining two interactions. Since both of these are to be calculated from different target matrix elements of the same operator $\hat{U}$, we shall study the microscopic content of that operator as a guide to the approximations which must be made for practical calculations. We note that the final distorting potential $P_* \hat{U} P_*$ is diagonal with respect to the excited state of the target and will be similar to the final state optical potential in that it should be dominated by the shape of the (final) nuclear density. We observe that Eq. (3.5) for $\hat{U}$ differs from Eq. (3.3) for U only in the appearance of the extra projector Q_*. In the elastic case a projection off the target ground state is required to avoid inclusion in the optical potential of those intermediate scatterings already accounted for by the scattering equation, Eq. (3.2), which produces the elastic wave function. In the present inelastic case, the extra projection in Eq. (3.5) is a direct consequence of the appearance of both initial and final distorted waves in the matrix element. The intermediate states already accounted for in Eq. (3.4), which produces the final distorted wave function, must also be removed from the interaction used to describe the transition.[1]

The operator $\hat{U}$ of Eq. (3.5) may be expanded in analogy to the usual Watson multiple scattering series[12] as

$$\hat{U} = \sum_{i=1}^{A} \hat{\tau}_i + \sum_{i \neq j}^{A} \hat{\tau}_i G_0 Q Q_* \hat{\tau}_j + \sum_{i \neq j \neq k}^{A} \cdots , \tag{3.6}$$

where

$$\hat{\tau}_i = v_{0i} + v_{0i} G_0 Q Q_* \hat{\tau}_i . \tag{3.7}$$

Here $\hat{\tau}_i$ is a scattering operator for the projectile and the ith nucleon of the target; it is not a two-body quantity both because of the projection operators in its definition and because of the many-body nature of the Green's function G_0. The first term of Eq. (3.6) is the sum of all terms of $\hat{U}$ in which the projectile interacts with one target nucleon at a time.[13] We focus our attention on this single scattering term and, accordingly, take

$$\hat{U} \simeq \sum_{i=1}^{A} \hat{\tau}_i . \tag{3.8}$$

We now seek to relate this single scattering approximation to the "impulse" approximation in which $\hat{\tau}_i$ is eliminated in favor of a free projectile-nucleon t matrix.

We define the many-body operator t_i to be given by

$$\begin{aligned} t_i &= v_{0i} + v_{0i} G_0 t_i \\ &= v_{0i} + t_i G_0 v_{0i} \\ &= (1 + t_i G_0) v_{0i} , \end{aligned} \tag{3.9}$$

so that if we multiply Eq. (3.7) from the left by the factor $(1 + t_i G_0)$ we get

$$(1 + t_i G_0) \hat{\tau}_i = t_i + t_i G_0 Q Q_* \hat{\tau}_i , \tag{3.10}$$

or, through the use of the identity $Q Q_* = 1 - P - P_*$,

$$\hat{\tau}_i = t_i - t_i G_0 (P + P_*) \hat{\tau}_i . \tag{3.11}$$

Equation (3.11) is the basic relation between $\hat{\tau}_i$ and t_i. We note, of course, that t_i is also a many-body operator and that further discussion is required in order to establish the circumstances under which t_i may be reasonably well approximated by some two-body transition operator. That familiar question will not be raised at this point of the argument.

The operators of interest to us are $P_* \hat{\tau}_i P_*$ and $P_* \hat{\tau}_i P$, since in the approximation of Eq. (3.8), that is all we need to obtain $P_* \hat{U} P_*$ and $P_* \hat{U} P$. From Eq. (3.11) we immediately see that $P_* \hat{\tau}_i P$ can be obtained as the solution of the pair of coupled equations

$$\begin{aligned} P_* \hat{\tau}_i P = {} & P_* t_i P - P_* t_i P_* G_0 P_* \hat{\tau}_i P \\ & - P_* t_i P G_0 P \hat{\tau}_i P \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} P \hat{\tau}_i P = {} & P t_i P - P t_i P_* G_0 P_* \hat{\tau}_i P \\ & - P t_i P G_0 P \hat{\tau}_i P . \end{aligned} \tag{3.13}$$

Similarly $P_* \hat{\tau}_i P_*$ can be obtained as the solution of the pair of coupled equations

$$\begin{aligned} P_* \hat{\tau}_i P_* = {} & P_* t_i P_* - P_* t_i P G_0 P \hat{\tau}_i P_* \\ & - P_* t_i P_* G_0 P_* \hat{\tau}_i P_* \end{aligned} \tag{3.14}$$

and

$$P\hat{\tau}_i P_* = Pt_i P_* - Pt_i P G_0 P \hat{\tau}_i P_* - Pt_i P_* G_0 P_* \hat{\tau}_i P_* \ . \tag{3.15}$$

The second and third terms on the right-hand side of Eqs. (3.12)–(3.15) remove from the first term, t, those intermediate states of the system which are already accounted for by the initial and final distorted waves. Accordingly the resulting operator $\hat{\tau}$ may be said to contain "orthogonality blocking" effects. These "blocked" states are not excluded from the scattering process, they have simply been included elsewhere.

On the assumption that t_i is well approximated by the free t matrix for projectile-nucleon scattering, and that we may factorize[14] the four "$t\rho$" terms in Eqs. (3.12)–(3.15), we may write

$$\langle \vec{k}'\phi_* \mid t_i \mid \vec{k}\phi \rangle \simeq t(\epsilon,q)\rho_{\rm tr}(q) \tag{3.16}$$

and

$$\langle \vec{k}',\phi \mid t_i \mid \vec{k},\phi \rangle \simeq t(\epsilon,q)\rho(q) \ , \tag{3.17}$$

where $\vec{q}=\vec{k}'-\vec{k}$ is the momentum transfer, $\rho(q)$ is the one-body target ground-state diagonal density, and $\rho_{\rm tr}(q)$ is the one-body transition density. Here ϵ is a suitable two-body relative energy which for definiteness we might take to be half of the laboratory energy of the nucleon projectile. The quantities given in Eqs. (3.16) and (3.17) can serve as the input to the coupled equations, Eqs. (3.12) and (3.13). Similarly, the transition and diagonal excited state density when folded in with t provide us with the necessary information for Eqs. (3.14) and (3.15). We suppress the spin and isospin dependence of t and ρ so as not to obscure our main point.

In order to be explicit, we introduce the notation

$$\langle \vec{k}'\phi_* \mid P_*\hat{\tau}P \mid \phi,\vec{k} \rangle \equiv \hat{w}(\vec{k}',\vec{k}) \tag{3.18}$$

and

$$\langle \vec{k}'\phi \mid P\hat{\tau}P \mid \phi,\vec{k} \rangle = \hat{x}(\vec{k}',\vec{k}) \ , \tag{3.19}$$

so that Eqs. (3.12) and (3.13) become, after use of the factorization approximation for target matrix elements of t,

$$\hat{w}(\vec{k}',\vec{k}) = t(\epsilon,q)\rho_{\rm tr}(q) - \int d^3k'' \frac{t(\epsilon, \mid \vec{k}'-\vec{k}'' \mid)\rho_*(\mid \vec{k}'-\vec{k}'' \mid)\hat{w}(\vec{k}'',\vec{k})}{E(k_0)+i\eta-E_x-E(k'')} - \int d^3k'' \frac{t(\epsilon, \mid \vec{k}'-\vec{k}'' \mid)\rho_{\rm tr}(\mid \vec{k}'-\vec{k}'' \mid)\hat{x}(\vec{k}'',\vec{k})}{E(k_0)+i\eta-E(k'')} \tag{3.20}$$

and

$$\hat{x}(\vec{k}',\vec{k}) = t(\epsilon,q)\rho(q) - \int d^3k'' \frac{t(\epsilon, \mid \vec{k}'-\vec{k}'' \mid)\rho_{\rm tr}(\mid \vec{k}'-\vec{k}'' \mid)\hat{w}(\vec{k}'',\vec{k})}{E(k_0)+i\eta-E_x-E(k'')} - \int d^3k'' \frac{t(\epsilon, \mid \vec{k}'-\vec{k}'' \mid)\rho(\mid \vec{k}'-\vec{k}'' \mid)\hat{x}(\vec{k}'',\vec{k})}{E(k_0)+i\eta-E(k'')} \ . \tag{3.21}$$

