# Nonlocal potentials and inadequacies of the Jost function in the description of nuclear phenomena

B. Mulligan and S. B. Qadri
*Department of Physics, The Ohio State University, Columbus, Ohio 43210*
(Received 29 September 1980)

This paper considers the conceptual problem of describing the physics of a many body system in terms of a two body $S$ matrix. Projection of the many body problem onto a two body channel results in a potential which is nonlocal. We demonstrate that for a nonlocal potential the analytic structure of the Jost function differs from that associated with a local potential. Moreover, the Jost function, and thus the $S$ matrix, is shown to be an incomplete description of the information in the two body channel. This is accomplished by considering Fredholm determinants associated with scattering integral equations.

[NUCLEAR REACTIONS Scattering by a nonlocal potential, Jost functions, $S$ matrices, and Fredholm determinants and their analytic character in the complex plane.]

## I. INTRODUCTION

An operator $S$ relating the initial and final stages of the nuclear scattering process was introduced in 1937 by Wheeler.[1] In 1943 Heisenberg[2] set forth a program for deriving all observable quantities in nuclear reactions (the cross sections for all possible collision processes, the energy levels of bound states, the lifetimes and decay energies of unstable states, etc.) in terms of the properties of $S$. Since that time considerable progress has been made in developing nuclear physics along these lines. Summaries of this progress have been given, for example, by Nussenzveig[3] and by Newton.[4] Extensive information is now available about the relationship between nuclear processes and the analytic properties of the $S$ matrix.

In much of the above work, emphasis has been placed on the connection between the analytic properties of the $S$ matrix in the complex $k$ plane and nuclear phenomena which can be described by a Schrödinger equation with a central local potential $\mathcal{U}$. For such a potential it is the radial equation for the partial wave $l$ which provides the link between the $S$ matrix for that partial wave and the potential. Considerations in the present paper are restricted to $l = 0$, for which the radial equation with the potential $\mathcal{U}$ is

$$\left[\frac{d^2}{dr^2} + k^2\right] u(r) = \mathcal{U}(r) u(r) \ . \tag{1}$$

Throughout this paper we also restrict the analysis to potentials which are real.

A consequence of Eq. (1) is that $S(k)$ can be written in terms of the Jost functions $\mathcal{L}^{\pm}(k)$. That is,

$$S(k) = \mathcal{L}^{-}(k)/\mathcal{L}^{+}(k) \ , \tag{2}$$

where the Jost functions $\mathcal{L}^{\pm}(k)$ are the Jost solutions $f^{\pm}(k,r)$ of Eq. (1) evaluated at $r = 0$,

$$\mathcal{L}^{\pm}(k) = f^{\pm}(k,r)\,|_{r=0} \ . \tag{3}$$

Thus, for a central local potential a discussion of the analytic properties of $S$ can be phrased in terms of a discussion of the analytic properties of $\mathcal{L}^{\pm}(k)$. Furthermore, Jost and Pais[5] have shown that $\mathcal{L}^{\pm}(k)$ are identical to the Fredholm determinants $D^{\pm}(k)$ associated with the kernel of the integral equation for the physical solution of Eq. (1) and its conjugate. Therefore, any discussion of a physical system described by Eq. (1) in terms of the analytic properties of $S(k)$ can equally well be carried out in terms of the analytic properties of the Fredholm determinants $D^{\pm}(k)$.

In the case of nucleon-nucleus scattering, Feshbach[6] has derived a formalism in which the many body Schrödinger equation is reduced to an effective two body equation. When formulating nuclear reaction theory in this manner, the effective interaction in the two body Schrödinger equation is nonlocal. In Feshbach's formalism, nonlocality of the potential results not only from the many body character of the complete interaction but also follows from the identity of the incident nucleon with nucleons in the target.

It thus becomes necessary in the context of nuclear physics to investigate the properties of the $S$ matrix for a radial equation similar to Eq. (1), but for which the potential is nonlocal. In the description of nuclear phenomena, significant differences may occur when a nonlocal potential comes into play as compared with the situation with a local potential. As mentioned above, for a local potential there is a simple relationship between the $S$ matrix, Jost functions, and Fredholm determinants. For example, in the case of a local potential only the Fredholm determinants $D^{\pm}(k)$ are of interest; the Fredholm determinants associated with kernels of integral equations for other solutions of Eq. (1) either are directly related to $D^{\pm}(k)$ or are unity.[7] For a nonlocal potential, however, a system of Fredholm determinants must be considered.[8,9] Also, when an interaction is nonlocal, the Jost functions $\mathcal{L}^{\pm}(k)$ are no longer equivalent to the Fredholm determinants $D^{\pm}(k)$ generated by the nonlocal interaction.[10] In the case of a symmetric nonlocal potential the Jost functions are the ratio of $D^{\pm}(k)$ to $D(k)$,[10] the Fredholm determinant associated with the kernel of the integral equation for the regular solution. Furthermore, the Jost functions for a nonlocal potential do not contain as complete a set of information about the nuclear system as do the determinants $D^{\pm}(k)$ and $D(k)$, since zeros and poles of $D^{\pm}(k)$ can be canceled by zeros and poles of $D(k)$ in taking the required ratio to find $\mathcal{L}^{\pm}(k)$.[11]

For this reason, we initiate here a study of the analytic properties of the Fredholm determinants $D^{\pm}(k)$ and $D(k)$ in the complex $k$ plane. Before undertaking this study, integral equations incorporating various boundary conditions for scattering by a nonlocal potential are discussed. These equations are presented in Sec. II, along with their associated Fredholm determinants. The Jost functions and Jost solutions are discussed in particular.

