Bound and unbound particles in Griffin model state densities

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Errors in earlier derivations of the density of unbound particle-hole states in the Griffin model are corrected. New expressions for the densities of states accessible in the two-body interactions are also derived. State densities calculated with the modified equations are compared with results of exact counting. Particle emission spectra calculated within the Griffin model are essentially unchanged, but the equal *a priori* probability assumption, found to be violated in the earlier work, is verified to a high degree of accuracy when the revised state densities are used.

[NUCLEAR REACTIONS Rederived densities of unbound particle-hole states and two-body interaction rates in Griffin preequilibrium statistical model. Verified equal *a priori* probability assumption.]

I. INTRODUCTION

A pair of recent papers,[1,2] hereafter referred to as I and II, dealt with the phenomenology of continuum angular distributions. The second of these papers, II, described modifications to the Griffin (or exciton) preequilibrium model which allowed the calculated reaction spectra to be divided into their multistep direct (MSD) and multistep compound (MSC) components. In this context, MSD processes are ones in which the system passes exclusively through a series of unbound configurations. All processes involving bound states are MSC. The present report is designed as an addendum to II, correcting several points of physics in the derivation of the densities of unbound states and of the transition state densities. State densities calculated with the new formulas are compared with the results of exact counting for some simple cases. The implications of the changes for the reaction calculations are discussed, and the validity of the equal *a priori* probabilities assumption is reexamined.

II. DERIVATION OF STATE DENSITIES

A. Density of unbound states

The first difficulty in the derivation of the unbound state densities is one of bookkeeping in the quantity $A_{p-1,h}$. Its parent quantity $A_{p,h}$ appears in the total state density and is defined in II as

$$A_{p,h} = \frac{{p_m}^2}{g_0} - \frac{p^2 + h^2 + n}{4g_0} \quad . \tag{1}$$

Here $p_m = \max(p,h)$ and g_0 is the single-particle state density in the equispacing model. The quantities p and h are the numbers of particle and hole degrees of freedom, respectively, and $n = p + h$. The first term in (1) represents the minimum energy required for the configuration by the Pauli exclusion principle and includes the requirements of "passive" or spectator particles or holes at the Fermi surface. It is based on the fact that the total number of particles (active + passive) must equal the total number of holes. When one of the particles is removed from consideration by being constrained to be unbound, this is no longer true. Therefore, (1) should *not* be used to evaluate $A_{p-1,h}$ in the unbound state density as was done in II. Instead we replace it with the quantity

$$C_{p-i,h} = \frac{{p_m}^2 + (p_m - i)^2}{2g_0} - \frac{(p-i)^2 + h^2 + (n-i)}{4g_0} . \tag{2}$$

The change of $A_{p-1,h}$ to $C_{p-1,h}$ effectively reduces the unbound state density for the usual situation of $p > h$. While the effect of this change is relatively small in the early stages of the reaction, it becomes quite significant for the more complex states of the system populated in later stages.

It should here be pointed out that while (2) is to replace $A_{p-i,h}$ in the unbound state densities and in the transition state densities considered below, the quantity $A_{p-i,h}$ evaluated using (1) should continue to be used in calculating the total particle-hole state densities for the residual nuclei which are used in particle emission rates. In this case the reduction in p is due to particle emission and results in a shift of the Fermi level so that once again the total numbers of particles and holes (active + passive) are equal. Note also that $C_{p-i,h}$ reduces to $A_{p,h}$ when $i = 0$ so that $C_{p,h} \equiv A_{p,h}$.

In addition to this bookkeeping error, two errors dealing with distinguishability were made in II in the derivation of the density of unbound states for a given particle-hole configuration.

First, in introducing the quantities $e_m(p)$ and $e_{mu}(p)$ in II, it was implicitly assumed that the particle with the highest excitation energy could be identified. This is, in fact, not possible since the energies of individual excitons or degrees of freedom are not included in the bookkeeping of the Griffin model. The quantities $e_m(p)$ and $e_{mu}(p)$ should be replaced by $A_{p,h} - C_{p-1,h}$ and S, respectively. The average energy S required for a particle to be unbound is defined in II.