Because of the identity of target particles, the particle index i has been dropped as unnecessary, and the sum over i replaced by multiplication by A. Thus, from Eq. (3.8) we have

$$P_*\hat{U}P = P_* \sum_i \hat{\tau}_i P = AP_*\hat{\tau}P \ , \tag{3.22}$$

from which we obtain the inelastic transition potential, $\hat{u}(\vec{k}',\vec{k})$, as

$$\hat{u}(\vec{k}',\vec{k}) = A\hat{w}(\vec{k}',\vec{k}) \ . \tag{3.23}$$

A similar treatment of the diagonal excited state matrix element $P_*\hat{U}P_*$ is easily obtained. This is the matrix element required for the diagonal distorting potential from which we may calculate the final state distorted wave. We postpone consideration of the final distorting potential until the next section. Now we wish to consider whether there exist circumstances under which the last term in Eq. (3.20) becomes negligible. If that term can safely be ignored then Eq. (3.20) is no longer coupled to Eq. (3.21) and we need only solve a single channel, uncoupled integral equation. Completely equivalent remarks can, of course, be made about the more general form of these equations given in Eqs. (3.12) and (3.13). We work with these operator relations because we feel the skeleton of the argument is most clearly revealed in this form. Thus we examine Eq. (3.12) for $P_*\hat{\tau}P$, which we partially solve as

$$P_* \hat{\tau} P = \frac{1}{1+P_* t P_* G_0} P_* t P - \frac{1}{1+P_* t P_* G_0} P_* t P G_0 P \hat{\tau} P \ . \quad (3.24)$$

The neglect of the last term on the right-hand side of Eq. (3.12) is equivalent to the neglect of the last term on the right of Eq. (3.24). However, the structure of Eq. (3.24) shows clearly that we may not neglect that last term. If we write Eq. (3.24) as

$$P_* \hat{\tau} P = \frac{1}{1+P_* t P_* G_0} P_* t P (1 - G_0 P \hat{\tau} P) \ , \quad (3.25)$$

we see that we may no more replace the factor $(1-G_0 P \hat{\tau} P)$ by unity than we may replace the factor $[1+P_* t P_* G_0]^{-1}$ by unity. In either case this implies neglect of terms which are in every way equivalent to terms which we conventionally keep in the evaluation of the distorted waves in the standard first order multiple scattering approximation. Thus we contend that we must either solve the coupled equations (3.12) and (3.13) for the transition potential $AP_* \hat{\tau} P$, or else take the attitude that the transition potential is $AP_* tP$ and that the two factors in Eq. (3.25) diagonal in P and P_*, respectively, have to be absorbed into the distorted waves. The latter attitude, we shall see, leads to a generalization of the KMT argument. The point we now stress, however, is that a consistent DWIA in which we have $AP_* tP$ as the transition operator to be evaluated between distorted waves requires very careful consideration as to the nature of the distorted waves to be used in the evaluation of the inelastic scattering transition matrix element.

A possible attitude for us to adopt, vis-a-vis the use of the DWA, is that we seek to determine the operator $P_* \hat{U} P$, in some acceptable truncation of the full microscopic expression. This effective transition operator, sandwiched between distorted waves, will yield the desired transition matrix element given in Eq. (3.1). The distorted waves could be considered separately from the effective transition operator. Since the distorted wave on the right in this matrix element is generated from PUP, the optical potential for elastic scattering, we might want to obtain $|\chi^{(+)}\rangle$ from a phenomenologically constructed optical potential. The distorted wave on the left is generated from $P_* \hat{U} P_*$, which is not an optical potential. Solution of the Lippmann-Schwinger equation

$$P_* \hat{T} P_* = P_* \hat{U} P_* + P_* \hat{U} P_* G_0 P_* \hat{T} P_*$$

does *not* yield the effective transition operator for elastic scattering from the excited state. Some further work is thus required to allow us to determine the distorting potential $P_* \hat{U} P_*$ from phenomenologically available information. Section IV of this paper concerns itself with the relation of this distorting potential to the optical potential for scattering from the excited target state as well as with the relation to the optical potential for scattering from the target ground state.

In accord with this attitude we have presented in this section the distorted wave single scattering approximation, defined by

$$P_* \hat{U} P \simeq A P_* \hat{\tau} P \ . \quad (3.26)$$

To calculate $P_* \hat{\tau} P$ we must solve a pair of coupled integral equations as discussed above. We reemphasize that further truncations or adjustments of the theory are necessary to obtain the usual DWIA.

If we are able to convince ourselves that the fundamental external interaction V only couples the ground and excited states very weakly then the distorted wave Born approximation recommends itself. Equation (3.5) for $\hat{U}$ tells us that

$$P_* \hat{U} P = P_* V P + P_* V G_0 Q Q_* \hat{U} P \ . \quad (3.27)$$

The second term on the right side of Eq. (3.27) is of higher order in the nondiagonal projection of V than is the first term. In the weak coupling limit this term may be dropped, leaving the truncation

$$P_* \hat{U} P \simeq P_* V P \ . \quad (3.28)$$

Equation (3.28) defines the distorted wave Born approximation, just as Eq. (3.26) defines the distorted wave single scattering approximation. In Sec. V, we shall explore these truncations further. There we adopt a completely microscopic viewpoint and closely correlate our treatment of the distorted waves with that of the effective transition operator.

IV. THE FINAL DISTORTING POTENTIAL

The final distorted wave to be used in the matrix element for inelastic scattering given in Eq. (3.1) is defined by Eq. (3.4) in terms of the distorting potential $P_* \hat{U} P_*$. It is important to note that the potential $P_* \hat{U} P_*$ is *not* the optical potential corresponding to elastic scattering from the target in the excited state because the operator $\hat{U}$ has both the excited state ϕ_* and the ground state ϕ removed from its intermediate states, whereas a proper "excited state optical potential" would only have the excited state

ϕ_* projected out. Actually $P_*\hat{U}P_*$ is the "reduced" final channel optical potential that would appear in a two-channel description of the coupling between elastic and inelastic scattering. The other three potentials appearing in such a formulation would be $P_*\hat{U}P$, $P\hat{U}P_*$, and $P\hat{U}P$. The absorptive parts of these potentials arise from the flux emerging in all reaction channels but the elastic channel and the inelastic channel. The unitarity properties of the operators $\hat{U}$ and U have been explicated previously.[1]