An important aspect of these considerations is that many authors, as part of their derivation of relations for Jost functions, have assumed that Jost solutions exist. It is now known that this assumption may not be true. Recently, an anomaly[12–14] associated with nonlocal potentials (as compared with local potentials) has been discussed by Mulligan *et al.*[8,9] If for a nonsingular local potential the initial conditions imposed on the radial part of the wave function $u(r)$ were $u(0)=u'(0)=0$, the solution would be zero everywhere. Mulligan *et al.* showed that for nonlocal potentials a nontrivial solution with $u(0)=u'(0)=0$ can exist. A (nonnormalizable) state whose wave function satisfies the condition $u(0)=u'(0)=0$ has been named a "spurious state." It is found[8] that Jost solutions do not exist at the energy at which a spurious state occurs. A spurious state also results in an ambiguity in the definition of the phase shift at the energy of the spurious state.[9]

Other conditions occur under which Jost solutions may not exist. In discussing states excluded by the Pauli principle, Swan[12] noted that these states appear as solutions of the scattering integral equations at all scattering energies. Such redundant states are simply continuum bound states[8,15] which appear in the spectrum of the scattering solution at every positive energy,[16] and can be expected in any scattering calculations which take into account antisymmetrization between the incident particle and a target described by antisymmetrized single particle states. Such states are a consequence of the fact that any row of a Slater determinant can be added to any other row of the determinant without changing the value of the determinant. Thus, the scattering solution is nonunique because any arbitrary amount of filled states can be added to the scattering state. It has been shown that under these conditions Jost solutions do not exist at any energy, although Jost functions can be defined at all energies.[11]

## II. INTEGRAL EQUATIONS, FREDHOLM DETERMINANTS, AND JOST FUNCTIONS

A nonlocal potential is an integral operator which enters into the $l=0$ radial equation in the following way

$$\left[\frac{d^2}{dr^2}+k^2\right]u(r)=\int_0^\infty V(r,r')u(r')dr' \ . \tag{4}$$

This integrodifferential equation can be converted into integral equations by the use of Green's functions which satisfy appropriate boundary conditions. Several integral equations, their solutions, and related quantities are defined in analogy with the local potential case.

The regular solution $\varphi(k,r)$ is defined by the boundary condtions

$$\varphi(k,0)=0;\ \varphi'(k,0)=1\ . \tag{5}$$

The integral equation for the regular solution $\varphi(k,r)$ is

$$\varphi(k,r)=k^{-1}\sin kr + \int_0^r \int_0^\infty G(k,r,r')V(r',s)\varphi(k,s)ds\,dr', \tag{6}$$

where $G(k,r,r')$ is the Green's function and is equal to $k^{-1}\sin k(r-r')$. The Fredholm determinant associated with the kernel of this integral equation is denoted by $D(k)$.

The physical solution $\psi^+(k,r)$ is defined by the mixed boundary conditions that $\psi^+(k,r)$ have the asymptotic form

$$\psi^+(k,r)\rightarrow \frac{i}{2}[e^{-ikr}-S^+(k)e^{ikr}] \tag{7}$$

as $r\rightarrow\infty$, and that $\psi^+(k,r)$ be regular at $r=0$. $S^+(k)$ is the $s$-wave scattering matrix element. The physical solution $\psi^+(k,r)$ and its conjugate $\psi^-(k,r)$ satisfy the integral equations

$$\psi^\pm(k,r)=\sin kr + \int_0^\infty \int_0^\infty G^\pm(k,r,r')V(r',s)\psi^\pm(k,s)ds\,dr', \tag{8}$$

where $G^\pm(k,r,r')=-k^{-1}e^{\pm ikr_>}\sin kr_<$ with $r_>=\max(r,r')$ and $r_<=\min(r,r')$. The asymptotic form of $\psi^-(k,r)$ for large $r$ is given by Eq. (7) with $S^+$ replaced by $S^-(k)=[S^+(k)]^*$, where * represents the complex conjugate. The Fredholm determinants associated with the kernels of Eq. (8) are denoted by $D^\pm(k)$.

The integral equations for the Jost solutions are

$$f^\pm(k,r)=e^{\pm ikr} - \int_r^\infty \int_0^\infty G(k,r,r')V(r',s)f^\pm(k,s)ds\,dr'. \tag{9}$$

The Fredholm determinant associated with the kernel of Eq. (9) is denoted by $\Delta(k)$.

The Fredholm determinants have the following properties: $\mathrm{Re}D^\pm(k)$, $D(k)$, and $\Delta(k)$ are even functions of $k$ while $\mathrm{Im}D^\pm(k)$ is an odd function of $k$. Both $D(k)$ and $\Delta(k)$ are real for $k$ real, and $D(k)=\Delta(k)$ for a symmetric nonlocal potential.[17]

In the case of a local potential, no ambiguity results from defining the Jost functions $\mathfrak{L}^\pm(k)$ in the standard manner indicated in Eq. (3). It is known that for a local potential $\mathfrak{L}^\pm(k)\neq 0$ for real $k\neq 0$.[4] It can also be demonstrated that for a local potential $\mathfrak{L}^\pm(k)$ is analytic in $k$ in the upper half plane.[3,4,12] Many of the important properties associated with a local potential can be shown to follow from these facts.

As discussed in the previous section, for a local potential the Jost functions $\mathfrak{L}^\pm(k)$ and the Fredholm determinants $D^\pm(k)$ are equal,

$$\mathfrak{L}^\pm(k)=D^\pm(k)\ \text{(local potential)}\ . \tag{10}$$

Therefore, for a local potential all properties associated with $\mathfrak{L}^\pm(k)$ are equally true of $D^\pm(k)$, and vice versa. Also, with the exception of Eq. (8), the integral equations for a local potential are Volterra integral equations. This is the reason the Fredholm determinants $D(k)$ and $\Delta(k)$ must be unity in the case of local potential. It is also consistent with the point of view that for a local potential the scattering information is contained in the Fredholm determinants $D^\pm(k)$.