While the maximum energy particle degree of freedom cannot be designated, the unbound particle degrees of freedom can. This distinguishability between bound and unbound particles was neglected in the state density derivations of II. When it is included, the $p!$ in the denominator of the unbound state density is replaced by $1!\,(p-1)!$, thus effectively introducing a factor of p. Additional corrections are needed if multiply unbound states are possible. When all of this is taken into account, the unbound state density has much the same mathematical form as the finite well depth correction of Běták and Dobeš.[3]

The final expression for the density of unbound states containing p particle and h hole degrees of freedom and excitation energy E may now be written in a somewhat simplified form as

$$\omega^{(u)}(p,h,E) = \frac{g_p(p)g_u^{n-1}(p)(E - C_{p-1,h} - S)^{n-1}}{(p-1)!h!(n-1)!} F(p) \quad . \tag{3}$$

For $h \leq 2$ we have

$$F(p) = \frac{1}{p} \sum_{i=1}^{p} \sum_{j=0}^{h} (-1)^{i+j+1} \begin{bmatrix} p \\ i \end{bmatrix} \begin{bmatrix} h \\ j \end{bmatrix} \Theta(E - iS - jV) \left[\frac{E - iS - jV}{E - S} \right]^{n-1} , \tag{4}$$

while for $h > 2$ only the $j = 0$ terms are included. In these equations, g_p and g_u are effective single-particle state densities for the unbound particle and the remaining excitons, respectively, and are defined in II. The potential well depth V is measured from the Fermi surface. Finally, Θ is the heaviside function which is zero for a negative argument and unity for a positive one. Note that $F(p)$ has a different meaning than in II, although in both cases it contains the finite well depth corrections.

B. Transition state densities

The rates for the various residual two-body interactions depend on the average density of final states accessible from a single initial state. The final-state densities are designated by the subscripts $+$, 0, and $-$ denoting pair creation, exciton scattering, and pair annihilation, respectively. The superscripts u and b are used to denote the unbound or bound character of the initial and final states. Thus $\omega_+^{(ub)}(p,h,E)$ denotes the average final-state density for pair creation from unbound states specified by p, and h, and E to bound states specified by $p + 1$, $h + 1$, and E.

The same errors which occurred in II in the derivation of the unbound state densities occurred also in the derivation of the transition state densities, except that the distinguishability of bound from unbound states was included in the probability of having the necessary excitons to initiate the transition. Replacing e_m and e_{mu}, considering the distinguishability of bound and unbound particles consistently, and using $C_{p-i,h}$ in place of $A_{p-i,h}$ lead to new expressions for the transition state densities. As in II, the quantities $x_1(p) = E - A_{p,h}$, $x_2(p) = E - C_{p-1,h} - S$, and $x_3(p) = (E - C_{p-2,h} - 2S)\,\Theta(E - C_{p-2,h} - 2S)$ are used as an abbreviated notation in expressing the transition state densities. The effective single-particle state densities g, g_u, and g_p are evaluated as in II.

The final-state density for pair creation from unbound to unbound states is now

$$\omega_+^{(uu)}(p,h,E) = \frac{g_a(p+1)g_u{}^2(p+1)}{2n} \frac{x_2^{n+1}(p+1)\left[\frac{1}{n}F_+(p+1)+\frac{n-1}{n}f(p+1)\right]}{x_2^{n-1}(p)}$$

$$+ \frac{m(p)}{1-m(p)}\omega_+^{(ub)}(p,h,E) \ . \tag{5}$$

Here $f(p)$ is the finite well depth correction for the total state density defined in II. The probability $m(p)$ that an unbound state will have two or more unbound particles here takes the form

$$m(p) = \frac{\sum_{i=2}^{p}\sum_{j=0}^{h}(-1)^{i+j}\binom{p}{i}\binom{h}{j}\Theta(E-iS-jV)(E-iS-jV)^{n-1}}{\sum_{i=1}^{p}\sum_{j=0}^{h}(-1)^{i+j+1}\binom{p}{i}\binom{h}{j}\Theta(E-iS-jV)(E-iS-jV)^{n-1}} \quad \text{for } h \leq 2 \ , \tag{6}$$