In principle, the multiple scattering expansion for $\hat{U}$ given in the preceding section could be used to calculate $P_*\hat{U}P_*$ under the same conditions that similar fundamental calculations of elastic optical potentials would be reliable. Alternatively, we may try to use knowledge of the microscopic content of the operator $\hat{U}$ to relate $P_*\hat{U}P_*$ to the elastic optical potential. We consider first the special case wherein the elastic scattering from the residual nucleus is physically accessible, as, for example, may be the case where the excited final state of the residual nucleus is the isobaric analog of a ground state of another nucleus. We assume that final state optical potentials are available or can be deduced from isospin symmetry considerations. We shall now investigate what adjustment must be made in order to use such an optical potential as the final distorting potential. The final "excited" state optical potential is defined as $P_*U_*P_*$, where

$$U_* = V + VG_0Q_*U_* \ , \tag{4.1}$$

in analogy to U. The optical potential $P_*U_*P_*$ gives the wave function for elastic scattering from a target in the excited state. This is not the same as the final distorted wave called for by Eq. (3.1) and defined in Eq. (3.4). To go from one to the other, Eq. (3.4) can be written as

$$\langle \hat{\chi}_*^{(-)} | = \langle \vec{k}', \phi_* | \frac{1}{1-\hat{U}P_*G_0} \tag{4.2}$$

or

$$\langle \hat{\chi}_*^{(-)} | = \langle \chi_*^{(-)} | (1-U_*P_*G_0)\frac{1}{1-\hat{U}P_*G_0} \ , \tag{4.3}$$

where we have introduced

$$\langle \chi_*^{(-)} | = \langle \vec{k}', \phi_* | \frac{1}{1-U_*P_*G_0} \tag{4.4}$$

as the state vector corresponding to the final state elastic scattering optical potential $P_*U_*P_*$. Substituting Eq. (4.3) into Eq. (3.1) for the elastic transition amplitude, we have[15]

$$T_{fi}^{\text{inel}} = \langle \chi_*^{(-)} | P_*\hat{\hat{U}}P | \chi^{(+)} \rangle \ , \tag{4.5}$$

where

$$\hat{\hat{U}} = (1-U_*P_*G_0)\frac{1}{1-\hat{U}P_*G_0}\hat{U} \ . \tag{4.6}$$

We now seek to express $\hat{\hat{U}}$ in terms of $\hat{U}$ by relating U_* to $\hat{U}$. We have

$$\hat{U} = V + VG_0Q_*Q\hat{U} \tag{4.7}$$

or

$$\hat{U} = V + VG_0Q_*\hat{U} - VG_0P\hat{U} \ , \tag{4.8}$$

where we have used $Q_*Q = Q_* - P$. Equation (4.8) can be written as

$$\hat{U} = \frac{1}{1-VG_0Q_*}V - \frac{1}{1-VG_0Q_*}VG_0P\hat{U} \ , \tag{4.9}$$

which, after the identification of U_* through Eq. (4.1), becomes

$$\hat{U} = U_* - U_*G_0P\hat{U} = U_* - \hat{U}G_0PU_* \ , \tag{4.10}$$

or

$$U_* = \frac{1}{1-\hat{U}PG_0}\hat{U} \ . \tag{4.11}$$

Substitution of Eq. (4.11) into Eq. (4.6) gives, after straightforward manipulation,

$$P_*\hat{\hat{U}}P = P_*\hat{U}P - P_*\hat{U}P\frac{1}{G_0^{-1}-P\hat{U}P}P\hat{U}P_* \times \frac{1}{G_0^{-1}-P_*\hat{U}P_*}P_*\hat{U}P \ . \tag{4.12}$$

From Eq. (4.12) it can be seen that the difference between $P_*\hat{\hat{U}}P$ and $P_*\hat{U}P$ is a term which is of third order in off-diagonal matrix elements of $\hat{U}$. In the weak coupling limit, i.e., where the off diagonal target matrix elements are inherently much smaller than diagonal target matrix elements of $\hat{U}$, the neglect of the second term of Eq. (4.12) may be pragmatically justified. Then Eq. (4.5) becomes

$$T_{fi}^{\text{inel}} \simeq \langle \chi_*^{(-)} | P_*\hat{U}P | \chi^{(+)} \rangle \ , \tag{4.13}$$

where the final distorted wave describes "elastic scattering" from the residual nucleus and is given by

$$\langle \chi_*^{(-)} | = \langle \vec{k}', \phi_* | + \langle \chi_*^{(-)} | P_*U_*P_*G_0 \ . \tag{4.14}$$

When the coupling is not weak enough for us to neglect the second term of Eq. (4.12), the advantage of having the particular final distorted wave in Eq. (4.5) may be far outweighed by the complexity and the redundancy built into $P_* \hat{U} P$.

The more usual situation is that in which elastic scattering from the residual nucleus is not experimentally accessible. There is then no particular advantage to using Eq. (4.5) and Eq. (3.1) is more appropriate. We assume here that a phenomenological optical potential for elastic scattering from the target is available and gives a faithful description of both PUP and the elastic distorted wave even in the nuclear interior. Calculations of T_{fi}^{inel} from Eq. (3.1) require that Eq. (3.4) be solved for the final distorted wave. Hence we need the potential $P_* \hat{U} P_*$, whereas we only have direct experimental information relevant to PUP. From Eqs. (3.3) and (3.5) for U and $\hat{U}$, respectively, we recall that U has the ground state of the target projected out, whereas $\hat{U}$ has both the ground state and the excited state of interest projected out. Since in principle an infinity of states (bound and continuum) of the target system remain for the intermediate states of both operators, one might expect that these operators are not significantly different. The difficulty with this argument can be seen if one relates $\hat{U}$ to U through Eqs. (3.3) and (3.5), viz.,

$$\hat{U} = V + VG_0 Q \hat{U} - VG_0 P_* \hat{U} \tag{4.15}$$

or

$$\hat{U} = \frac{1}{1 - VG_0 Q} V - \frac{1}{1 - VG_0 Q} VG_0 P_* \hat{U} , \tag{4.16}$$

which immediately yields

$$\hat{U} = U - UG_0 P_* \hat{U} . \tag{4.17}$$

For $P_* \hat{U} P_*$, Eq. (4.17) becomes

$$P_* \hat{U} P_* = P_* U P_* - P_* U P_* G_0 P_* \hat{U} P_* , \tag{4.18}$$

or equivalently

$$[-P_* \hat{U} P_*] = [-P_* U P_*] + [-P_* U P_*] G_0 [-P_* \hat{U} P_*] . \tag{4.19}$$

This equation involves only diagonal elements and we see that $-P_* \hat{U} P_*$ is related to $-P_* U P_*$ in the same way that a transition operator is related to a potential for projectile-nucleus "elastic" scattering. Thus $P_* \hat{U} P_*$ can only be approximated by $P_* U P_*$ under roughly the same circumstances that the Born approximation is valid for elastic scattering.

Let us now consider how Eq. (4.18) [or Eq. (4.19)] may be used to calculate $P_* \hat{U} P_*$, and hence the final distorted wave, once $P_* U P_*$ is known. Equation (4.19) can be converted to the equivalent Schrödinger differential equation in position space if we may consider $P_* U P_*$ to be local, and the "phase shifts" corresponding to the potential $[-P_* U P_*]$ can be obtained in the usual way. Summation over partial waves will then yield the "on-shell T matrix" which is the on-shell momentum matrix element of $[-P_* \hat{U} P_*]$. The on-shell condition means that the latter is a function of momentum transfer (and energy) only. The usual spherical Bessel transform will then yield a local function in position space. After a sign change this should be a good representation of the local equivalent to the inherently nonlocal potential $\langle \vec{r}', \phi_* | P_* \hat{U} P_* | \phi_* \vec{r} \rangle$. This local potential could then be used to calculate the final distorted wave in the usual fashion. The above "local" procedure may well introduce errors in the higher momentum transfer components of the resulting potential, a point which needs investigation. However to our knowledge practical prescriptions for the final distorted wave which are reasonably well based upon a consistent underlying theory have not previously been developed even to the level of the above discussion.