For a symmetric nonlocal potential, it has been shown[10,18–20] that $D^\pm(k)$ and $\mathfrak{L}^\pm(k)$ are related by the expression

$$\mathfrak{L}^\pm(k)=\frac{D^\pm(k)}{D(k)}\ \text{(symmetric nonlocal potential)}\ . \tag{11}$$

Thus the local potential result (10) is a special case of Eq. (11). For a nonlocal potential, the integral equations with which $D(k)$ and $\Delta(k)$ are associated are Fredholm integral equations rather than Volterra integral equations. The Fredholm determinants $D(k)$ and $\Delta(k)$ are not unity in such a case and may have zeros for real or complex values of $k$, whereas in the case of a local potential the Fredholm determinants $D^\pm(k)$ do not have zeros for real values of $k$. Since for real $k$, $\mathfrak{L}^\pm(k)\neq 0$ for $k\neq 0$, it is clear from Eq. (11) that for $k$ real the zeros of the Fredholm determinants $D^\pm(k)$ and $D(k)$ are not independent. Whenever there is a zero of $D^\pm(k)$ for real $k$, $D(k)$ is also zero at that same real value of $k$.

A zero of $D^\pm(k)$ for real $k\neq 0$ is associated[15,21] with a continuum bound state, often referred to by the abbreviation CBS. From the above discussion it is clear that a continuum bound state is characterized by simultaneous zeros of both $D^\pm(k)$ and $D(k)$. The existence of a continuum bound state results from the fact that solutions of the homogeneous integral equations associated with Eq. (8), namely

$$\psi_h^{\pm}(k,r) = \int_0^\infty \int_0^\infty G^{\pm}(k,r,r')V(r',s)\psi_h^{\pm}(k,s)ds\,dr' , \tag{12}$$

exist and are normalizable solutions for real $k \neq 0$ when $D^{\pm}(k)=0$. Trivial solutions $\psi_h^{\pm}(k,r)=0$ are the only solutions allowed when $D^{\pm}(k)\neq 0$.

A zero of $D(k)$ for real $k \neq 0$ is associated with a spurious state.[8,14] Since $D(k)$ is also zero for a continuum bound state, we maintain here the nomenclature that a CBS occurs when at a real value of $k$ both $D^{\pm}(k)$ and $D(k)$ are zero, and a spurious state occurs when at a real value of $k$ $D(k)$ is a zero and $D^{\pm}(k)$ is nonzero. It is demonstrated in Ref. 8 that at a spurious state the regular solution $\varphi(k,r)$ does not exist, although it is possible to obtain a solution of Eq. (4) regular at the origin. Also, as discussed earlier, the Jost solutions $f^{\pm}(k,r)$ do not exist at a spurious state. On the other hand, $\psi^{+}(k,r)$ and $\psi^{-}(k,r)$ do exist at a spurious state. At a continuum bound state $\psi^{\pm}(k,r)$ and $\varphi(k,r)$ always exist while $f^{\pm}(k,r)$ may or may not exist.

Problems with the existence of Jost solutions clearly are of importance in discussing the analytic character of the Jost functions $\mathcal{L}^{\pm}(k)$ for a nonlocal potential. The derivation of Eq. (11), which was based on the existence of $f^{\pm}(k,r)$, will break down when $f^{\pm}(k,r)$ do not exist. Correspondingly, the definition of the Jost functions as given in Eq. (3) will no longer hold if $f^{\pm}(k,r)$ do not exist. This is especially a problem in the case of redundant states, since it would mean a failure of the existence of the Jost functions at any energy. However, the results in Ref. 11 demonstrate that the Jost functions $\mathcal{L}^{\pm}(k)$ can be directly related to the kernels of integral equations by means of Fredholm determinants without referring to the solutions of the integral equations involved. It is in this way that in defining the Jost functions the question of the existence of Jost solutions at a CBS or in the presence of a redundant state can be avoided. Thus, in what follows we take Eq. (11) as the definition of $\mathcal{L}^{\pm}(k)$.

## III. FUNCTIONAL ANALYSIS AND ANALYTIC PROPERTIES OF FREDHOLM DETERMINANTS

For a real potential the analytic character of $\mathcal{L}^{-}(k)$ is related to that of $\mathcal{L}^{+}(k)$ by complex conjugation. Thus only the character of $\mathcal{L}^{+}(k)$ will be explored here. In comparing $\mathcal{L}^{+}(k)$ for a nonlocal potential with $\mathcal{L}^{+}(k)$ for a local potential, we will see that for a nonlocal potential zeros of $D(k)$ [and thus poles of the Jost function $\mathcal{L}^{+}(k)$] are found on or above the real axis in some cases. For this reason, the techniques from complex analysis used to obtain the character of the Fredholm determinant $D^{+}(k)$ and the Jost function $\mathcal{L}^{+}(k)$ for a local potential are not sufficient for a general discussion of $D(k)$, $D^{+}(k)$, and $\mathcal{L}^{+}(k)$ in the case of a nonlocal potential. Consequently, we turn to functional analysis in order to discuss the existence of solutions of the scattering integral equations and the analytic properties of the Fredholm determinants associated with them.

The kernels of integral Eqs. (6), (8), and (9) are not square integrable. Thus, the usual derivations[22] of the Fredholm alternative do not apply. However, Riesz[23] and Schauder[24] have shown that the Fredholm alternative easily can be extended to integral equations for which the operators are compact. Their proof is a "determinant-free" proof, and the conclusions of the Riesz-Schauder theory [25] are the "determinant-free" Fredholm theorems.

To make use of these results, the following concepts from functional analysis are needed. A transformation is defined as completely continuous if it transforms every bounded set in a Banach space into a compact set in a Banach space.[26] For this reason, completely continuous operators are also known as compact operators. An operator $T$ is said to be bounded with respect to the set of functions $\phi_n(x)$ if, for all $\phi_n$, $||T\phi_n|| \leq N||\phi_n||$, where $||f||$ denotes the norm of $f$ chosen subject to the conditions of a norm for a Banach space.

