Additional calculations of triton moments

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The formalism of hyperspherical harmonics is used to calculate several moments of the triton photoeffect for a Volkov potential with Serber exchange. The accuracy of Clare's calculations of moments σ_0 and σ_1 is improved by including more terms in the hyperspherical harmonic expansion of the potential and of the ground state wave function. The moment $\sigma_2=8.9\times10^4$ MeV^{3}mb is calculated using one term in the hyperspherical harmonic expansions of the potential and wave function. We invert four moments and find reasonable agreement with Gorbunov's measurements of the ^{3}He photoeffect.

[NUCLEAR REACTIONS Triton photoeffect, hyperspherical harmonics, moments of photoeffect, inversion of moments.]

Sum rules were used by Clare[1] (CL) to find σ_{-1}, σ_0, and σ_1 for the triton for a Volkov spin-independent potential with Serber exchange character. Only the first term in the hyperspherical harmonic (hh) expansions of the potential and wave function were used in the CL calculations. A Laguerre inversion technique was developed to calculate the dipole cross section from its moments, σ_p.

This work is concerned with calculating more accurate values of σ_0 and σ_1. Our accuracy is improved by including terms up to grand orbital $L=2$ and $L=4$, respectively, in the hh expansion of the potential and wave function. The calculation of the moment σ_2 is then made using sum rules, using only the first term in the hh expansion of potential and wave function. Moment inversion is used, using four moments σ_{-1}, σ_0, σ_1, and σ_2 and the resulting cross section compared with experiment.

We use the potential

$$V=\sum_{ij}V(r_{ij})(1-x+xP_{ij}) , \quad (1)$$

where $(1-x)$ is the fraction of Wigner exchange force, and x the fraction of Majorana exchange. We give numerical results for a Serber force, with $x=\frac{1}{2}$. We have a ground state i completely symmetric for space exchange, so

$$P_{ij}\,|\,i\,\rangle=|\,i\,\rangle . \quad (2)$$

The moment σ_0 is given by

$$\sigma_0=(2\pi^2/\hbar c)\langle i\,|\,[D,[H,D]]\,|\,i\,\rangle . \quad (3)$$

Here D is the operator for electric dipole transitions and $H=T+V$ is the nuclear Hamiltonian. Using Eqs. (2) and (3) we find

$$\sigma_0=(4\pi^2/3)(\hbar^2/M) -(4\pi^2\alpha x/3)\langle i\,|\,V_{12}\xi^2\,|\,i\,\rangle . \quad (4)$$

We use the first three terms in the hh expansion[2] of $V_{12}=V(r_{12})$,

$$V_{12}=\pi/4\sum_{k=0}^{\infty}(k+1)^{(2)}P_{2k}^{00}(\phi)V_{2k}(r) . \quad (5)$$

The hh expansion of the ground state wave function is truncated at two terms, giving

$$r^{5/2}\langle r,\Omega\,|\,i\,\rangle=u_0(r){y_0}^{(0)}(\Omega) +u_4(r){y_4}^{(0)}(\Omega) . \quad (6)$$

The hyperradial wave functions $u_0(r)$ and $u_4(r)$ are given by Ballot[3] in graphical form. We use tabulated values from Fabre de la Ripelle (private communication). Explicit expressions for the hh ${Y_L}^{(0)}$ are given in Lally.[4] Substituting Eqs. (5) and (6) in (4) and using the orthonormality properties of hh yields the result

$$\sigma_0=(4\pi^2/3)\alpha(\hbar^2/M)-(4\pi^2\alpha x/3)\Big[(\tfrac{1}{2})\int_0^\infty r^2u_0{}^2V_0dr -(\tfrac{1}{2})\int_0^\infty r^2u_0{}^2V_2dr+(\tfrac{1}{2})\int_0^\infty r^2u_4^2V_0dr -(3)^{-1/2}\int_0^\infty r^2u_0u_4V_2dr+(3)^{1/2}\int_0^\infty r^2u_0u_4V_4dr\Big] . \quad (7)$$

The integrals are evaluated using Simpson's rule, and numerical values for the radial wave functions $u_0(r)$ and $u_4(r)$ provided by Ballot and Fabre. We find

$$\sigma_0=57.0 \text{ MeV mb} . \quad (8)$$

This value is 14% less than Clare's results [which included only the first integral in Eq. (7)], and it is 3% lower than the integrated cross section found by Fitzgibbon[5] from the cross section for the same potential.