We suggest here yet another way to calculate the final distorted wave [Eq. (3.4)] directly from $P_* U P_*$. This method, which eliminates the need to introduce the local equivalent discussed above, entails the solution of only one scattering equation and is to be recommended from a practical point of view. From Eq. (4.18), $P_* U P_*$ can be thought of as the "elastic transition operator" associated with the potential $P_* \hat{U} P_*$. The corresponding wave function can therefore be calculated directly from the "transition operator" $P_* U P_*$. To illustrate this, Eq. (4.18) can be written in the form

$$P_* \hat{U} P_* = (1 - P_* \hat{U} P_* G_0) P_* U P_* . \tag{4.20}$$

Thus

$$\langle \hat{\chi}_*^{(-)} | P_* \hat{U} P_* = \langle \hat{\chi}_*^{(-)} | (1 - P_* \hat{U} P_* G_0) P_* U P_* \tag{4.21}$$

$$= \langle \vec{k}', \phi_* | P_* U P_* , \tag{4.22}$$

where we have used Eq. (3.4) which defines $\langle \hat{\chi}_*^{(-)} |$. A second use of Eq. (3.4) gives

$$\langle \hat{\chi}_*^{(-)} | = \langle \vec{k}', \phi_* | + \langle \vec{k}', \phi_* | P_* U P_* G_0 . \tag{4.23}$$

Equation (4.23) indicates that we could calculate $\langle\hat{\chi}_*^{(-)}|$ *exactly* by using P_*UP_* as the potential and solving for the associated scattering wave function in first Born approximation. With P_*UP_* given as a local function in position space this prescription could easily be carried out by performing an integration in one variable.[16] It is encouraging to be able to present procedures as practical and straightforward as those we have described which can, nevertheless, incorporate the subleties associated with the final distorted wave, thereby to establish the inelastic DWA on a firm theoretical foundation.

We conclude this section by discussing how P_*UP_*, the required input to the above procedure, might be deduced. If we assume for the present that

$$\langle\vec{k}',\phi\,|\,PUP\,|\,\phi,\vec{k}\rangle\simeq At(\epsilon,q)\rho(q)\ , \tag{4.24}$$

and likewise that

$$\langle\vec{k}',\phi_*\,|\,P_*UP_*\,|\,\phi_*,\vec{k}\rangle\simeq At(\epsilon,q)\rho_*(q)\ , \tag{4.25}$$

then

$$\langle\vec{k}',\phi_*\,|\,P_*UP_*\,|\,\phi_*,\vec{k}\rangle\simeq\langle\vec{k}',\phi\,|\,PUP\,|\,\phi,\vec{k}\rangle + At(\epsilon,q)[(\rho_*(q)-\rho(q)]\ , \tag{4.26}$$

where the densities are normalized to unity.

For a single particle excitation of the target the difference between the initial and final nuclear densities comes from the initial and final state occupied by the struck nucleon. In that circumstance, we might estimate the difference $\rho_*(q)-\rho(q)$ to be

$$\rho_*(q)-\rho(q)\simeq\rho_*^{n'l'j'}(q)-\rho^{nlj}(q)\ , \tag{4.27}$$

where $n'l'j(nlj)$ are the final (initial) single particle quantum numbers in a shell model basis, and $\rho_*^{n'l'j}(\rho^{nlj})$ is the contribution to the unit-normalized final (initial) nuclear density from the indicated single particle state. Given a microscopic model of the transition density, there is sufficient information to construct the right-hand side of Eq. (4.27). Equation (4.26) then becomes

$$\langle\vec{k}',\phi_*\,|\,P_*UP_*\,|\,\phi_*\vec{k}\rangle\simeq\langle\vec{k}',\phi\,|\,PUP\,|\,\phi,\vec{k}\rangle + At(\epsilon,q)[\rho_*(q)-\rho(q)]_{\text{s.p.}}\ , \tag{4.28}$$

where the s.p. label indicates a single particle shell model without configuration mixing determination.

We remark that Eq. (4.28) may represent a better approximation than the above presentation suggests, and the use of a phenomenological elastic optical potential as the first term of Eq. (4.28) recommends itself.

V. A CONSISTENT MICROSCOPIC TREATMENT

In previous sections we have segregated our discussions of the transition potential and the initial and final distorting potentials and have not taken full advantage of the underlying theoretical framework which links them. This segregation is appropriate when one has in mind calculations employing distorted waves obtained from phenomenologically determined distorting potentials. It was for this reason that we chose to derive our distorted wave expression for inelastic scattering so that the initial channel distorted wave is the exact elastic scattering state vector. However, we are aware that phenomenological optical potentials are constrained to fit only the asymptotic form (i.e., elastic phase shifts) of the elastic scattering wave function, while the interior portion of the wave function depends sensitively upon the assumptions made about shape, strength, locality, and other details. We derive here a consistent treatment wherein the distorting potentials are described by single scattering truncations which are coordinated with the single scattering truncation of the transition potential.

In the case where the off-diagonal coupling is very weak, we have discussed the approximation to the effective transition operator as

$$\begin{aligned}P_*\hat{U}P&=P_*\frac{1}{1-VQQ_*G_0}VP\\&=P_*VP+P_*VQQ_*\frac{1}{G_0{}^{-1}-VQQ_*}VP\\&\simeq P_*VP\ .\end{aligned} \tag{5.1}$$

If we may discard the off-diagonal matrix elements in Eq. (5.1), then we may likewise do so in the expression for $P_*\hat{U}P_*$:

$$P_*\hat{U}P_*=P_*\frac{1}{1-VQQ_*G_0}VP_*\simeq P_*VP_*\ . \tag{5.2}$$

The expression for PUP may similarly be truncated to

$$\begin{aligned}PUP&=P\frac{1}{1-VQG_0}VP\\&=PVP+PVQ\frac{1}{G_0{}^{-1}-VQ}VP\simeq PVP\ .\end{aligned} \tag{5.3}$$

Thus a consistent distorted wave Born approximation is given by

$$T_{fi}^{\text{inel}}(\text{DWBA}) = \langle \bar{\chi}_*^{(-)} | P_* V P | \bar{\chi}^{(+)} \rangle , \quad (5.4)$$

where

$$| \bar{\chi}^{(+)} \rangle = | \vec{k}, \phi \rangle + G_0 P V P | \bar{\chi}^{(+)} \rangle \quad (5.5)$$

and

$$\langle \bar{\chi}_*^{(-)} | = \langle \vec{k}', \phi_* | + \langle \bar{\chi}_*^{(-)} | P_* V P_* G_0 . \quad (5.6)$$

In the study of Coulomb excitation, where the weak coupling approximation is applicable, Eqs. (5.4)–(5.6) are indeed an acceptable theoretical starting point. Even in that case, however, it must be remembered that giant resonances require consideration of some of the off-diagonal terms neglected in Eqs. (5.4)–(5.6).

For intermediate energy proton inelastic scattering there is substantial evidence that the interior behavior of the distorted waves has a strong bearing on the outcome of calculations, especially for low-spin natural parity excitations where the transition density is significant in the nuclear interior. In particular, recent DWIA calculations[17] of such excitations by proton inelastic scattering from ^{16}O produce improved fits to data when the distorting potentials are obtained from first order microscopic calculations using the same nucleon-nucleon t matrix as is used to calculate the transition potential.

From the microscopic inelastic scattering formulation that we have developed, we now apply first order scattering truncations to all three of the required potentials to obtain a consistent distorted wave impulse approximation. In this circumstance, significant practical advantages accrue from reabsorbing certain portions of the first order transition potential into the distorted waves. The result is, we believe, the most consistent and practical prescription for carrying out a complete first order microscopic calculation.