Following the approach of Iwasaki and Mulligan,[27] we apply the theorem which states that a bounded operator takes a compact sequence over into a compact sequence.[28] Let $f(x)$ be a function from the set $C_k(0,\infty)$ and let

$$Tf(y) = \int_0^\infty t(y,x)f(x)dx \quad . \tag{13}$$

In order that $T$ be a compact operator the following conditions must be satisfied:

$$\lim_{y\to\infty} \int_0^\infty |t(y,x)|dx = 0 \quad , \tag{14}$$

$$\sup_{y\geq 0} \int_0^\infty |t(y,x)|dx < \infty \quad , \tag{15}$$

and

$$\lim_{\delta\to 0} \sup_{\substack{|y_1-y_2| \le \delta \\ y_1,y_2>0}} \int_0^\infty |t(y_1,x)-t(y_2,x)|\,dx = 0 \ . \tag{16}$$

The integral Eqs. (6), (8), and (9) all can be written in the form

$$\chi(r) = \chi_0(r) + \int_0^\infty K(r,s)\chi(s)ds \ , \tag{17}$$

where

$$K(r,s) = \int_0^\infty g(r,r')V(r',s)dr' \ . \tag{18}$$

In order to establish complete continuity of the kernel $K$, it is convenient to rewrite Eq. (18) as

$$K(r,s) = \int_0^\infty g(r,r')\mu^{-1}(r')\mu(r')V(r',s)dr' \ . \tag{19}$$

We then pick $\mu^{-1}(r)$ to be a continuous function which will bound the kernel $g$ on the infinite interval. Defining a kernel $\mathcal{G}$ as

$$\mathcal{G}(r,r') = g(r,r')\mu^{-1}(r') \ , \tag{20}$$

we require that there exist an $A$ such that

$$\int_0^\infty |\mathcal{G}(r,r')|\,dr' \le A \ . \tag{21}$$

The conditions for compactness of $K$ then will be satisfied if the operator $U$, defined by

$$U(r,s) = \mu(r)V(r,s) \ , \tag{22}$$

is compact. Thus only potentials for which

$$\lim_{r\to\infty} |\mu(r)| \int_0^\infty |V(r,s)|\,ds = 0 \tag{23}$$

will be considered.

For a completely continuous operator $T$ satisfying the function equation

$$\phi - \lambda T\phi = \psi \tag{24}$$

($\psi$ given, $\phi$ to be determined), Riesz and Schauder showed that principles completely analogous to those for Fredholm's integral equation hold. That is,[29] the resolvent of Eq. (24) exists and can be written as

$$S_\lambda = \frac{D_\lambda}{\Delta(\lambda)} \ , \tag{25}$$

where $\Delta(\lambda)$ is analytic in the $\lambda$ plane. Thus the existence and analytic properties of the Fredholm determinants associated with the integral Eqs. (6), (8), and (9) can be related directly to the complete continuity of the kernels of those integral equations.

Using this approach, we examine the question of complete continuity for the kernel of the physical integral equation. For this Green's function we get

$$\int_0^\infty |\mathcal{G}(r,r')|\,dr' = \int_0^r |-k^{-1}e^{ikr}\sin kr'\mu^{-1}(r')|\,dr' + \int_r^\infty |-k^{-1}\sin kr\, e^{ikr'}\mu^{-1}(r')|\,dr' \ . \tag{26}$$

From the inequality

$$|\sin z| < |z|\exp(\mathrm{Im} z)$$

we find that

$$\int_0^\infty |\mathcal{G}(r,r')|\,dr' \le e^{-(\mathrm{Im}k)r}\int_0^r r' e^{(\mathrm{Im}k)r'}|\mu^{-1}(r')|\,dr' + r\,e^{(\mathrm{Im}k)r}\int_r^\infty e^{-(\mathrm{Im}k)r'}|\mu^{-1}(r')|\,dr' \ . \tag{27}$$

Since for $\mathrm{Im}k \ge 0$,

$$e^{-(\mathrm{Im}k)r}e^{(\mathrm{Im}k)r'} < 1, \quad r > r' \tag{28}$$

and

$$e^{(\mathrm{Im}k)r}e^{-(\mathrm{Im}k)r'} < 1, \quad r < r' \tag{29}$$

we get the inequality

$$\int_0^\infty |\mathcal{G}(r,r')|\,dr' \le \int_0^\infty r'|\mu^{-1}(r')|\,dr' \ . \tag{30}$$

The integral $\int_0^\infty r'|\mu^{-1}(r')|\,dr'$ is bounded if as $r\to\infty$ the function $|\mu^{-1}(r)|$ goes to zero faster than $1/r^2$. Equation (23) thus implies that as $r\to\infty$ if $\int_0^\infty |V(r,s)|\,ds$ goes to zero faster than $1/r^2$, the operator $K$ will be completely continuous for $\mathrm{Im}k \ge 0$. Under these circumstances, the Fredholm determinant $D^+(k)$ will be analytic for $\mathrm{Im}k \ge 0$.

Considering now the integral equation for the regular solution we have

$$\int_0^\infty |\mathcal{G}(r,r')|\,dr' = \int_0^r |k^{-1}\sin k(r-r')|\,|\mu^{-1}(r')|\,dr' \ . \tag{31}$$

This yields the inequality

$$\int_0^\infty |\mathcal{G}(r,r')|\,dr' \geq \frac{1}{2|k|}\int_0^r |e^{-(\mathrm{Im}k)(r-r')} - e^{(\mathrm{Im}k)(r-r')}| \times|\mu^{-1}(r')|\,dr' \tag{32}$$

or

$$\int_0^\infty |\mathcal{G}(r,r')|\,dr' \geq \frac{1}{2|k|}e^{(\mathrm{Im}k)r}\int_0^r e^{(\mathrm{Im}k)r'}|\mu^{-1}(r')|\,dr' - \frac{1}{2|k|}e^{-(\mathrm{Im}k)r}\int_0^r e^{(\mathrm{Im}k)r'}|\mu^{-1}(r')|\,dr' \ . \tag{33}$$

For $\mathrm{Im}k = 0$, the right-hand side of Eq. (33) becomes zero. However, for $\mathrm{Im}k > 0$ or $\mathrm{Im}k < 0$ the right-hand side of Eq. (33) is unbounded as $r \to \infty$, independent of the choice of $\mu(r)$.