The next moment

$$\sigma_1=\sigma_1^{(0)}+\sigma_1^{(x)}+\sigma_1^{(xx)} , \quad (9)$$

where

$$\sigma_1^{(0)}=-(4\pi^2/\hbar c)\langle i\mid[T,D]^2\mid i\rangle=(8\pi^2/9)\alpha(\hbar^2/M)\langle i\mid T\mid i\rangle=590 \text{ MeV}^2\text{ mb} , \quad (10)$$

$$\sigma_i^{(x)}=-(4\pi^2/\hbar c)\langle i\mid[T,D][V,D]+[V,D][T,D]\mid i\rangle , \quad (11)$$

$$\sigma_1^{(xx)}=-(4\pi^2/\hbar c)\langle i\mid[V,D]^2\mid i\rangle . \quad (12)$$

We evaluate the term proportional to x as follows[4]:

$$\sigma_1^{(x)}=(8\pi^2/3^{1/2})\alpha(\hbar^2/M)(x)\{\langle i\mid\partial/\partial\eta_z[V_{13}(z_1-z_3)P_{13}+V_{23}(z_2-z_3)P_{23}]\mid i\rangle + \langle i\mid[V_{13}(z_1-z_3)P_{13}+V_{23}(z_2-z_3)P_{23}]\partial/\partial\eta_z\mid i\rangle\} .$$

$$=(8\pi^2/3)\alpha(\hbar^2/M)(x)\{\langle i\mid[V_{13}(z_1-z_3)+V_{23}(z_2-z_3)]\mid_{\eta_z}\mid i\rangle\} . \quad (13)$$

Here the subscript η_z means that the partial derivative with respect to the Jacobi coordinate η_z is taken only for the quantitites in the square bracket.

We evaluate Eq. (12) as follows[4]:

$$\sigma_1^{(xx)}=4\pi^2\alpha x^2[\langle i\mid V_{13}^2(z_1-z_3)^2\mid i\rangle+\langle i\mid V_{13}V_{12}(z_1-z_3)(z_1-z_2)\mid i\rangle + \langle i\mid V_{23}V_{12}(z_2-z_3)(z_1-z_2)\mid i\rangle+\langle i\mid V_{23}^2(z_2-z_3)^2\mid i\rangle] . \quad (14)$$

We evaluate (13) and (14) using the first two terms in Fabre's expansion[2]

$$V_{ij}=V(r_{ij})=[1/\Gamma(D/2)]\sum_{k=0}^{\infty}(2k+D/2-1)[\Gamma(k+D/2-1)/k!] \times[{}_2F_1(-k,k+D/2-1,\tfrac{3}{2},r_{ij}^2/r^2)]V_{2k}^{D,0}(r) . \quad (15)$$

Making use of the orthonormality of the hh gives us the results

$$\sigma_1^{(x)}=-(8\pi^2\alpha x)(\hbar^2/M)[\int_0^\infty u_0{}^2(r)V_0(r)dr+(\tfrac{1}{6})\int_0^\infty ru_0{}^2(r)(dV_0/dr)dr -(\tfrac{143}{96})\int_0^\infty u_0{}^2(r)V_2(r)dr+(\tfrac{5}{64})\int_0^\infty ru_0{}^2(r)(dV_2/dr)dr]$$

$$=579 \text{ MeV}^2\text{ mb} , \quad (16)$$

and

$$\sigma_1^{(xx)}=4\pi^2\alpha x^2[\tfrac{1}{2}\int_0^\infty r^2u_0^2(r)V_0^2(r)dr+(\tfrac{13}{64})\int_0^\infty r^2u_0^2(r)V_0(r)V_2(r)dr + \int_0^\infty r^2u_0^2(r)V_2^2(r)dr] = 551\ \mathrm{MeV}^2\,\mathrm{mb}\ . \quad (17)$$