We begin from Eq. (3.1), which is

$$T_{fi}^{\text{inel}} = \langle \hat{\chi}_*^{(-)} | P_* \hat{U} P | \chi^{(+)} \rangle , \quad (5.7)$$

and from the multiple scattering expansion given in Eqs. (3.6) and (3.7) for $\hat{U}$ and the analogous equations for U. We retain only the first terms which correspond to the projectile interacting with one target nucleon at a time. That is

$$PUP \simeq \sum_{i=1}^{A} P \tau_i P , \quad (5.8)$$

$$P_* \hat{U} P_* \simeq \sum_{i=1}^{A} P_* \hat{\tau}_i P_* , \quad (5.9)$$

and

$$P_* \hat{U} P \simeq \sum_{i=1}^{A} P_* \hat{\tau}_i P , \quad (5.10)$$

where

$$\tau_i = v_{0i} + v_{0i} G_0 Q \tau_i , \quad (5.11)$$

and $\hat{\tau}_i$ is given by Eq. (3.7). Our meaning here of first order does not imply the further approximation of τ_i and $\hat{\tau}_i$ by free projectile-nucleon t matrices. The initial (elastic) distorted wave then is given by

$$| \chi^{(+)} \rangle = | \vec{k}, \phi \rangle + G_0 A P \tau P | \chi^{(+)} \rangle \quad (5.12)$$

and the final distorted wave is given by

$$\langle \hat{\chi}_*^{(-)} | = \langle \vec{k}', \phi_* | + \langle \hat{\chi}_*^{(-)} | A P_* \hat{\tau} P_* G_0 , \quad (5.13)$$

where we have used the identity of target nucleons. We now use the relation

$$\hat{\tau}_i = \tau_i - \tau_i G_0 P_* \hat{\tau}_i , \quad (5.14)$$

which follows when v_{0i} is eliminated from Eqs. (3.7) and (5.11), to express Eq. (5.10) as

$$P_* \hat{U} P \simeq A P_* \hat{\tau} P \equiv \frac{1}{1 + P_* \tau P_* G_0} A P_* \tau P . \quad (5.15)$$

We also need the relation

$$\tau_i = t_i - t_i G_0 P \tau_i = t_i - \tau_i G_0 P t_i , \quad (5.16)$$

which follows when v_{0i} is eliminated between Eqs. (3.9) and (5.11), to write

$$P_* \tau P = P_* t P \frac{1}{1 + G_0 P t P} . \quad (5.17)$$

Combining Eqs. (5.15) and (5.17), we have

$$P_* \hat{U} P \simeq A P_* \hat{\tau} P = \frac{1}{1 + P_* \tau P_* G_0} \times A P_* t P \frac{1}{1 + G_0 P t P} . \quad (5.18)$$

The factors on either side of $AP_* tP$ in Eq. (5.18) may be thought of as Moeller wave operators for scattering from "potentials" $[-P_* \tau P_*]$ and $[-PtP]$. When Eq. (5.18) is substituted into Eq. (5.7) these factors, immediately to the left and right of $AP_* tP$, can be absorbed into the distorted waves $\langle \hat{\chi}_*^{(-)} |$ and $| \chi^{(+)} \rangle$ to produce new distorted waves which, as we shall see, are much simpler to calculate than are the original distorted waves given in Eqs. (5.12) and (5.13). We now have represented the complete first order truncation of Eq. (5.7) for T_{fi}^{inel} as

$$T_{fi}^{\text{inel}}(\text{first order}) = \langle \hat{\xi}_*^{(-)} | AP_* tP | \xi^{(+)} \rangle \ , \tag{5.19}$$

where

$$| \xi^{(+)} \rangle = \frac{1}{1+G_0 PtP} | \chi^{(+)} \rangle \tag{5.20}$$

and

$$\langle \hat{\xi}_*^{(-)} | = \langle \hat{\chi}_*^{(-)} | \frac{1}{1+P_* \tau P_* G_0} \ . \tag{5.21}$$

These new distorted waves can be explicated through combination of Eq. (5.12) with Eq. (5.20) and Eq. (5.13) with Eq. (5.21). For $| \xi^{(+)} \rangle$ this gives

$$\begin{aligned} | \xi^{(+)} \rangle &= \frac{1}{1+G_0 PtP} \frac{1}{1-G_0 AP\tau P} | \vec{k}, \phi \rangle \\ &= [(1-G_0 AP\tau P)(1+G_0 PtP)]^{-1} | \vec{k}, \phi \rangle \\ &= [\, 1-G_0 (AP\tau P + AP\tau PG_0 PtP \\ &\quad - PtP)]^{-1} | \vec{k}, \phi \rangle \\ &= [1-G_0 (APtP - PtP)]^{-1} | \vec{k}, \phi \rangle \ , \end{aligned} \tag{5.22}$$

where in the last step in Eq. (5.22), we have used Eq. (5.16) to identify $P\tau P + P\tau PG_0 PtP$ as PtP. The result is then that

$$| \xi^{(+)} \rangle = \frac{1}{1-G_0 (A-1) PtP} | \vec{k}, \phi \rangle \ . \tag{5.23}$$

The corresponding procedure for $\langle \hat{\xi}_*^{(-)} |$ gives

$$\begin{aligned} \langle \hat{\xi}_*^{(-)} | &= \langle \vec{k}', \phi_* | \frac{1}{1-AP_* \hat{\tau} P_* G_0} \frac{1}{1+P_* \tau P_* G_0} \\ &= \langle \vec{k}', \phi_* | [\, (1+P_* \tau P_* G_0) \\ &\quad \times (1-AP_* \hat{\tau} P_* G_0)]^{-1} \end{aligned} \tag{5.24}$$

or

$$\langle \hat{\xi}_*^{(-)} | = \langle \vec{k}', \phi_* | \frac{1}{1-(A-1) P_* \tau P_* G_0} \ , \tag{5.25}$$

where in the last step we have used Eq. (5.14) to eliminate $\hat{\tau}$ in favor of τ.

From Eq. (5.23), the new initial distorted wave can be immediately identified as the wave function obtained from the first order KMT auxiliary optical potential $(A-1)PtP$. Explicitly then, we have

$$| \xi^{(+)} \rangle \equiv | \chi_{\text{KMT}}^{(+)} \rangle \ , \tag{5.26}$$

where

$$| \chi_{\text{KMT}}^{(+)} \rangle = | \vec{k}, \phi \rangle + G_0 (A-1) PtP | \chi_{\text{KMT}}^{(+)} \rangle \ . \tag{5.27}$$

In a similar fashion, Eq. (5.25) identifies $\langle \xi_*^{(-)} |$ as the KMT-type wave function obtained with $(A-1)P_* \tau P_*$ as the first order auxiliary potential for the final channel scattering. In order to emphasize the KMT-type structure of $\langle \hat{\xi}_*^{(-)} |$, we write

$$\langle \hat{\xi}_*^{(-)} | \equiv \langle \hat{\chi}_{*\,\text{KMT}}^{(-)} | \ , \tag{5.28}$$

where

$$\langle \hat{\chi}_{*\,\text{KMT}}^{(-)} | = \langle \vec{k}', \phi_* | + \langle \hat{\chi}_{*\,\text{KMT}}^{(-)} | (A-1) P_* \tau P_* G_0 \ . \tag{5.29}$$

With these definitions, the consistent microscopic first order matrix element for inelastic scattering is

$$T_{fi}^{\text{inel}}(\text{first order}) = \langle \hat{\chi}_{*\,\text{KMT}}^{(-)} | AP_* tP | \chi_{\text{KMT}}^{(+)} \rangle \ . \tag{5.30}$$

It is interesting to note that the appearance of KMT distorted waves is *not* an additional approximation to the consistent single scattering truncation, but is a *natural consequence* of the factorization of the full single scattering transition potential $AP_* \hat{\tau} P$, given in Eq. (5.18) along with the absorption of the left and right factors into the distorted waves. This means that even if we make the standard step of approximating the operator t by the free projectile-nucleon t matrix, then Eq. (5.30), despite appearances, is *not* the usual DWIA result. Equation (5.30) automatically includes (although implicitly now) the "orthogonality blocking" effects present in $P_* \hat{\tau} P$ as described in Sec. III. Also, the final distorted wave in Eq. (5.30) is not the standard KMT prescription because the operator τ appears instead of t in Eq. (5.29). The origin of this difference is also attributable to "orthogonality blocking" effects. What we learn from Eq. (5.30) is that such effects are intimately connected with the type of distorted wave chosen. This result suggests that extreme cau-

tion must be exercised in the use of a distorted wave formalism to investigate the influence of the nuclear medium upon the "effective" projectile-nucleon t matrix. At the very least, careful attention must be paid to the interior properties of the distorted waves employed. At intermediate energies, where microscopic theories of the optical potential are most applicable, it is perhaps safest to begin investigations with a completely microscopic and consistent first order expression for inelastic scattering such as Eq. (5.30).