This means that the technique used to discuss complete continuity for the kernel of the integral equation for the regular solution, that of breaking the kernel down into the product of a bounded operator and a completely continuous operator, can work in the case of the regular solution only on the real $k$ axis. This implies that for $\mathrm{Im}k \neq 0$ the operator may not be completely continuous and hence the Fredholm determinant may not be analytic. As we shall see, $D(k)$ normally will have poles in both the upper and lower halves of the $k$ plane; it would, therefore, be expected that any proof to the contrary would fail.

On the real $k$ axis, however,

$$\int_0^\infty |\mathcal{G}(r,r')|\,dr' \leq r\int_0^r |\mu^{-1}(r')|\,dr' - \int_0^r r'|\mu^{-1}(r')|\,dr' \ . \tag{34}$$

The right-hand side is bounded if as $r \to \infty$ the function $|\mu^{-1}(r)|$ goes to zero faster than $1/r^2$. Therefore, under the condition that as $r \to \infty$ the integral $\int_0^\infty |V(r,s)|\,ds$ goes to zero faster than $1/r^2$, the kernel of the integral equation for the regular solution will be completely continuous for real $k$. Using the same reasoning as that employed in the discussion of $D^+(k)$, we conclude that the Fredholm determinant $D(k)$ will be analytic on the real axis.

Complete continuity of the kernels of the integral Eqs. (6) and (8) is crucial to establishing the existence of the Fredholm determinants $D^+(k)$ and $D(k)$ for a nonlocal potential. Since these kernels are completely continuous for $k$ real, it necessarily follows that the Fredholm determinants $D^+(k)$ and $D(k)$ exist for $k$ real. Fredholm determinants can then be defined for complex $k$ by extending these definitions, established on the real axis, to the complex plane. In this context, the above discussions of the character of $D^+(k)$ and $D(k)$ in the upper half plane show that $D^+(k)$ when so extended will be analytic in that portion of the $k$ plane, while $D(k)$ may not.

Complete analysis of the character of a physical system described by a symmetric nonlocal potential requires a discussion of the poles and zeros of $D^+(k)$ and $D(k)$ in both the upper and lower half planes. We restrict our discussion here to the upper half plane only. The character of $D^+(k)$ and $D(k)$ in the lower half plane will be discussed in a separate paper.[30]

## IV. ZEROS OF $D^+(k)$ AND $D(k)$ IN THE UPPER HALF PLANE

In the case of a local potential, there is a well-known relation between the zeros of $D^+(k)$ and negative energy bound states. As discussed earlier, a nonlocal potential can exhibit not only negative energy bound states but also continuum bound states and spurious states. Such states are related to zeros of $D^+(k)$ and $D(k)$.

The discussion of zeros of $D^+(k)$ for a nonlocal potential follows the same lines as that for a local potential. Consider the following equation:

$$\left[\frac{d^2}{dr^2} + k^2\right]\psi(r) = \int_0^\infty V(r,r')\psi(r')dr' \ , \tag{35}$$

where $V$ is a real symmetric nonlocal potential. The notation

$$H = -\frac{d^2}{dr^2} + \int_0^\infty dr'V(r,r') \tag{36}$$

brings Eq. (35) into the form

$$H\psi = k^2\psi \ , \tag{37}$$

identifying $k^2$ as the eigenvalue of the Hamiltonian $H$. Equation (37) can also be written as an integral equation for the solutions $\psi$. If the physical Green's function $G^+$ defined in connection with Eq. (8) is used in constructing this integral equation, then for $\mathrm{Im}k \geq 0$ only the homogeneous form of this integral equation will have bounded solutions.[31] It is well known from the theory of integral equations that the zeros of the Fredholm determinant associated with any homogeneous integral equation are the eigenvalues of the integral operator of that equation and are the only values of the Fredholm determinant for which the equation has solutions. Thus, in the upper half plane the zeros of $D^+(k)$, the Fredholm determinant associated with the physical solution, correspond to the eigenvalues of the Hamiltonian $H$. Physical considerations dictate that the Hamiltonian $H$ be Hermitian with respect to bounded solutions of Eq. (37). Therefore in the upper half plane the eigenvalues of $H$, and thus the zeros of $D^+(k)$, can occur only for values of $k^2$ which are real.

It follows that the zeros of $D^+(k)$ in the upper half of the complex $k$ plane must lie either on the real axis or on the positive imaginary axis. A negative energy bound state is defined as occurring when the following conditions are satisfied:

$$k = i\gamma \ (\gamma \text{ real}), \quad \psi(k,r) \sim e^{-\gamma r} \text{ as } r \to \infty$$

and

$$D^+(k) = 0 \ . \tag{38}$$

Since bounded solutions of Eq. (37) follow only from Eq. (38), a negative energy bound state corresponds to a zero of $D^+(k)$ at $k = i\gamma$, as is the case with a local potential. Also, for a local potential the possibility of a zero of $D^+(k)$ for $k$ real, not excluded by the above discussion, can be excluded on other grounds.[8] Since for a local potential $\mathfrak{L}^+(k) \neq 0$ for real $k \neq 0$,[4] it follows that for a local potential $D^+(k)$ must not be zero for real $k \neq 0$. On the other hand, the modified definition of $\mathfrak{L}^+(k)$ in the case of a nonlocal potential, given by Eq. (11), allows for a zero of $D^+(k)$ on the real $k$ axis as long as such a zero is accompanied by a corresponding zero of $D(k)$, resulting in a bound state in the continuum. It should be noted that since the eigenvalue of $H$ is $k^2$, zeros of $D^+(k)$ and of $D(k)$ on the real $k$ axis will occur in pairs, symmetric about the imaginary axis.