Then the moment

$$\sigma_1=1720\ \mathrm{MeV}^2\,\mathrm{mb}\ . \quad (18)$$

Our result is 12% lower than the CL result of 1950 MeV^2 mb and 10% higher than Fitzgibbon's result of 1566 MeV^2 mb.

The moment σ_2 is given by

$$\sigma_2=(4\pi^2/\hbar c)\sum_f E_\gamma^3\,|\langle i\,|D\,|f\rangle|^2 =(2\pi^2/\hbar c)\langle i\,|\,[[H,[H,D]],[H,D]],\,|\,i\rangle\ . \quad (19)$$

The commutator $[H,D]$ is

$$[H,D]=-(2e\hbar^2/M\sqrt{3})\partial/\partial\eta_z+exV_0(r)[(z_1-z_3)P_{13}+(z_2-z_3)P_{23}]\ . \quad (20)$$

Evaluation of Eqs. (19) and (20) gives the long expression

$$\begin{aligned}\sigma_2=&\ (4\pi^2\alpha/3)(\hbar^4/M^2)\int_0^\infty u_0^2\nabla^2V_0dr+(8\pi^2\alpha x/3)(\hbar^2/M)(8.48)\int_0^\infty u_0^2V_0dr\\&+8\pi^2\alpha x(\hbar^2/M^2)\int_0^\infty u_0^2V_0^2dr-(20\pi^2\alpha x/3)(\hbar^4/M^2)\int_0^\infty r^{-1}u_0^2\frac{dV_0}{dr}dr\\&+(8\pi^2\alpha x/3)(\hbar^4/M^2)\int_0^\infty u_0\frac{du_0}{dr}\frac{dV_0}{dr}dr-(8\pi^2\alpha x)(\hbar^2/M)\int_0^\infty ru_0^2V_0\frac{dV_0}{dr}dr\\&+(34\pi^2\alpha x^2)(\hbar^2/M)\int_0^\infty u_0^2V_0^2dr+(14\pi^2\alpha x^2)(\hbar^2/M)\int_0^\infty ru_0^2V_0\frac{dV_0}{dr}dr\\&-(2\pi^2\alpha x^2)(\hbar^2/M)\int_0^\infty r^2u_0^2V_0(\nabla^2V_0)dr-(4\pi^2\alpha x^2)(\hbar^2/M)\int_0^\infty r^2u_0V_0\frac{dV_0}{dr}\frac{du_0}{dr}dr\\&-(4\pi^2\alpha x^2)(\hbar^2/M)\int_0^\infty ru_0V_0^2\frac{du_0}{dr}dr+(10\pi^2\alpha x^2)(\hbar^2/M)\int_0^\infty u_0^2V_0^2dr\\&-6\pi^2\alpha x^3\int_0^\infty r^2u_0^2V_0^3dr=8.93\times10^4\ \mathrm{MeV}^3\,\mathrm{mb}\ . \quad (21)\end{aligned}$$

This result is 657% higher than Maleki's[6] calculation for a Volkov force of pure Wigner character, and is 48% higher than Fitzgibbon's value.[5]

We use the CL moment inversion technique.[1,7] Also see related work by Langhoff.[8] The triton photoeffect cross section is given in terms of coefficients Λ_n, Laguerre polynomials, and a decreasing exponential, as follows:

$$\sigma(E/D)=(E+B)D^{-1}\exp(-E/D)\times\sum_n\Lambda_nL_n(E/D)\ . \quad (22)$$

CL developed this equation on the assumption that at high energies the cross section is the product of a polynomial and a decreasing exponential. The polynomial is expressed using Laguerre polynomials, since they forn an orthonormal set with this interval and weighting function.