The only ingredient whose calculation requires further comment here is $(A-1)P_*\tau P_*$, the potential determining the final distorted wave through Eq. (5.29). [The transition potential and initial distorted wave in Eq. (5.30) clearly involve standard calculations.] From Eq. (5.16) we have

$$P_*\tau P_* = P_* tP_* - P_* tPG_0 P\tau P_* \ , \tag{5.31}$$

and

$$P\tau P_* = PtP_* - PtPG_0 P\tau P_* \ . \tag{5.32}$$

Ideally then, these coupled two-body integral equations should be solved for $\langle \vec{k}', \phi_* \mid P_*\tau P_* \mid \phi_*, \vec{k} \rangle$.

The formal solution of the coupled equations [Eqs. (5.31) and (5.32)], is easily seen to be

$$P_*\tau P_* = P_* tP_* - P_* tP \frac{1}{G_0{}^{-1} + PtP} PtP_* \ . \tag{5.33}$$

We observe that $P_*\tau P_*$ differs from $P_* tP_*$ by a term which is second order in the off-diagonal projection of t. Where such off diagonal transitions are weak relative to diagonal transitions, and in the circumstance that the energy denominator in Eq. (5.33) is nonresonant, we may expect $P_*\tau P_*$ to be well represented by $P_* tP_*$. In that case we would have $\langle \chi^{(-)}_{*\,\mathrm{KMT}} \mid$ as an approximation to $\langle \hat{\chi}^{(-)}_{*\,\mathrm{KMT}} \mid$, where

$$\langle \chi^{(-)}_{*\,\mathrm{KMT}} \mid = \langle \phi_*, \vec{k}' \mid + \langle \chi^{(-)}_{*\,\mathrm{KMT}} \mid (A-1)P_* tP_* G_0 \tag{5.34}$$

is the standard first order KMT distortion for scattering from the excited state. This indicates that we may successfully approximate T^{inel}_{fi} by

$$T^{\mathrm{inel}}_{fi}(\text{first order}) \simeq \langle \chi^{(-)}_{*\,\mathrm{KMT}} \mid AP_* tP \mid \chi^{(+)}_{\mathrm{KMT}} \rangle \ , \tag{5.35}$$

which is both easy to calculate and has a well defined theoretical foundation.

It would be convenient to be able to approximate $P_* tP_*$ in Eq. (5.34) by PtP, so that the same first order KMT auxiliary distorting potential could be used for both distortions. Under some circumstances this may also prove to be an acceptable approximation. We stress, though, that there is little practical need for such an approximation in this first order KMT treatment.

VI. CONCLUSIONS

Several different aspects of the inelastic scattering problem have been discussed in this paper. We first turn our attention to the intermediate energy regime where impulsive treatments, which have been so useful for elastic scattering, suggest themselves. A most practical calculational prescription of this kind has been presented in Sec. V. The result given in Sec. V represents the exact lowest order matrix elements in a spectator expansion. The reason this fact is stressed is that it implies that corrections to the first order truncation are also capable of eventual numerical computation, so that this very practical theoretical development can be applied to the eventual investigation of so-called correlative effects. A further reason for the emphasis on the result of Sec. V as a first order spectator truncation is that the full effect of the Pauli principle has been shown to be included in the spectator treatment in an extraordinarily simple and convenient manner.[1] All that is required at the level of the first order result given in Sec. V is that the two-body transition operator, t, be properly antisymmetric in its own two-body variables, and that the target states be antisymmetric in their coordinates. This requirement is hardly a restriction. We have emphasized that serious errors can arise in the absence of a carefully constructed theoretical framework in which special effort is made to identify and avoid possible redundancies at each level of approximation. It is thus important to be able to present a relatively uncomplicated calculational prescription which satisfies the criteria we have considered physically essential. The distorted wave impulse result under discussion is given in Eq. (5.30) as

$$T^{\mathrm{inel}}_{fi}(\text{first order}) = \langle \hat{\chi}^{(-)}_{*\,\mathrm{KMT}} \mid AP_* tP \mid \chi^{(+)}_{\mathrm{KMT}} \rangle \ , \tag{6.1}$$

where the first order KMT-like distorted waves are those corresponding to distorting potentials $(A-1)P_*\tau P_*$ and $(A-1)PtP$, as defined and discussed in Sec. V. The only troublesome element in this prescription is the excited state KMT-type auxiliary distorting $(A-1)P_*\tau P_*$ which is of nonstandard form. Calculation of this two-body potential

requires a knowledge of the structure of the excited state and the operator τ, given in Eq. (5.16) as

$$\tau = t - tG_0 P\tau \ . \tag{6.2}$$

We have shown that $P_*\tau P_*$ may be calculated directly from Eq. (6.2) as the solution of the pair of coupled equations given in Eqs. (5.31) and (5.32), wherein the quantities P_*tP_*, P_*tP, PtP, and PtP_* serve as input.

It has been pointed out that there are circumstances under which $P_*\tau P_*$ may be adequately approximated by P_*tP_*, in which case solution of the coupled equations can be avoided. Various other, less severe, approximations to $P_*\tau P_*$ have also been examined at sufficient length to make clear that a number of different options are open for the modification and extension of computational programs already in existence.

The other aspect of the inelastic scattering problem that has been emphasized is the general structure of the exact matrix element in a theory constructed so that the initial distorted wave is the exact microscopically defined two-body elastic scattering state. There is general agreement that the effective two-body state is

$$|\chi^{(+)}\rangle = P\,|\Psi^{(+)}\rangle \ , \tag{6.3}$$

where $|\Psi^{(+)}\rangle$ is the complete many-body, many-channel, scattering state corresponding to the usual elastic scattering incident boundary conditions and P is the projector onto the ground state(s) of the target (and the projectile). This distorted wave is that which is obtained from the usual microscopic optical potential for elastic scattering,[11] PUP, where

$$U = V + VG_0(1-P)U \ . \tag{6.4}$$

This provides the microscopic framework for the presentation of Secs. II–IV. The exact matrix element for inelastic scattering is given in Eq. (2.45) as

$$T_{fi}^{\text{inel}} = \langle \hat{\chi}_*^{(-)}\,|\,P_*\hat{U}P\,|\,\chi^{(+)}\rangle \ , \tag{6.5}$$

in which the transition operator and left hand distortion are both controlled by $\hat{U}$, given in Eq. (2.45) as

$$\hat{U} = V + VG_0(1-P-P_*)\hat{U} \ . \tag{6.6}$$

The left hand "excited" distorted wave is generated by means of the two-body potential $P_*\hat{U}P_*$ and the transition operator is $P_*\hat{U}P$.