That such a zero of $D(k)$ is possible has been established by many examples.[8] In general, because the boundary conditions associated with the equation for the regular solution are not the same as those associated with the physical solution, it would not be expected that $D(k)$ would be zero for the same values of $k$ as is $D^+(k)$. Indeed, for a local potential $D(k)$ is unity at all zeros of $D^+(k)$. Thus, as mentioned earlier, a zero of $D(k)$ at the same value of $k$ as that for which $D^+(k)$ is zero is a special circumstance; usually zeros of $D(k)$ can be expected to occur at values of $k$ for which $D^+(k)$ is not zero.

To establish the character of $D(k)$ we note that the integral equation for the regular solution is a function of $k^2$. Changing $k$ to $-k$ in Eq. (6) yields

$$\varphi(-k,r) = k^{-1}\sin kr + \int_0^r \int_0^\infty G(k,r,r')V(r',s)\varphi(-k,s)ds\, dr, \tag{39}$$

where we have used the fact that

$$G(k,r,r') = G(-k,r,r') \ . \tag{40}$$

Equation (39) is the same as that with which we started. Therefore $D(k) = D(-k)$, which implies $D$ is a function of $k^2$. Thus if $D(k)$ is zero for $k = k_0$, where $k_0$ is real and positive, $D(k)$ is zero also for $k = -k_0$.

For $k$ complex, it follows that if $D(k)$ is zero for $k = \kappa + i\gamma$, then $D(k)$ is zero for $k = -\kappa - i\gamma$. Another symmetry property also exists with respect to zeros of $D(k)$ in the complex plane. Replacing $k$ in Eq. (6) by $k^*$ yields

$$\varphi(k^*,r) = (k^*)^{-1}\sin k^* r + \int_0^r \int_0^\infty G(k^*,r,r')V(r',s)\varphi(k^*,s)ds\, dr' \ . \tag{41}$$

Taking the complex conjugate of Eq. (41) gives

$$[\varphi(k^*,r)]^* = k^{-1}\sin kr + \int_0^r \int_0^\infty G(k,r,r')V(r',s)[\varphi(k^*,s)]^* ds\, dr' . \tag{42}$$

It follows that

$$[D(k^*)]^* = D(k) \ . \tag{43}$$

Therefore, if $D(k)$ is zero for $k = \kappa + i\gamma$, then $D(k)$ is also zero for $k = \kappa - i\gamma$. This, combined with the above result for zeros of $D(k)$ for complex $k$, establishes that zeros of $D(k)$ are symmetric with respect to both the real and imaginary $k$ axes.

Since $D(k)$ is a ratio of two polynomials of the same order, the number of zeros and number of poles in the entire complex plane are equal. We already have demonstrated that $D(k)$ will not have poles on the real axis. Its poles will be symmetrically distributed in the upper and lower half planes. If a pair of zeros of $D$ happen to occur on the real axis, then the number of poles in either of the half planes, excluding the real axis, exceeds the number of zeros by 1. If $2m$ zeros of $D(k)$ lie on the real axis symmetrically located about the imaginary axis, then in either half plane the number of poles exceeds the number of zeros by $m$, excluding the real axis.

## V. EXAMPLES

The behavior of the zeros and poles of $D^+(k)$ and $D(k)$ in the upper half plane in the case of a nonlocal potential will now be demonstrated using two examples of potentials designed to describe the nucleon-nucleon interaction. Both examples will make use of nonlocal potentials of the form

$$V(r,r') = \lambda g(r) g(r') \ . \tag{44}$$

### A. Potential with Yamaguchi form factor

In 1954 Yamaguchi[32] introduced a nonlocal potential of the form (44), with

$$g(r) = e^{-\alpha r} \ , \tag{45}$$

to describe nucleon-nucleon scattering. The expressions for $D^+(k)$ and $D(k)$ for this potential are given in Ref. 8. They can be written in the form

$$D^+(k) = N^+(k)/M^+(k) \ , \tag{46}$$

where

$$N^+(k) = 2\alpha k^2 + i4\alpha^2 k - (\lambda + 2\alpha^3) \tag{47}$$

and

$$M^+(k) = 2\alpha(k + i\alpha)^2 \ , \tag{48}$$

and

$$D(k) = N(k)/M(k) \ , \tag{49}$$

where

$$N(k) = 2\alpha k^2 - (\lambda - 2\alpha^3) \tag{50}$$

and

$$M(k) = 2\alpha(k - i\alpha)(k + i\alpha) \ . \tag{51}$$

From these expressions it is clear that the roots of $N^+(k)$ are

$$k_{1,2} = -i\alpha \pm \left[\frac{\lambda}{2\alpha}\right]^{1/2} \ . \tag{52}$$

No values of $\lambda,\alpha$ will make $D^+(k)$ zero for real values of $k$; therefore no continuum bound state can be associated with the Yamaguchi form factor. The roots of $N(k)$ are

$$k_{1,2} = \pm \left[\frac{\lambda}{2\alpha} - \alpha^2\right]^{1/2} \ . \tag{53}$$

Thus $D(k)$ can be zero for a wide range of values of $\lambda$ and $\alpha$. Although the values of $\lambda$ and $\alpha$ used by Yamaguchi do not generate a spurious state at any energy, if $\lambda > 2\alpha^3$ a spurious state will occur.