The coefficients Λ_n are calculated, as algebraic expressions[1,7] involving the moments. E is the energy above threshold, and B is the (three-body) threshold energy of 8.48 MeV. The scale parameter D is introduced to make the arguments of L_n and the exponential function dimensionless. We

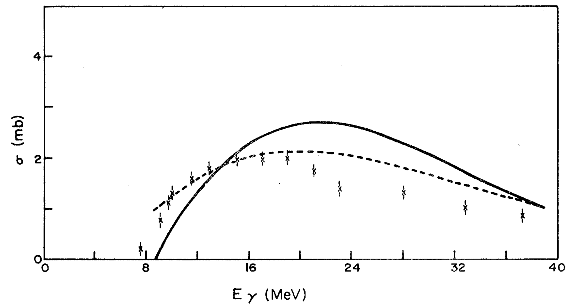

FIG. 1. The solid curve gives the cross section vs photon energy for Laguerre inversion using four moments, with parameter $D=12.4$ MeV. The dashed curve shows Clare's inversion (Ref. 1) using three moments with $D=8.8$ MeV. The x's with errors show Gorbunov's data.

use σ_{-1}, σ_0, σ_1, and σ_2 from Eq. (21), so we use four terms in the sum in Eq. (22).

Two criteria are applied to find the best values for the parameter D: the cross section given by Eq. (22) should be non-negative, and it should be zero at threshold. The figure shows our inversion, choosing $D=12.4$ MeV, as a solid curve. We chose this value of D so that the cross section would be zero at threshold; but the cross section does go slightly negative at an energy of about 50 MeV above threshold. Our inversion gives reasonable agreement with Gorbunov's data,[9] but not as good agreement as that found by CL (dashed curve) using only three moments, with $D=8.8$ MeV.

The large difference between the solid and dashed curves suggests that Laguerre inversion may not be a satisfactory technique to find the cross section from a small number of moments. Perhaps the *assumption* of an exponential decrease with energy of the cross section at high energies is invalid. There are indications[10] that the asymptotic form is $\exp(-E^{1/2}/D^{1/2})$. We are developing alternative inversion techniques on this assumption. Of course the values of the moments found in this paper will be useful using alternative inversion techniques. Two other sources of disagreement with Gorbunov's data are (i) our use of the *spin-independent* Volkov potential, and (ii) our calculation is for ^{3}H, the data for ^{3}He.

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[1]R. B. Clare and J. S. Levinger, Phys. Rev. C 23, 657 (1981).

[2]M. Fabre de la Ripelle, Proceedings of the International School on Nuclear Theory Physics, Predeal, 1969.

[3]J. Ballot, M. Beiner, and M. Fabre de la Ripello, in *Proceedings on the International Symposium on the Present Status and Novel Developments in Many-Body Problems, Rome, 1972,* edited by V. Calogero and C. Ciofi degli Atti (Editrice Compositori, Bologna, 1974), p. 565.

[4]D. F. Lally, MS thesis, Rensselaer Polytechnic Institute, 1981 (unpublished).

[5]J. S. Levinger and R. Fitzgibbon, Phys. Rev. C 18, 56 (1978).

[6]S. Maleki and J. S. Levinger, Phys. Rev. C 19, 565 (1979).

[7]R. B. Clare, MS thesis, Rensselaer Polytechnic Institute, 1980 (unpublished).

[8]C. Corcoran and P. Langhoff, J. Math. Phys. 18, 651 (1977).

[9]A. N. Gorbunov in Proceedings of the P. N. Lebedev Physics Institute, Nauka, Moscow 1974, Vol. 71, p. 1; *Photonuclear and Photomesic Processes,* Trudy, edited by D. V. Skobel'tsyn (Consultants Bureau, New York, 1976), Vol. 71, p. 1 (translation).

[10]J. S. Levinger, I. Elminyawi, and D. Lally, Contribution to International Conference on High Energy Physics and Nuclear Structure, 1981, Versailles.