while for $h > 2$ only the $j = 0$ terms are included. The function $F_+(p)$ is

$$\begin{aligned} F_+(p) &= 1 - \left[\frac{E-S-V}{E-S}\right]^{n-1}\Theta(E-S-V) - \frac{1}{2}\left[\frac{E-2S}{E-S}\right]^{n-1}\Theta(E-2S) \\ &\quad + \frac{1}{2}\left[\frac{E-2S-V}{E-S}\right]^{n-1}\Theta(E-2S-V) \quad \text{for } h \leq 2 \\ &= 1 - \frac{1}{2}\left[\frac{E-2S}{E-S}\right]^{n-1}\Theta(E-2S) \quad \text{for } h > 2 \ . \end{aligned} \tag{7}$$

The remaining final-state densities for pair creation are

$$\begin{aligned} \omega_+^{(ub)}(p,h,E) = \frac{1-m(p)}{2n(n+1)}\Bigg[& g^3(p+1)f(p+1)\left\{\frac{n^2}{2}[x_1(p+1)-x_2(p)]^2 + \frac{n}{2}[x_1{}^2(p+1)-x_2{}^2(p)]\right. \\ & \left. + x_1(p+1)x_2(p)\right\} \\ & - 2g_p(p+1)g_u{}^2(p+1)F_+(p+1)\frac{x_2^{n+1}(p+1)}{x_2^{n-1}(p)}\Bigg] \ , \end{aligned} \tag{8}$$

$$\begin{aligned} \omega_+^{(bu)}(p,h,E) = {} & \frac{g_p(p+1)g_u{}^2(p+1)h}{x_1^{n-1}(p)f(p) - p\,x_2^{n-1}(p)F(p)} \\ & \times \Bigg[\frac{x_2^{n+1}(p+1)-px_3^{n+1}(p+1)}{2n(n+1)} - \frac{1}{4n(n+1)} \\ & \times \{G_1^{n-1}[n(n-1)G_1{}^2 - 2(n+1)(n-1)G_1x_2(p+1) + n(n+1)x_2{}^2(p+1)] \\ & - pG_2^{n-1}[n(n-1)G_2{}^2 - 2(n+1)(n-1)G_2x_3(p+1) + n(n+1)x_3{}^2(p+1)]\}\Bigg] \end{aligned} \tag{9}$$

with $G_1 = (E-V)\,\Theta(E-V)$ and $G_2 = (E-S-V)\,\Theta(E-S-V)$ as in II, and finally,

$$\omega_{+}^{(bb)}(p,h,E) = \frac{1}{2n(n+1)} \frac{1}{x_1^{n-1}(p)f(p) - px_2^{n-1}(p)F(p)}$$
$$\times \Bigg[ng^3(p+1)x_1^{n+1}(p+1)f(p+1) - p(n-1)g_u{}^3(p+1)x_2^{n+1}(p+1)F(p+1)$$
$$- pg^3(p+1)x_2^{n-1}(p)F(p) \Bigg\{ \frac{n^2}{2}[x_1(p+1) - x_2(p)]^2$$
$$+ \frac{n}{2}[x_1{}^2(p+1) - x_2{}^2(p)] + x_1(p+1)x_2(p) \Bigg\} \Bigg]$$
$$- \omega_{+}^{(bu)}(p,h,E) \quad . \tag{10}$$