As pointed out earlier, both $D^+(k)$ and $D(k)$ must be dimensionless and must reduce to unity as $k$ becomes real and large. Thus both $D^+(k)$ and $D(k)$ must be ratios of polynomials of equal order. The zeros of $M^+(k)$ are

$$k_{1,2} = -i\alpha \ . \tag{54}$$

The zeros of $M(k)$ are

$$k_{1,2} = \pm i\alpha \ . \tag{55}$$

As predicted, the poles of $D^+(k)$ are confined to the lower half plane, while those of $D(k)$ are equally divided between the upper and lower half planes. This potential will not exhibit a bound state unless $\lambda$ is negative, and unless $|\lambda| > 2\alpha^3$. The zeros of $D^+(k)$ are otherwise restricted to the lower half of the $k$ plane. The zeros of $D(k)$, on the other hand, may be equidistant from the origin on either the negative and positive real axes or on the negative and positive imaginary axes.

From this example we see that a zero of $D(k)$ can occur on the real axis or in the upper half plane and that a pole of $D(k)$ can occur in the upper half plane.

### B. Beregi potential

Beregi[33] has suggested a nucleon-nucleon one-term separable nonlocal potential of the form given by Eq. (44) with

$$g(r) = e^{-\alpha_1 r} - a e^{-\alpha_2 r} \ . \tag{56}$$

This potential yields a continuum bound state at 259.3 MeV and a bound state at $-2.225$ MeV.[9] The parameters for the Beregi potential are

$$\lambda = -302.73 \text{ fm}^{-3} \ ,$$

$$\alpha_1 = 2.67 \text{ fm}^{-1} \ ,$$

$$\alpha_2 = 5.34 \text{ fm}^{-1} \ ,$$

$$a = 3.0854 \ .$$

The Fredholm determinants for this potential are given in Ref. 9. They can be written as

$$D^+(k) = N^+(k)/M^+(k) \ , \tag{57}$$

where

$$N^+(k) = k^4[2\alpha_1\alpha_2(\alpha_1+\alpha_2)] + ik^3(\alpha_1+\alpha_2)[4\alpha_1^2\alpha_2 + 4\alpha_1\alpha_2^2]$$
$$+ k^2[(\alpha_1+\alpha_2)(-2\alpha_1^3\alpha_2 - 2\alpha_1\alpha_2^3 - \lambda\alpha_2 - \lambda a^2\alpha_1 - 8\alpha_1^2\alpha_2^2) + 4\lambda a\alpha_1\alpha_2]$$
$$+ ik[(\alpha_1+\alpha_2)(-4\lambda a\alpha_1\alpha_2 - 2\lambda a^2\alpha_1^2 - 4\alpha_1^2\alpha_2^3 - 4\alpha_1^3\alpha_2^2 - 2\lambda\alpha_2^2) + 8\lambda a\alpha_1^2\alpha_2 + 8\lambda a\alpha_1\alpha_2^2]$$
$$+ [(\alpha_1+\alpha_2)(2\alpha_1^3\alpha_2^3 + \lambda\alpha_2^3 + \lambda a^2\alpha_1^3) - 4\lambda a\alpha_1^2\alpha_2^2] \quad (58)$$

and

$$M^+(k) = 2\alpha_1\alpha_2(\alpha_1+\alpha_2)(k+i\alpha_1)^2(k+i\alpha_2)^2 \ , \quad (59)$$

and

$$D(k) = N(k)/M(k) \ , \quad (60)$$

where

$$N(k) = k^4[2\alpha_1\alpha_2(\alpha_1+\alpha_2)] + k^2[(\alpha_1+\alpha_2)(2\alpha_1\alpha_2^3 + 2\alpha_1^3\alpha_2 - \lambda\alpha_2 - \lambda a^2\alpha_1) + 4\alpha_1\alpha_2\lambda a]$$
$$+ (\alpha_1+\alpha_2)[2\alpha_1^3\alpha_2^3 - \lambda\alpha_2^3 + 2\alpha_1\alpha_2\lambda a(\alpha_1+\alpha_2) - \lambda a^2\alpha_1^3] - 4\alpha_1^2\alpha_2^2\lambda a \quad (61)$$

and

$$M(k) = 2\alpha_1\alpha_2(\alpha_1+\alpha_2)(\alpha_1^2+k^2)(\alpha_2^2+k^2) \ . \quad (62)$$

The zeros of $N^+(k)$ and $M^+(k)$ are tabulated in Table I, while the zeros of $N(k)$ and $M(k)$ are tabulated in Table II. Again we find that $D^+(k)$ has no poles in the upper half plane, and has zeros in the upper half plane only on the positive imaginary axis (corresponding to the bound state at $-2.225$ MeV) or on the positive and negative real axes (corresponding to the continuum bound state at 259.3 MeV). The zeros of $D(k)$ in the upper half plane are on the positive and negative real axes (corresponding to the continuum bound state) and on the positive and negative imaginary axes. The poles of $D^+(k)$ [the zeros of $M^+(k)$] are confined to the lower half plane, while the poles of $D(k)$ [the zeros of $M(k)$] are equally distributed between the positive and negative imaginary axes.

## VI. CONCLUSIONS

Based on general considerations from functional analysis, we have demonstrated that for a nonlocal potential the behavior of the zeros of $D^+(k)$ is different than that for a local potential, since zeros can appear in pairs, one on the positive real axis and one on the negative real axis. With nonlocal potentials, as with local potentials, poles of $D^+(k)$ are confined to the bottom half of the complex plane. For a local potential the Fredholm determinant $D(k)$ is unity. For a nonlocal potential, $D(k)$ will have zeros and poles symmetrically distributed with respect to the origin, thus occurring in both the upper and lower halves of the complex plane.