The analogy between Eqs. (6.6) and (6.4) is explored at length in Sec. II, so that it may become intuitively obvious that just as intermediate scatterings to the ground state must be excluded in the definition of the ground state optical potential operator U, so must intermediate scatterings to both states be excluded from the operator $\hat{U}$. This feature is seen to be required to produce a consistent, nonredundant theory. It is also seen to be of practical importance as well. Much of the material in Secs. III and IV is concerned with the establishment of the practical importance of the exclusion of these intermediate scattering events.

Because the diagonal operator, $P_*\hat{U}P_*$, and its off-diagonal counterpart, $P_*\hat{U}P$, may be very different in character, the discussion of the final distortion potential has been somewhat artificially separated from the study of the transition potential in Sec. IV. The somewhat extended exploration of some of the possible ways that one can exploit these quantities is largely intended to convince the reader that a consistent theoretical approach does not preclude the usual phenomenological exploration, but does serve to guide and constrain them.

Throughout this paper we have had in mind the inelastic scattering of nucleons from nuclei but have not explicitly incorporated the Pauli effects arising from the identity of the projectile with the nuclear constituents. We emphasize that the essential features of the results obtained herein persist when Pauli symmetries are included.[1]

APPENDIX A

In Sec. II, Eqs. (2.1)–(2.21), we have developed the transition matrix element for inelastic scattering in the two-potential form beginning with the "prior" form of T_{fi}^{inel} [cf. Eq. (2.1)]. Here we show the alternative development of T_{fi}^{inel} beginning with the "post" form

$$T_{fi}^{\text{inel}} = \langle \Psi_*^{(-)}\,|\,V\,|\,\vec{k},\phi\rangle \ , \tag{A1}$$

where, in correspondence to Eq. (2.5), we now have

$$\langle \Psi_*^{(-)}| = \langle \vec{k}',\phi_*| + \langle \Psi_*^{(-)}|\,VG_0 \ . \tag{A2}$$

Here $\langle \Psi_*^{(-)}|$ is the full state vector for the $(A+1)$-body system and has the boundary conditions of an incident plane wave of a nucleon with momentum $\vec{k}'$ relative to the residual nucleus in the excited state ϕ_* and incoming spherical waves in all possible open channels. In exact parallel to the development in Sec. II, we introduce a two-body distorting potential W for the *initial* channel and write

$$V = W + (V-W) \ , \tag{A3}$$

where W is also diagonal in the basis of target eigenstates ($P_* WP = 0$). Then, following the same procedure that led to Eq. (2.19), we obtain from Eq. (A1) the form

$$T_{fi}^{\text{inel}} = \langle \Psi_*^{(-)} | (V - W)P | \eta^{(+)} \rangle , \tag{A4}$$

where now $| \eta^{(+)} \rangle$ is defined by

$$| \eta^{(+)} \rangle = | \vec{k}, \phi \rangle + G_0 W | \eta^{(+)} \rangle . \tag{A5}$$

Here, because W has been required to be diagonal in the basis of target eigenstates,

$$| \eta^{(+)} \rangle = P | \eta^{(+)} \rangle , \tag{A6}$$

and $| \eta^{(+)} \rangle$ is a product of the target ground state and a projectile-target relative motion wave function. Insertion of unity in the form $1 = P_* + Q_*$ into Eq. (A4), then gives

$$T_{fi}^{\text{inel}} = \langle \Psi_*^{(-)} | P_* VP | \eta^{(+)} \rangle + \langle \Psi_*^{(-)} | Q_* (V - W)P | \eta^{(+)} \rangle , \tag{A7}$$

in direct analogy with the "prior" form result, Eq. (2.21).

APPENDIX B

Here we show what is implied for the optical potential for elastic scattering if the distorted wave corresponding to that potential is taken to be

$$| \chi^{(+)} \rangle = P | \Psi^{(+)} \rangle , \tag{B1}$$

where $| \Psi^{(+)} \rangle$ is the many-body outgoing wave state vector corresponding to the elastic scattering boundary conditions, viz.,

$$| \Psi^{(+)} \rangle = | \vec{k}, \phi \rangle + G_0 V | \Psi^{(+)} \rangle . \tag{B2}$$

With the usual definitions of P as projector onto the target ground state and Q the conjugate projector, we find from Eq. (B2) that

$$Q | \Psi^{(+)} \rangle = G_0 QV(P + Q) | \Psi^{(+)} \rangle , \tag{B3}$$

so that

$$Q | \Psi^{(+)} \rangle = \frac{1}{1 - G_0 QV} G_0 QVP | \Psi^{(+)} \rangle . \tag{B4}$$

Thus we may write $P | \Psi^{(+)} \rangle$ as

$$\begin{aligned} P | \Psi^{(+)} \rangle &= | \vec{k}, \phi \rangle + G_0 PVP | \Psi^{(+)} \rangle + G_0 PVQ | \Psi^{(+)} \rangle \\ &= | \vec{k}, \phi \rangle + G_0 PVP | \Psi^{(+)} \rangle \\ &\quad + G_0 PV \frac{1}{1 - G_0 QV} G_0 QVP | \Psi^{(+)} \rangle , \end{aligned} \tag{B5}$$

or with the definition $| \chi^{(+)} \rangle \equiv P | \Psi^{(+)} \rangle$

$$| \chi^{(+)} \rangle = | \vec{k}, \phi \rangle + G_0 PUP | \chi^{(+)} \rangle , \tag{B6}$$

where

$$\begin{aligned} U &= V + V \frac{1}{1 - G_0 QV} G_0 QV \\ &= V + VQG_0 \frac{1}{1 - VQG_0} V . \end{aligned} \tag{B7}$$

Equation (B7) can also be rewritten as

$$\begin{aligned} U &= (1 - VQG_0) \frac{1}{1 - VQG_0} V \\ &\quad + VQG_0 \frac{1}{1 - VQG_0} V \\ &= \frac{1}{1 - VQG_0} V , \end{aligned} \tag{B8}$$

which immediately yields the integral equation

$$U = V + VQG_0 U , \tag{B9}$$

whose solution is Eq. (B7).

APPENDIX C

In this appendix, we derive the result given in Eq. (2.43) without the explicit use of the two-potential approach. We do so in order to be able to derive a distorted wave formula which is equally applicable to both elastic and inelastic scattering. Our motivation for this is that we can then most dramatically emphasize how important it is to avoid "double counting." We begin with the formula applicable to elastic and inelastic scattering,

$$T_{fi} = \langle \vec{k}', \phi_f | V | \Psi^{(+)} \rangle , \tag{C1}$$

where all the familiar quantities are defined as in the text and the new quantity ϕ_f introduced above is the final nuclear state, which for elastic scattering is ϕ and for inelastic scattering is ϕ_*. We then write the Lippmann-Schwinger equation for $| \Psi^{(+)} \rangle$ as

$$| \Psi^{(+)} \rangle = | \vec{k}, \phi \rangle + G_0 V | \Psi^{(+)} \rangle , \tag{C2}$$

and project onto the Q space to obtain

$$Q | \Psi^{(+)} \rangle = \frac{1}{1 - G_0 QV} G_0 QVP | \Psi^{(+)} \rangle . \tag{C3}$$

Adding $P | \Psi^{(+)} \rangle$ to both sides of Eq. (C3), we obtain

$$|\Psi^{(+)}\rangle = \frac{1}{1-G_0QV}P|\Psi^{(+)}\rangle$$
$$= \frac{1}{1-G_0QV}|\chi^{(+)}\rangle , \quad \text{(C4)}$$

where the elastic distorted state vector $|\chi^{(+)}\rangle \equiv P|\Psi^{(+)}\rangle$ is just the one discussed at length in the text. Equation (C4) permits us to reexpress the matrix element Eq. (C1) as

$$T_{fi} = \left\langle \vec{k}', \phi_f \left| \frac{1}{1-VG_0Q} V \right| \chi^{(+)} \right\rangle . \quad \text{(C5)}$$