For exciton scattering the final-state densities are

$$\omega_0^{(ub)}(p,h,E) = \frac{[1-m(p)]g_0{}^2}{n} \Bigg\{ \frac{p+2h-1}{2} n[x_1(p) - x_2(p)]f(p)$$
$$+ (p-1)x_2(p)[f(p) - 2F_0(p)] \Bigg\} \tag{11}$$

with

$$F_0(p) = 1 - \frac{1}{2}\left[\frac{E-2S}{E-S}\right]^{n-1} \Theta(E-2S) \quad , \tag{12}$$

and

$$\omega_0^{(bu)}(p,h,E) = \frac{g_u(p)g_p(p)}{2n} \frac{p}{x_1^{n-1}(p)f(p) - px_2^{n-1}(p)F(p)}$$
$$\times \big((p+2h-1)\{2x_2^n(p) + n[x_1(p) - x_2(p)]x_2^{n-1}(p)\}f(p) - 4(n-1)x_2^n(p)F(p)$$
$$- (p-1)(p+2h-1)\{2x_3^n(p) + n[x_2(p) - x_3(p)]x_3^{n-1}(p)\}F(p)$$
$$+ 4(p-1)(n-2)x_3^n(p)F_u(p)\big) \tag{13}$$

with the correction function for doubly unbound states given by

$$F_u(p) = \frac{2F(p)\,m(p)}{(p-1)}\left[\frac{E-S}{E-2S}\right]^{n-1} \quad \text{for } p \geq 2 \text{ and } E > 2S \quad , \tag{14}$$

and is unity otherwise. The pair annihillation transition state densities are

$$\omega_{-}^{(ub)}(p,h,E) = [1-m(p)]\, g_h h(h-1)/2 \tag{15}$$

as in II and

$$\omega_{-}^{(bu)}(p,h,E) = \frac{g_p(p-1)hp(p-1)}{4} \frac{1}{x_1^{n-1}(p)f(p) - px_2^{n-1}(p)F(p)}$$

$$\times \{ x_2^{n-3}(p-1)[(n-2)(n-3)x_2^2(p-1) - 2(n-1)(n-3)x_2(p-1)x_1(p)$$

$$+ (n-1)(n-2)x_1^2(p)]f(p)$$

$$- (p-2)x_3^{n-3}(p-1)[(n-2)(n-3)x_3^2(p-1) - 2(n-1)(n-3)x_3(p-1)x_2(p)$$

$$+ (n-1)(n-2)x_2^2(p)]F(p)$$

$$- 4[x_2^{n-1}(p)F(p) - (p-2)x_3^{n-1}(p)F_u(p)] \} \quad . \tag{16}$$

III. COMPARISONS WITH EXACT COUNTING

Because the density of unbound states is a much larger fraction of the total in this work than in II, the calculations of the transition state densities sometimes involve finding small differences between large numbers. It thus becomes vital that the correction functions be handled appropriately. For this reason state densities calculated using the revised expressions have been compared with values obtained from the direct counting of states in a few simple cases.

The sample system chosen has $g_0E = 21$. All calculations are done in the strict equispacing model with the effective g values assumed equal to g_0. The total number of states for this case, $\omega(p,h,E)/g_0$, is found from exact counting to be 90 for $(p,h) = (2,1)$ and 9 for $(p,h) = (2,0)$. The formula from II yields values of 90.25 and 9.25, respectively, in excellent agreement with the exact counting numbers. On the other hand, the results of II seriously underestimate the densities of unbound states. The discrepancy amounts to nearly a factor of 2 for (2,0) states with g_0S ranging from 2 – 14 and for (2,1) states with $g_0S = 8$ and $g_0V = 4$ to 12. For an infinite well depth, the agreement for (2,1) states improves as S decreases and the unbound state density approaches the total one. By contrast, the revised expressions of this work in all cases agree with exact counting to within 15%, and the general level of agreement is 2% or better.

For the transition state density calculations, g_0S was fixed at 5, and it was assumed that $g_0V > 21$. The results obtained from (5) to (16) agree quite well with the numbers of states found from exact counting. The worst discrepancy is about 17%, and in general the agreement is to within 10% or less.

IV. REACTION MODEL CALCULATIONS

The revised equations given here have been incorporated into the computer code PRECO – D, and calculations have been run to test the effect of the changes on actual reaction calculations. As in II, results are shown for protons incident on ^{54}Fe to form a ^{55}Co composite nucleus at 43.3 MeV of excitation. All input quantities are unchanged from II.