Part of the motivation for studying the Fredholm determinants $D^+(k)$ and $D(k)$ for a nonlocal potential, rather than the Jost function $\mathfrak{L}^+(k)$, is the assertion that for a nonlocal potential $\mathfrak{L}^+(k)$ contains a less complete set of information about the nuclear system than does the combination of $D^+(k)$ and $D(k)$. This can be demonstrated in terms of the calculations presented in the examples. Using Eq. (11) for $\mathfrak{L}^+(k)$ this Jost function for the Yamaguchi potential is

$$\mathfrak{L}^+(k) = \frac{D^+(k)}{D(k)} = \frac{(k-i\alpha)[2\alpha k^2 + i4\alpha^2 k - (\lambda + 2\alpha^3)]}{(k+i\alpha)[2\alpha k^2 - (\lambda - 2\alpha^3)]} \ . \quad (63)$$

TABLE I. Zeros of $N^+(k)$ and $M^+(k)$.

| Zeros of $N^+(k)$ | Zeros of $M^+(k)$ |
|---|---|
| (1.77,0) | (0,−2.67) |
| (−1.77,0) | (0,−2.67) |
| (0,0.23) | (0,−5.34) |
| (0,−16.25) | (0,−5.34) |

TABLE II. Zeros of $N(k)$ and $M(k)$.

| Zeros of $N(k)$ | Zeros of $M(k)$ |
|---|---|
| (1.77,0) | (0,2.67) |
| (−1.77,0) | (0,−2.67) |
| (0,11.49) | (0,5.34) |
| (0,−11.49) | (0,−5.34) |

In forming the ratio $D^+(k)/D(k)$ the pole of $D^+(k)$ at $k=-i\alpha$ has canceled the pole of $D(k)$ at that same value of $k$.

The same situation occurs when using Eq. (11) to calculate $\mathcal{L}^+(k)$ for the Beregi potential. From Tables I and II it is clear that in forming $\mathcal{L}^+(k)$ the poles of $D^+(k)$ and $D(k)$ at $(0,-2.67)$ and $(0,-5.34)$ as well as the zeros of $D^+(k)$ and $D(k)$ at $(1.77,0)$ and $(-1.77,0)$ have canceled one another.

On this basis, we conclude that standard analyses of nuclear scattering processes may be incomplete if based directly on the analytic properties of Jost functions or the $S$ matrix. Due to the intrinsic nonlocality of many nuclear interactions, it seems advisable to study further the properties of the Fredholm determinants $D^{\pm}(k)$ and $D(k)$ in the complex plane. In particular, a fruitful line of investigation might be to understand the analytic properties of $\mathcal{L}^{\pm}(k)$ and $S(k)$ under circumstances in which the potential is known in principle to be nonlocal, but for which a local potential description of the nuclear process is known to be adequate empirically.


[1]J. Wheeler, Phys. Rev. 52, 1107 (1937).
[2]W. Heisenberg, Z. Phys. 120, 513 (1943).
[3]H. M. Nussenzveig, *Causality and Dispersion Relations* (Academic, New York, 1972).
[4]R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).
[5]R. Jost and A. Pais, Phys. Rev. 82, 840 (1951).
[6]H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958); 19, 287 (1962); 43, 410 (1967). See also H. Feshbach, in *Reaction Dynamics* (Gordon and Breach, New York, 1973), pp. 171–216.
[7]See Ref. 4, Chap. 12.
[8]B. Mulligan, L. G. Arnold, B. Bagchi, and T. O. Krause, Phys. Rev. C 13, 2131 (1976).
[9]B. Bagchi, T. O. Krause, and B. Mulligan, Phys. Rev. C 15, 1623 (1977).
[10]C. S. Warke and R. K. Bhaduri, Nucl. Phys. A162, 289 (1971).
[11]B. Bagchi, B. Mulligan, and S. B. Qadri, Phys. Rev. C 20, 1251 (1979).
[12]P. Swan, Proc. R. Soc. A228, 10 (1955).
[13]M. Coz, L. G. Arnold, and A. D. MacKellar, Ann. Phys. (N.Y.) 59, 219 (1970).
[14]L. G. Arnold and A. D. MacKellar, Phys. Rev. C 3, 1095 (1971).
[15]A. Martin, Nuovo Cimento 7, 607 (1958).
[16]T. O. Krause and B. Mulligan, Ann. Phys. (N.Y.) 94, 31 (1975).
[17]B. Bagchi and B. Mulligan, Phys. Rev. C 20, 1973 (1979); L. G. Arnold, B. Bagchi, and B. Mulligan (unpublished).
[18]M. Bertero, G. Talenti, and G. A. Viano, Nuovo Cimento 46, 337 (1966).
[19]Y. Singh and C. S. Warke, Can. J. Phys. 49, 1029 (1971).
[20]S. S. Ahmed, Nuovo Cimento 23A, 362 (1974).
[21]M. Gourdin and A. Martin, Nuovo Cimento 6, 757 (1957).
[22]F. Smithies, *Integral Equations* (Cambridge University, Cambridge, England, 1958).
[23]F. Riesz, Acta. Math. Acad. Sci. Hung. 41, 71 (1918).
[24]J. Schauder, Stud. Math. 2, 183 (1930).
[25]For a summary of Riesz-Schauder theory see, for example, K. Yosida, *Functional Analysis* (Springer, Berlin, 1974), pp. 279–286.
[26]F. Riesz and B. Sz-Nagy, *Functional Analysis* (Unger, New York, 1955), p. 206.
[27]M. Iwasaki and B. Mulligan, Bull. Am. Phys. Soc. 22, 1030 (1977); 23, 629 (1978); and (unpublished).
[28]T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1966), p. 158.
[29]J. Radon, Sitzungsber. Akad. Wiss. Wien, Math. Naturwiss. Kl. Abt. 2A: 128, 1083 (1919).
[30]B. Bagchi and B. Mulligan (unpublished).
[31]B. Mulligan and A. D. Wolfe (unpublished).
[32]Y. Yamaguchi, Phys. Rev. 95, 1628 (1954).
[33]P. Beregi, Nucl. Phys. A206, 217 (1973).