Again following the line pursued in the text, we define the state vector $\langle \hat{\psi}_f^{(-)}|$ to be

$$\langle \hat{\Psi}_f^{(-)}| = \langle \vec{k}', \phi_f | \frac{1}{1-VQG_0} , \quad \text{(C6)}$$

or equivalently

$$\langle \hat{\Psi}_f^{(-)}| = \langle \vec{k}', \phi_f | + \langle \hat{\Psi}_f^{(-)}| VQG_0 . \quad \text{(C7)}$$

We now define the projector P_f to be

$$P_f = |\phi_f\rangle\langle\phi_f| \quad \text{(C8)}$$

and take Q_f to be $1-P_f$. For inelastic scattering P_f will be P_*, whereas for elastic scattering P_f will be P. The use of the projector Q_f in Eq. (C7) then gives

$$\langle \hat{\Psi}_f^{(-)}| Q_f = \langle \hat{\Psi}_f^{(-)}| P_f VQQ_f G_0 \frac{1}{1-VQQ_fG_0} , \quad \text{(C9)}$$

so that by adding $\langle \hat{\Psi}_f^{(-)}| P_f$ to both sides of Eq. (C9) we obtain

$$\langle \Psi_f^{(-)}| = \langle \hat{\Psi}_f^{(-)}| P_f \frac{1}{1-VQQ_fG_0}$$
$$\equiv \langle \hat{\chi}_f^{(-)}| \frac{1}{1-VQQ_fG_0} , \quad \text{(C10)}$$

where the last equality on the right of Eq. (C10) serves as a definition of $\langle \hat{\chi}_f^{(-)}|$ as $\langle \hat{\Psi}_f^{(-)}| P_f$. This then implies that

$$T_{fi} = \langle \hat{\chi}_f^{(-)}| \frac{1}{1-VQQ_fG_0} V|\chi^{(+)}\rangle , \quad \text{(C11)}$$

which is the generalization of Eq. (2.43) to include both elastic and inelastic scattering once we define the operator $\hat{U}$ to be

$$\hat{U} = \frac{1}{1-VQQ_fG_0} V , \quad \text{(C12)}$$

so as to be able to write

$$T_{fi} = \langle \hat{\chi}_f^{(-)}| P_f \hat{U} P|\chi^{(+)}\rangle , \quad \text{(C13)}$$

and likewise

$$\langle \hat{\chi}_f^{(-)}| = \langle \vec{k}'\phi_f | + \langle \hat{\chi}_f^{(-)}| \hat{U}QP_fG_0 . \quad \text{(C14)}$$

Equation (C14) follows from Eqs. (C7) and (C10). For inelastic scattering, Eq. (C14) is identical to Eq. (2.47) since $QP_f = QP_* = P_*$. For elastic scattering, however, $QP_f = QP = 0$ and Eq. (C14) reduces to

$$\langle \hat{\chi}_f^{(-)}| = \langle \vec{k}'\phi_f | = \langle \vec{k}'\phi | \text{(elastic)} . \quad \text{(C15)}$$

This result will serve to emphasize how important a role is played by the projected states. For elastic scattering the final state "distorted wave" is necessarily undistorted. To obtain this obvious result within the distorted wave framework requires careful treatment of the projectors. A cavalier remark to the effect that a given state is but one of an infinitude of states can lead to disastrous overcounting. While this remark appears entirely elementary, a not very dissimilar looseness may lead to very serious difficulties in the analysis of pion-charge exchange data.

[1]A. Picklesimer, P. C. Tandy, and R. M. Thaler, Phys. Rev. C 25, 1215 (1982), the preceding paper.

[2]S. T. Butler, Proc. R. Soc. London, Ser. A 208, 559 (1951); W. Tobocman, Phys. Rev. 94, 1655 (1954); C. A. Levinson and M. K. Banerjee, Ann. Phys. (N.Y.) 2, 471 (1957); 2, 499 (1957); 3, 67 (1958); N. K. Glendenning, Phys. Rev. 114, 1297 (1959); E. Rost and N. Austern, *ibid.* 120, 1375 (1960); M. H. MacFarlane and J. B. French, Rev. Mod. Phys. 32, 567 (1960).

[3]W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, London, 1961); N. Austern, *Direct Nuclear Reaction Theories* (Wiley, New York, 1970); I. E. McCarthy, *Introduction to Nuclear Theory* (Wiley, New York, 1968).

[4]A representative sample of practical calculations is to be found in Refs. 2–40 of Ref. 1 above.

[5]The fitting of data by the adjusting of parameters of the DWBA matrix element is itself a variational principle.

[6]For a recent review see *Lecture Notes in Physics 89, Microscopic Optical Potentials,* edited by H. V. v. Geramb,

(Springer, New York, 1979).

[7]A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N.Y.) 8, 551 (1959).

[8]Our results can be obtained without using the method of scattering from two potentials. Such a derivation is presented in Appendix C of this paper and also in Ref. 1 above.

[9]M. Gell-Mann and M. L. Goldberger, Phys. Rev. 91, 398 (1953).

[10]We do not mean to imply by our use of Eq. (2.30) that the formal expression therein must be used to calculate the initial distorting potential. We utilize Eq. (2.30) as a formal statement of how the microscopic content of $PWP = PUP$ relates to V, so that the implicit cancellations between V and W in Eq. (2.22) may be made explicit. We will then arrive at formal expressions which show how both the effective transition operator and the final distorted wave relate to V. This will leave us in a good position to see how these formal ingredients of the distorted wave matrix element can be translated into practical, microscopically acceptable, prescriptions for calculations.

[11]H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958); 19, 287 (1962).

[12]K. M. Watson, Phys. Rev. 89, 575 (1953). Note that the multiple scattering expansion given in Eqs. (3.6) and (3.7) differs from the Watson series for the optical potential in that the projector QQ_* takes the place of Q.

[13]E. R. Siciliano and R. M. Thaler, Phys. Rev. C 16, 1322 (1977).

[14]The approximations given in Eqs. (3.16) and (3.17) are not an integral part of the theory presented in this paper and their validity is not established for all circumstances. However, we utilize Eqs. (3.16) and (3.17) here to place our remarks in the context of modern DWIA calculations which employ, for example, the effective nucleon-nucleon t matrices developed by A. Picklesimer and G. E. Walker, Phys. Rev. C 17, 237 (1978), or W. G. Love and M. A. Franey, Phys. Rev. C 24, 1073 (1981). On shell, the use of these effective t matrices gives exactly the form of Eqs. (3.16) and (3.17).

[15]We have in Eq. (4.5) written an exact expression for the transition matrix element, T_{fi}^{inel}, in which the final and initial distortions are those which correspond to the incoming and outgoing wave scattering states appropriate to the elastic scattering excited state and ground state optical potentials. Much of the foregoing discussion has, of course, concerned itself with the general arguments which do not suggest expression of this matrix element in this form. The principal reasons that mitigate against the form given in Eq. (4.5) are that information concerning elastic scattering from excited states is not usually accessible, and that there is a redundancy built into the operator $\hat{\hat{U}}$ since part of its role [c.f., Eq. (4.6)] is to undo the double-counting inherent in use of $\langle \chi_*^{(-)} |$ in place of $\langle \hat{\chi}^{(-)} |$.

[16]Alternatively, Eq. (4.23) could be incorporated as an option in standard DWA computer codes. The numerical algorithm for solution of the standard homogeneous Schrödinger equation for the final distorted wave would need to be modified to produce a solution of an inhomogeneous Schrödinger equation where the source term is the plane wave times the potential $P_* U P_*$ as a local function in position space.

[17]J. Kelley *et al.*, Phys. Rev. Lett. 45, 2012 (1980).