State densities. The total and unbound state den-

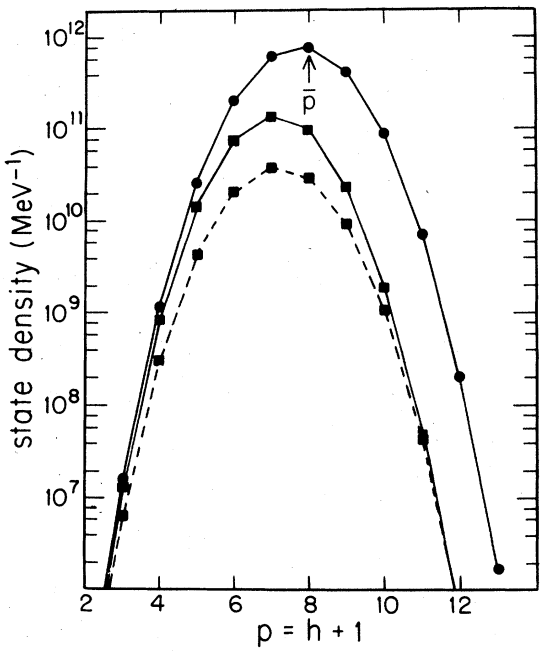

FIG. 1. State densities for ^{55}Co at 43.3 MeV of excitation as a function of particle and hole number. Circles and squares denote, respectively, the total and unbound state densities. The dashed line gives the corresponding uncorrected unbound state densities from II.

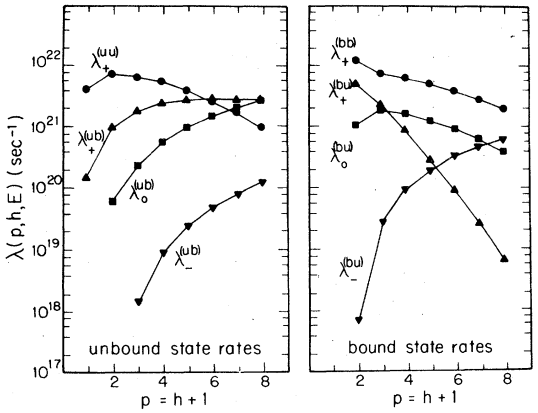

FIG. 2. Average two-body interaction rates for ^{55}Co at 43.3 MeV of excitation as a function of the particle and hole numbers of the initial state in the interaction. (Compare with Fig. 3 in II.)

sities are shown in Fig. 1 along with the unbound state densities of II. It is clear that while the number of unbound states has increased, the peak of the distribution falls at about the same particle number as in II, and qualitatively things are quite similar.

Quantitatively we find that there are several competing effects. The factor of p due to the distinguishability of the unbound particle increases the unbound state density. At low particle numbers this is partially offset by the factor $F(p)$ which contains corrections to the statistical factors due to multiply unbound states, and is less than unity. At higher particle numbers $F(p)$ approaches unity, but the replacement of $A_{p-1,h}$ and $C_{p-1,h}$ becomes an important factor in reducing the unbound state density. In the present case, it just compensates for the extra factor of p around $p = 11$ to 12.

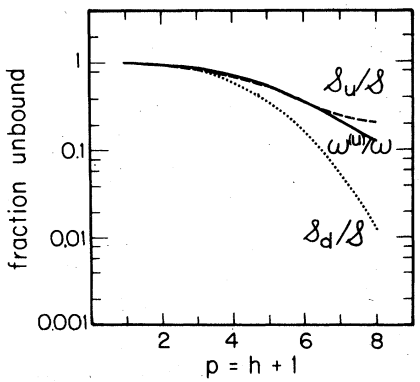

FIG. 3. Fraction of the strength in unbound states and fraction which is MSD compared with the fraction of the states which are unbound. Calculations are for ^{54}Fe + $p \rightarrow$ ^{55}Co at 43.3 MeV of excitation. (Compare with top part of Fig. 5 in II.)

Transition rates. The values for the transition rates obtained using (5)—(16) are shown in Fig. 2. The results are qualitatively similar to those from II. The biggest changes are an increase in $\omega_0^{(bu)}$ at low particle number and a decrease in $\omega_-^{(bu)}$ at high particle number.

Equal probabilities assumption. Figure 3 shows the fraction $\mathcal{S}_u/\mathcal{S}$ of the reaction strength which passes through unbound states at different particle-hole numbers compared with the expected ratio, $\omega^{(u)}/\omega$, if the equal *a priori* probabilities assumption for all states of a given class is valid. Whereas the corresponding figure in II indicated a serious violation of the assumption, the present work shows it to have a quite remarkable degree of validity. The violation of the assumption was, in II, attributed to the fact that unbound states preferentially populate unbound states in pair creation while bound states prefer other bound states. This is still true, but the degree of enhancement is now much less for unbound initial states than for bound ones, just oppo-

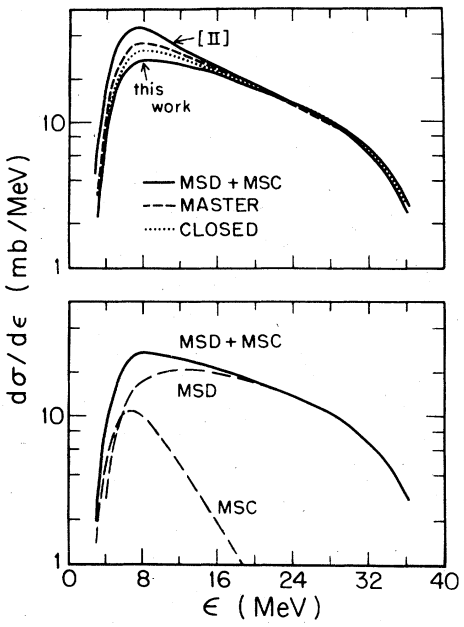

FIG. 4. Calculated preequilibrium proton spectra for ^{54}Fe + $p \rightarrow$ ^{55}Co at 43.3 MeV of excitation. In the top half of the figure, the two solid curves compare the present results obtained in the MSD/MSC formalism with the corresponding results from II. Also shown are the closed form and master equation results from an earlier formalsim where the MSD/MSC distinction was not made. The lower half of the figure shows the present preequilibrium spectrum separated into its MSD and MSC components and may be compared with the lower half of Fig. 6 in II.

site from the situation in II. Thus in the crucial initial stages of the reaction, unbound states, which are numerous, will slightly overpopulate unbound states of the next degree of complexity while bound states, which are much scarcer, will seriously overpopulate bound states. The two effects largely cancel. Finally, since the proportion of unbound states is greater, both the fraction of the strength which is unbound and the MSD fraction $\mathcal{S}_d/\mathcal{S}$ are larger here than in II.

Particle spectra. Figure 4 shows the calculated preequilibrium components of the proton spectrum. Whereas in II the MSD/MSC calculation showed *more* low energy particles than the closed form and master equation results of earlier formalisms, the present corrected calculation shows *fewer* low energy particles. This reduction occurred as strength was shifted from the MSC preequilibrium component to the MSC equilibrium component which has a similar spectral shape. The total (preequilibrium + equilibrium) MSC component and the MSD component are virtually unchanged from II, as is the total spectrum calculated. Only the division of the cross section into preequilibrium and equilibrium parts has been altered.

V. SUMMARY AND CONCLUSIONS

The unbound state densities and transition state densities have been corrected in three areas: (1) the previously assumed ability to designate the maximum energy particle has been removed, (2) the distinguishability of unbound from bound particles has been consistently included, and (3) a corrected expression for $A_{p-i,h}$ has been employed and designated $C_{p-i,h}$.

The results of these changes are an increase in the proportion of unbound states in the system, relatively little shift in the two-body interaction rates, and a restoration of the equal probabilities assumption. Although the fraction of the cross section which is preequilibrium is somewhat decreased, the MSD component and the total spectrum are nearly unchanged. As was pointed out in II, the MSD/MSC division of the cross section is less sensitive to the vagaries of the method used in calculation than the preequilibrium/equilibrium division. Its stability in the present case means that the analysis of the systematics of continuum angular distributions in I should be unaffected by the results of this work.

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